On Assigning Referees to Tournament Schedules

Jeffrey H. Dinitz
Mathematics and Statistics
University of Vermont
Burlington VT,
USA 05405

Douglas R. Stinson
School of Computer Science
University of Waterloo
Waterloo ON,
Canada N2L 3G1

Abstract

In this note, we discuss a tournament scheduling problem that connects Room squares and balanced tournament designs. The problem involves assigning referees to tournament schedules, and it can be solved using certain types of Room squares called "maximum empty subarray Room squares". We also point out that these Room squares can be "gravity-transformed" into balanced tournament designs.

1 Introduction and Background

We begin with some definitions. Let n be an odd integer and let S be a set of size $n+1$ called *symbols*. A *Room square* of side n (denoted RS(n)) based on symbol set S is an $n \times n$ array, F, which satisfies the following properties:

1. every cell of F either is empty or contains an unordered pair of symbols from S,

2. every symbol $x \in S$ occurs once in every row and once in every column of F,

3. every unordered pair of symbols occurs in exactly one cell of F.

Figure 1: A Room square of side 7

01		45	67			23
57	02				13	46
	56	03	12		47	
	37		04	26		15
36	14	27		05		
24			35	17	06	
		16		34	25	07

It is known that a Room square of side n exists if and only if n is an odd integer, $n \geq 1$, $n \neq 3, 5$. See Mullin and Wallis [5] for a proof. For additional information about Room squares and related structures, see Dinitz and Stinson [2] or *The CRC Handbook of Combinatorial Designs* [1]. A Room square of side 7 is given in Figure 1.

A Room square of side n, say F, can be used to schedule a round robin tournament for $n+1$ teams. The rows of F are indexed by n playing fields, and the columns of F are indexed by n rounds. The round robin tournament satisfies the following properties:

1. every team plays once on every field and every team plays once in every round,

2. every pair of teams plays together exactly once during the tournament.

It is easily seen that there are exactly $(n+1)/2$ games in every round. Therefore we need at least $(n+1)/2$ referees so that every game has a referee assigned to it. In order to eliminate possible bias of referees, we would like to assign referees to games in such a way that every team receives each referee roughly the same number of times. More precisely, for every team T and for every referee R, it should be the case that R is assigned to exactly one or two games involving team T. A Room square for which referees can be assigned in this way will be called a *referee-minimal Room square*, denoted RMRS(n).

Given a Room square F of side n, a *column-transversal* in F is a set of n filled cells with the property that no two cells are in the same column and no symbol occurs more than twice in these cells. F will be an RMRS(n) if and only if there exists a set of $(n+1)/2$ disjoint column-transversals in F (each column transveral corresponds to an assigned referee).

Referee-minimal Room squares have a nice three-dimensional interpretation. This interpretation makes use of structures called "balanced tournament designs", which we define now.

Let m be an integer and let S be a set of size $2m$ called *symbols*. A *balanced tournament design* of order m (denoted BTD(m)) based on symbol set S is an $m \times (2m-1)$ array, G, which satisfies the following properties:

1. every cell of G contains an unordered pair of symbols from S,
2. every symbol $x \in S$ occurs either once or twice in every row of G, and once in every column of G,
3. every unordered pair of symbols occurs in exactly one cell of G.

It is known that a balanced tournament design of order m exists if and only if m is a positive integer such that $m \neq 2$. See Schellenberg, van Rees and Vanstone [6] for a proof of this fact. As well, a brief survey of balanced tournament designs can be found in [1, pp. 238–241].

Now, it is not hard to see that an RMRS(n) is equivalent to a three-dimensional "brick", having dimensions $n \times n \times \frac{n+1}{2}$, that satisfies certain conditions. Suppose that we think of the three dimensions of the brick as corresponding to fields, rounds, and referees, respectively. If we collapse the third dimension (i.e., project onto the first two dimensions), then we obtain a Room square of side n. If we collapse the first dimension, then we obtain a BTD($(n+1)/2$).

2 A Construction for Referee-minimal Room Squares

We now describe a method of constructing RMRS(n) for almost all odd integers $n \geq 9$. We make of use a special type of Room square called a *maximum empty subarray Room square*, denoted MESRS(n), which was first defined by Stinson [7]. An MESRS(n) is an RS(n) containing an $\frac{n-1}{2} \times \frac{n-1}{2}$ subarray of empty cells. We present an MESRS(9) in Figure 2.

Figure 2: A maximum empty subarray Room square of side 9

37					28	59	$4X$	16
	56				$1X$	47	29	38
		$2X$			67	18	35	49
			48		39	26	17	$5X$
				19	45	$3X$	68	27
12	$8X$	57	69	34				
46	13	89	$7X$	25				
58	79	14	23	$6X$				
$9X$	24	36	15	78				

It is not hard to see that the rows and columns of an MESRS(n), say F, can be permuted so that F has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A has dimensions $\frac{n+1}{2} \times \frac{n+1}{2}$, B has dimensions $\frac{n+1}{2} \times \frac{n-1}{2}$, C has dimensions $\frac{n-1}{2} \times \frac{n+1}{2}$, D has dimensions $\frac{n-1}{2} \times \frac{n-1}{2}$, B and C are filled, D is empty, and the only filled cells in A are the diagonal cells. An MESRS(n) that is displayed in this fashion is said to be in *standard form*. Note that the MESRS(9) in Figure 2 is in standard form.

In constucting RMRS(n), we will make use an existence result for MESRS which is found in Lamken [3]. (Theorem 2.1 follows directly from [3] and the fact that an MESRS($2n-1$) is equivalent to a special type of BTD(n) called a *partitioned* balanced tournament design; see [4].)

Theorem 2.1 *Suppose $n \geq 9$ is an odd integer, and $n \neq 17, 21, 29$. Then there exists an MESRS(n).*

Now, suppose we have an MESRS(n) in standard form. Let $m = (n+1)/2$. Suppose that the rows of B are denoted $B_1, \ldots, B_m$, and denote the rows of C by $C_1, \ldots, C_{m-1}$. Let the diagonal of A be denoted A_d. We now assign referees for the games. For $1 \leq i \leq m-1$, referee R_i is assigned to all the games in the cells in $B_i \cup C_i$. Referee R_m is assigned to the games in the cells in $A_d \cup B_m$.

We show that this assignment of referees to games yields an RMRS(n). Every set of cells C_i contains every team exactly once, and every set of cells B_i contains every team at most once. Therefore referees $R_1, \ldots, R_{m-1}$ are assigned to each team either once or twice. In addition, it is not hard to see that the set of cells A_d contains every team exactly once, and hence the desired property holds also for referee R_m. Hence, we have proven the following.

Theorem 2.2 *Suppose $n \geq 9$ is an odd integer, and suppose $n \neq 17, 21, 29$. Then there exists an RMRS(n).*

3 Referee Field Changes

A round robin tournament based on a Room square is set up so that every team plays on a different field during each round. However, there may be no reason why the referees should be required to change fields so often. On the contrary, it might be desirable for the referees to change fields as infrequently as possible.

Here is a small example to illustrate.

Example 3.1 *Consider the RMRS(9) constructed as we have described in Section 2. Referees R_1, R_2, R_3 and R_4 each change fields once, and referee R_5 changes fields four times. The total number of field changes is therefore eight. We will show a bit later that the total number of referee field changes in this example is small as posssible.* □

In general, if we construct an RMRS(n) as described in Section 2, then the total number of referee field changes is $n-1$. We now show that it is impossible to construct an RMRS(n) in which the total number of referee field changes is less than $n-1$.

Theorem 3.2 *For any RMRS(n), the total number of referee field changes is at least $n-1$.*

Proof: For $1 \leq i \leq n$, let f_i denote the number of referees that are assigned to at least one game on the ith playing field. If $f_i = f_j = 1$ for some $i \neq j$, then the referee assigned to field i must be different from the referee assigned to field j (otherwise a referee would be assigned to more than n games, an impossibility). Hence, there at most $(n+1)/2$ fields having $f_i = 1$ and at least $(n-1)/2$ fields having $f_i \geq 2$. Consequently,

$$\sum_{i=1}^{n} f_i \geq \frac{n+1}{2} + 2 \times \frac{n-1}{2} = \frac{3n-1}{2}.$$

Now, it is not difficult to see that the total number of field changes is

$$\sum_{i=1}^{n} f_i - \frac{n+1}{2} \geq \frac{3n-1}{2} - \frac{n+1}{2} = n-1.$$

This completes the proof. □

Summarizing, we obtain the following result about referee field changes in RMRS(n).

Theorem 3.3 *Suppose $n \geq 9$ is an odd integer, and suppose $n \neq 17, 21, 29$. Then there exists an RMRS(n) in which the total number of referee field changes is equal to $n-1$. Moreover, for any odd integer n, there does not exist an RMRS(n) in which the total number of referee field changes is less than $n-1$.*

4 Gravity-transformed Room Squares

Suppose that F is a Room square of side n. Think of lifting F so that it is vertical, and imagine that the contents of all the filled cells are affected by a gravitational force. Then the cells in the bottom $(n+1)/2$ rows are completely filled, and the cells in the the top $(n-1)/2$ rows are all empty. The $\frac{n+1}{2} \times n$ array of filled cells in this structure will be called a *gravity-transformed Room square*, which we denote by GTRS(n). A gravity-transformed Room square of side 9, derived from the RS(9) from Figure 2, is presented in Figure 3.

It is not hard to see that the GTRS(9) presented in Figure 3 is in fact a BTD(5). This is no accident, in view of the following theorem, which we state without proof.

Theorem 4.1 *Suppose F is an MESRS(n) in standard form, and let G be the GTRS(n) derived from F. Then G is a BTD($(n+1)/2$).*

Hence, for every odd integer $n \geq 9$, (except possibly $n = 17, 21, 29$) there is a Room square of side n which can be gravity-transformed into a balanced tournament design.

Figure 3: A gravity-transformed Room square of side 9

37	56	$2X$	48	19	28	59	$4X$	16
12	$8X$	57	69	34	$1X$	47	29	38
46	13	89	$7X$	25	67	18	35	49
58	79	14	23	$6X$	39	26	17	$5X$
$9X$	24	36	15	78	45	$3X$	68	27

5 An Open Problem

We mentioned a three-dimensional interpretation of RMRS(n) in Section 1. In this interpretation, we have a three-dimensional brick such that one two-dimensional projection yields a Room square and another two-dimensional projection yields a balanced tournament design. It is conceivable that the third two-dimensional projection could also be a balanced tournament design; however, we do not have any examples where this occurs. Thus we pose the following open problem.

> For which odd integers n does there exist a three-dimensional brick B having dimensions $n \times n \times \frac{n+1}{2}$, such that every two-dimensional projection of B is either a BTD($(n+1)/2$) or an RS(n)?

The existence of such a brick is equivalent to the existence of a Room square of side n which contains $(n+1)/2$ disjoint transversals, where each transversal consists of n filled cells with the property that no two cells are in the same row or column and no symbol occurs more than twice in these cells.

Acknowledgements

D. R. Stinson's research is supported by the Natural Sciences and Engineering Research Council of Canada through the grant NSERC-RGPIN #203114-02.

References

[1] C. J. Colbourn and J. H. Dinitz, eds. *The CRC Handbook of Combinatorial Designs*, CRC Press, Inc., 1996.

[2] J. H. Dinitz and D. R. Stinson. Room squares and related designs. In *Contemporary Design Theory: A Collection of Surveys*, John Wiley & Sons, Inc., 1992, pp. 137–204.

[3] E. R. Lamken. A few more partitioned balanced tournament designs. *Ars Combinatoria* **43** (1996), 121–134.

[4] E. R. Lamken and S. A. Vanstone. Balanced tournament designs and related topics. *Discrete Math.* **77** (1989), 159–176.

[5] R. C. Mullin and W. D. Wallis. The existence of Room squares. *Aequationes Math.* **13** (1975), 1–7.

[6] P. J. Schellenberg, G. H. J. van Rees and S. A. Vanstone. The existence of balanced tournament designs. *Ars Combinatoria* **3** (1977), 303–318.

[7] D. R. Stinson. Room squares with maximum empty subarrays. *Ars Combinatoria* **20** (1985), 159–166.