Computing transverse t-designs

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Abstract

In this paper, we develop a computational method for constructing transverse t-designs. An algorithm is presented that computes the G-orbits of k-element subsets transverse to a partition $\mathcal{H}$, given that an automorphism group G is provided. We then use this method to investigate transverse Steiner quadruple systems. We also develop recursive constructions for transverse Steiner quadruple systems, and we provide a table of existence results for these designs when the number of points $v \leq 24$. Finally, some results on transverse t-designs with $t > 3$ are also presented.

1 Introduction

Given a partition $\mathcal{H} = \{H_1, H_2, \ldots, H_r\}$ of a set X, we say that a subset $T \subseteq X$ is *transverse* with respect to $\mathcal{H}$ if $|T \cap H_i| = 0$ or 1 for each $i = 1, 2, \ldots, r$. A *transverse t-design* with parameters t-(v, k, λ) is a triple $(X, \mathcal{H}, \mathcal{B})$ such that the following properties are satisfied:

1. X is a v-element set of *points*,
2. $\mathcal{H} = \{H_1, H_2, \ldots, H_r\}$ is a partition of X into subsets called *holes*, and
3. $\mathcal{B}$ is a collection of k-element subsets called *blocks*, each of which is transverse with respect to $\mathcal{H}$, such that every transverse t-element subset of points is in exactly λ blocks.

A transverse 2-$(v, k, 1)$ design is also called a *group divisible design*.

A transverse 3-$(12, 4, 1)$ design having holes $\mathcal{H} = \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{a, b, c, d\}\}$ is displayed in Figure 1.

Let $h_i = |H_i|$ be the size of the hole $H_i \in \mathcal{H}$. The *type* of a transverse t-design is the multi-set $\{h_1, h_2, \ldots, h_r\}$ of hole sizes. It is customary to write ${s_1}^{u_1}{s_2}^{u_2} \ldots {s_m}^{u_m}$ for the type of a transverse t-design with u_i holes of size s_i, $i = 1, 2, \ldots, m$. If all the holes have the same size, h, then the transverse t-design is said to be *uniform*. Such a design has type h^u for some u.

$\{3,5,7,a\}$	$\{3,5,6,b\}$	$\{3,4,7,d\}$	$\{3,4,6,c\}$	$\{2,5,7,c\}$	$\{2,5,6,d\}$	$\{2,4,7,b\}$	$\{2,4,6,a\}$
$\{1,5,7,b\}$	$\{1,5,6,c\}$	$\{1,4,7,a\}$	$\{1,4,6,d\}$	$\{1,3,7,c\}$	$\{1,3,6,a\}$	$\{1,3,5,d\}$	$\{1,3,4,b\}$
$\{1,2,7,d\}$	$\{1,2,6,b\}$	$\{1,2,5,a\}$	$\{1,2,4,c\}$	$\{0,5,7,d\}$	$\{0,5,6,a\}$	$\{0,4,7,c\}$	$\{0,4,6,b\}$
$\{0,3,7,b\}$	$\{0,3,6,d\}$	$\{0,3,5,c\}$	$\{0,3,4,a\}$	$\{0,2,7,a\}$	$\{0,2,6,c\}$	$\{0,2,5,b\}$	$\{0,2,4,d\}$

Figure 1: The 32 blocks of a transverse 3-$(12,4,1)$ design of type $2^4 4^1$, with holes $\{0,1\},\{2,3\},\{4,5\},\{6,7\},\{a,b,c,d\}$

The transverse t-(v,k,λ) designs of type 1^v are the the (ordinary) t-designs. A transverse t-$(v,k,1)$ design is a *transverse Steiner system*. A *transverse Steiner triple system* (or *transverse* STS) of type $h_1h_2\cdots h_r$ is a transverse 2-$(v,3,1)$ design of type $h_1h_2\cdots h_r$, and a *transverse Steiner quadruple system* (or *transverse* SQS) of type $h_1h_2\cdots h_r$ is a transverse 3-$(v,4,1)$ design of type $h_1h_2\cdots h_r$.

The remainder of this paper is organized as follows. Section 2 gives some basic constructions for transverse designs using other types of designs. In Section 3, we discuss how to construct transverse t-designs with a given automorphism group. In order to do this, an algorithm is required that finds the orbits of transverse subsets under the automorphism group. In Section 4, we focus on transverse SQS; we review old results and we give several new constructions for these designs. In Section 4.4, we give a table of existence results for transverse SQS on at most 24 points. Finally Section 5 concludes the paper.

2 Some basic constructions

A (uniform) transverse t-(kh,k,λ) design of type h^k is equivalent to an *orthogonal array* of order h, strength t, index λ and degree k, denoted $\mathrm{OA}_\lambda(t,k,h)$. It is well-known that an $\mathrm{OA}_1(t,t+1,h)$ exists for all t and h. Hence, we have the following result as a consequence.

Theorem 2.1 *There is a transverse t-$((t+1)h,t+1,1)$ design of type h^{t+1} for all integers $h\geq 1$ and $t\geq 2$.*

Proof: Define $X=\mathbb{Z}_h\times\{1,\ldots,t+1\}$, $\mathcal{H}=\{\mathbb{Z}_h\times\{i\}:1\leq i\leq t+1\}$, and

$$\mathcal{B}=\{\{(a_1,1),\ldots,(a_{t+1},t+1)\}:a_1+\cdots+a_{t+1}\equiv 0 \pmod h\}.$$

Then $(X,\mathcal{H},\mathcal{B})$ is the desired t-$((t+1)h,t+1,1)$ design of type h^{t+1}. ∎

If $(X,\mathcal{H},\mathcal{B})$ is a transverse t-(v,k,λ) design and $x\in H\in\mathcal{H}$, then $(X',\mathcal{H}',\mathcal{B}')$ is a transverse $(t-1)$-$(v-|H|,k-1,\lambda)$ design, where

$$\begin{aligned} X' &= X\setminus H,\\ \mathcal{H}' &= \mathcal{H}\setminus\{H\}, \text{ and}\\ \mathcal{B}' &= \{B\setminus\{x\}:x\in B\in\mathcal{B}\}.\end{aligned}$$

The design $(X',\mathcal{H}',\mathcal{B}')$ is called the *derived design* of $(X,\mathcal{H},\mathcal{B})$ with respect to x.

Suppose $\mathcal{H}$ is a partition of type h^u of a set X. Two transverse t-$(v,k,1)$ designs of type h^u having holes $\mathcal{H}$, say $(X,\mathcal{H},\mathcal{B})$ and $(X,\mathcal{H},\mathcal{B}')$, are said to be *disjoint* if $\mathcal{B}\cap\mathcal{B}'=\emptyset$. A collection $(X,\mathcal{H},\mathcal{B}_i)$

($i = 1, 2, \ldots, n$, where $n = h(u-t)/(k-t)$) of pairwise disjoint transverse t-$(v, k, 1)$ designs of type h^u having holes $\mathcal{H}$ is called a *large set* of transverse t-$(v, k, 1)$ designs of type h^u. Given any subset of k points transverse to $\mathcal{H}$, there is a unique design $(X, \mathcal{H}, \mathcal{B}_i)$ in the large set that contains the k given points as a block.

The following theorem shows a useful equivalence when $k = t + 1$.

Theorem 2.2 *There exists a large set of transverse* t-$(hu, t+1, 1)$ *designs of type* h^u *if and only if there exists a transverse* $(t+1)$-$(hu+n, t+2, 1)$ *design of type* $h^u n^1$*, where* $n = h(u-t)$.

Proof: Suppose $\mathcal{H}$ is a partition of type h^u of a set X. Suppose that $(X, \mathcal{H}, \mathcal{B}_i)$ ($i = 1, \ldots, n$) are the transverse t-$(hu, t+1, 1)$ designs of type h^u in a large set. Let $Y = \{y_1, \ldots, y_n\}$ be a set of n points disjoint from X. Define

$$\mathcal{B} = \{\{y_i\} \cup B : B \in \mathcal{B}_i, 1 \leq i \leq n\}.$$

Then $(X \cup Y, \mathcal{H} \cup \{Y\}, \mathcal{B})$ is a transverse $(t+1)$-$(hu+n, t+2, 1)$ design of type $h^u n^1$.

Conversely, suppose we start with a transverse $(t+1)$-$(hu+n, t+2, 1)$ design of type $h^u n^1$. It can be shown that every block contains a point in the hole of size n. If we construct the n derived designs through the points in the hole of size n, then we get the desired large set. ■

It is easy to see that $n = h$ in the above theorem if and only if $u = k = t + 1$. In this case, the designs are orthogonal arrays. We can extend Theorem 2.1 as follows:

Corollary 2.3 *There is a large set of transverse* t-$((t+1)h, t+1, 1)$ *designs of type* h^{t+1} *for all integers* $h \geq 1$ *and* $t \geq 2$.

Proof: ¿From Theorem 2.1, there exists a transverse $(t+1)$-$((t+2)h, t+2, 1)$ design of type h^{t+2}. Now apply Theorem 2.2. ■

Here is a simple "inflation" construction.

Theorem 2.4 *If a transverse* t-$(v, t+1, \lambda)$ *design of type* $h_1 h_2 \cdots h_r$ *exists, then a transverse* t-$(vw, t+1, \lambda)$ *design of type* $(wh_1)(wh_2)\cdots(wh_r)$ *exists for every integer* $w > 0$.

Proof: Take w copies of each point, and replace each block B of a transverse t-$(v, t+1, \lambda)$ design by the blocks in a transverse t-$(w(t+1), t+1, 1)$ design of type w^{t+1}, which exists by Theorem 2.1. ■

3 Constructing transverse t-designs having specifed automorphism groups

A permutation g on a set X acts on the subsets of X in a natural way. Given $S \subseteq X$, we define $g(S)$ by $g(S) = \{g(x) : x \in S\}$. A permutation g is an *automorphism* of the transverse design $(X, \mathcal{H}, \mathcal{B})$ provided that

1. $g \in \mathit{Sym}(X)$ (the symmetric group on X),
2. $g(H) \in \mathcal{H}$ for every hole $H \in \mathcal{H}$, and
3. $g(B) \in \mathcal{B}$ for every block $B \in \mathcal{B}$.

A collection of automorphisms of $(X, \mathcal{H}, \mathcal{B})$ that forms a group (under composition of permutations) is called an *automorphism group* of the transverse design $(X, \mathcal{H}, \mathcal{B})$. If G is an automorphism group of $(X, \mathcal{H}, \mathcal{B})$, then $\mathcal{H}$ and $\mathcal{B}$ are each unions of group orbits, where the *orbit* of any subset $S \subseteq X$ under the action of G is defined to be $G(S) = \{g(S) : g \in G\}$.

It is easy to see that the permutations

$$\begin{aligned} \alpha &= (0,2)(4,6)(1,3)(5,7)(a,b)(c,d), \\ \beta &= (0,4)(2,6)(1,5)(3,7)(a,c)(b,d), \text{ and} \\ \gamma &= (0,2,4)(1,3,5)(a,b,c) \end{aligned}$$

preserve the blocks and holes of the transverse design in Figure 1. Thus the group $G = \langle \alpha, \beta, \gamma \rangle$ generated by them is an automorphism group of the design.

Suppose that a subgroup G of *Sym*(X) preserves the holes $\mathcal{H} = \{H_1, H_2, \ldots, H_r\}$. The orbits of subsets transverse to $\mathcal{H}$ under the action of G can be computed if we have one representative for each orbit. The actual orbits can then be constructed by running through all the elements of G, applying them to the orbit representatives, and removing duplicates.

Orbits of transverse $(k+1)$-element subsets can be obtained by the following method. Let $\mathcal{R}$ be a set of orbit representatives for the orbits of transverse k-element subsets of X under the permutation group G. Given $A \in \mathcal{R}$, define $\mathcal{C}(A)$ by

$$\mathcal{C}(A) = \cup\{H_i : A \cap H_i = \emptyset, i = 1, 2, \ldots, r\},$$

and let

$$\mathcal{S} = \{A \cup \{x\} : A \in \mathcal{R} \text{ and } x \in \mathcal{C}(A)\}.$$

Let Γ be any orbit of transverse $(k+1)$-element subsets. Consider any representative $B' \in \Gamma$. Writing $B' = A' \cup \{x'\}$ for some k-set A', we see that $A' \in \Delta$ for some orbit Δ of transverse k-element subsets. Thus there is a $g \in G$ such that $g(A') = A \in \mathcal{R}$. Hence Γ has the orbit representative $A \cup \{x\}$, where $x = g(x')$. Thus, applying the permutations in the group G to each $A \cup \{x\}$, $x \in \mathcal{C}(A)$, and keeping the one that is minimum in lexicographical order, we will construct the desired list $\mathcal{S}$ of distinct orbit representatives of transverse $(k+1)$-element subsets. Pseudocode for this method is provided in Algorithm 3.1.

Algorithm 3.1: *TransReps*$(G, \mathcal{H}, \mathcal{R})$

$\mathcal{S} \leftarrow$ the empty list
for each A in the list $\mathcal{R}$
do
- **comment** compute $C = \mathcal{C}(A)$
- $C \leftarrow \emptyset$
- **for each** $H \in \mathcal{H}$ **do if** $A \cap H = \emptyset$ **then** $C \leftarrow C \cup H$
- **comment** compute orbit representatives containing A
- **for each** $x \in C$
  do
  - $B \leftarrow A \bigcup \{x\}$
  - **comment** find the minimum orbit representative $B^\star$ of $G(B)$
  - $B^\star \leftarrow B$
  - **for each** $g \in G$ **do if** $g(B) < B^\star$ in lexicographic order **then** $B^\star \leftarrow g(B)$
  - **if** $B^\star$ is not in the list $\mathcal{S}$ **then** insert $B^\star$ into the list $\mathcal{S}$

return $(\mathcal{S})$

If we can compute the number $N[k+1]$ of orbits of transverse $(k+1)$-element subsets prior to computing the orbit representatives of $(k+1)$-subsets, then we can possibly abort the computation in Algorithm 3.1 early. In order to do this, we would add the statement

$$\textbf{if } |\mathcal{S}| \geq N[k+1] \quad \textbf{then exit}$$

after $B^\star$ is inserted onto the list $\mathcal{S}$.

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_r\}$ be a partition of X into holes and let G be an automorphism group preserving $\mathcal{H}$. Then $N[k]$, the number of orbits of transverse k-element subsets, is given by the Cauchy-Frobenius-Burnside formula:

$$N[k] = \frac{1}{|G|} \sum_{g \in G} \mathit{Fix}(k, g),$$

where $\mathit{Fix}(k,g)$ is the number of transverse k-element subsets fixed by the permutation g. In order to use this formula, an efficient algorithm is required to compute $\mathit{Fix}(k,g)$. We develop such an algorithm now.

Let $g \in G$ and write g as a product of disjoint cycles:

$$g = C_0 C_1 C_2 \cdots C_{s-1}.$$

For $j = 0, 1, \ldots, s-1$, define $\overline{C_j}$ by

$$\overline{C_j} = \{i : x \in H_i \text{ for some } x \in C_j\}.$$

If K is a transverse k-element subset fixed by g, then

1. K is a union of cycles $C_{j_1}, C_{j_2}, \ldots, C_{j_\ell}$,
2. C_{j_h} is transverse to $\mathcal{H}$ for $h = 1, 2, \ldots, \ell$, and
3. $\overline{C_{j_h}} \cap \overline{C_{j_{h'}}} = \emptyset$ for all $1 \leq h < h' \leq \ell$.

Thus, we associate with each permutation $g \in G$ a graph $\mathcal{G}_g = (\mathcal{V}, \mathcal{E})$ whose vertices are the cycles C_j in g with C_j transverse to $\mathcal{H}$, and in which C_j is adjacent to $C_{j'}$ if and only if $\overline{C_j}$ and $\overline{C_{j'}}$ are disjoint. If $A \subseteq \mathcal{V}$ is a clique in $\mathcal{G}_g$, then g fixes a subset of size

$$\sum_{C_j \in A} \mathit{Len}(C_j),$$

where $\mathit{Len}(C_j)$ is the length of the cycle C_j. The fixed transverse subset K corresponding to the clique A is

$$K = \{x : x \in C_j \text{ and } C_j \in A\}.$$

Example 3.1. Consider the holes

$$\mathcal{H} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4,5,6\}, \{7,8,9\}\}$$

and the permutation

$$\alpha = (0,1)(2,3)(4)(5)(6)(7)(8)(9) = (0,1)(2,3),$$

which preserves $\mathcal{H}$. In Figure 2, the graph $\mathcal{G}_\alpha$ is displayed.

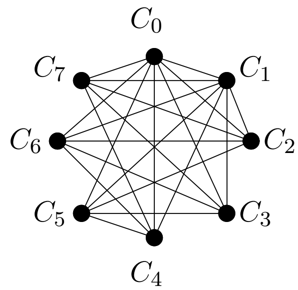

Figure 2: The graph $\mathcal{G}_\alpha$ where $\alpha = (0,1)(2,3)$.

The vertices in graph $\mathcal{G}_\alpha$ are $C_0 = (0,1), C_1 = (2,3), C_2 = (4), C_3 = (5), C_4 = (6), C_5 = (7), C_6 = (8)$, and $C_7 = (9)$. Note, for example, that C_3 is not adjacent to C_4 because $\overline{C_3} \cap \overline{C_4} = \{5\}$. One clique in this graph contains vertices C_0, C_1, C_4 and C_5. The size of this clique is four and $K = \{0,1,2,3,6,7\}$ is the corresponding fixed subset. The size of K is six. $\square$

This process of finding the number of transverse orbits is given by the pseudocode in Algorithm 3.2. Implementation details for Algorithms 3.1 and 3.2 described can be found in [5].

Algorithm 3.2: *TransNorb*(G)

```
for k ← 0 to |H| do N[k] ← 0
for each g ∈ G
      ⎧ construct G_g
      ⎪ for each clique A of G_g
  do  ⎨       ⎧ j ← 0
      ⎪   do  ⎨ for each cycle C ∈ A do j ← j + Len(C)
      ⎩       ⎩ N[j] ← N[j] + 1
for k ← 0 to |H| do N[k] ← N[k]/|G|
return (N)
```

Example 3.2. Let the holes be as in Example 3.1 and let the group $G = \langle \alpha, \beta, \gamma \rangle$, where

$$\begin{aligned} \alpha &= (0,1)(2,3) \\ \beta &= (0,2)(1,3) \\ \gamma &= (0,1,2)(4,5,6)(7,8,9). \end{aligned}$$

We tabulate the numbers *Fix*(k, g), for each $g \in G$, in Table 1. Each value $N[k]$ in this table is computed by summing the entries in the relevant column, and then dividing by $|G| = 12$.

$\square$

Once a possible automorphism group G has been chosen, we first find the orbit representatives, using the techniques described above. Then we determine the possible transverse t-designs having G as an automorphism group. The construction of the transverse t-designs is done using standard techniques, which we now summarize briefly.

Table 1: Computation of $N[k]$

	Fix(k,g)						
g	$k=0$	1	2	3	4	5	6
I	1	10	39	76	79	42	9
α	1	6	11	12	19	6	9
β	1	6	11	12	19	6	9
$\alpha\beta$	1	6	11	12	19	6	9
γ	1	1	0	1	1	0	0
γ^2	1	1	0	1	1	0	0
$\alpha\gamma$	1	1	0	1	1	0	0
$\beta\gamma$	1	1	0	1	1	0	0
$\gamma\beta$	1	1	0	1	1	0	0
$\alpha\gamma^2$	1	1	0	1	1	0	0
$\beta\gamma^2$	1	1	0	1	1	0	0
$\alpha\beta\gamma^2$	1	1	0	1	1	0	0
$N[k]$	1	3	6	10	12	5	3

Given any orbit Δ of transverse t-subsets of X and any orbit Γ of transverse k-subsets, the quantity

$$|K \in \Gamma : K \supseteq T|$$

is independent of the choice of orbit representative $T \in \Delta$ (a proof of this fact can for be found, for example, in [7]). This motivates the following definition.

Let $\mathcal{H}$ be a partition of X, let G be a subgroup of *Sym*(X) preserving $\mathcal{H}$, and suppose $0 \leq t \leq k \leq |X|$. Then the *orbit incidence matrix* for transverse t- versus k-subsets (with respect to the partition $\mathcal{H}$) is the $N[t]$ by $N[k]$ matrix A_{tk} such that

1. the rows of A_{tk} are labeled by the orbits $\Delta_1, \Delta_2, \ldots, \Delta_{N[t]}$ of transverse t-element subsets,
2. the columns of A_{tk} are labeled by the orbits $\Gamma_1, \Gamma_2, \ldots \Gamma_{N[k]}$ of transverse k-element subsets, and
3. the $[\Delta_i, \Gamma_j]$-entry is $A_{tk}[\Delta_i, \Gamma_j] = |\{K \in \Gamma_j : K \supseteq T\}|$ where $T \in \Delta_i$ is any fixed representative.

The following theorem, in the case of transverse designs of type 1^v (i.e., the ordinary t-designs), first appeared in [6]. The matrices in this case are often called the *Kramer-Mesner matrices*.

Theorem 3.1 *If there is a* $(0,1)$*-valued solution* U *to the matrix equation*

$$A_{tk}U = \lambda J,$$

where $J = [1, 1, \ldots, 1]^T$*, then there is a (simple) transverse* t*-*(v, k, λ) *design with holes* $\mathcal{H}$*.*

Proof: The k-element subsets in the union of the orbits Γ_j, where $U[j] = 1$, form the blocks of the desired transverse t-design. ■

We can solve the matrix equation $A_{tk}U = \lambda J$ for a $(0,1)$-valued solution U using a simple backtracking procedure if the size of the matrix A_{tk} is sufficiently small. Other methods are described in [7].

4 Transverse Steiner quadruple systems

4.1 Necessary conditions

An investigation of transverse STS (i.e., group-divisible designs with block size three) was done by Colbourn [1], who determined the existence of transverse STS of all possible types on at most 60 points. The following theorem establishes some necessary conditions for a transverse SQS to exist. Note that a transverse SQS with one or two holes trivially exists because there are no transverse triples. On the other hand, no transverse SQS with three holes can exist because the transverse triples cannot be covered by transverse quadruples.

Theorem 4.1 *Suppose that a transverse* SQS *of type* $h_1 h_2 \cdots h_n$ *exists, where* $n \geq 4$*. Then* $v = \sum_{i=1}^{n} h_i$ *and the following hold:*

1. $h_i + h_j \equiv v \pmod{2}$ *for all* $i \neq j$,
2. *There exists a transverse* STS *of type* $\prod_{i \neq \ell} h_i$ *for all* $\ell = 1, 2, \ldots, n$ *, and*
3. $\sum_{1 \leq i < j < k \leq n} h_i h_j h_k \equiv 0 \pmod{4}$.

Proof: Let $(X, \mathcal{H}, \mathcal{B})$ be a transverse SQS with holes $\mathcal{H} = \{H_1, \ldots, H_n\}$, where $|H_i| = h_i$ for $i = 1, 2, \ldots, n$. Suppose $i \neq j$ and let $x \in H_i$ and $y \in H_j$. Then, for each $z \in \bigcup_{\ell=1}^{n} H_\ell \backslash \{H_i \cup H_j\}$, there is a unique block $\{x, y, z, z'\}$ that contains x, y, z. This induces a pairing z, z' of the points not in $H_i \cup H_j$. Therefore

$$\sum_{\ell=1}^{n} |H_l| - |H_i| - |H_j| \equiv 0 \pmod{2}.$$

Hence,

$$h_i + h_j = |H_i| + |H_j| \equiv \sum_{\ell=1}^{n} |H_\ell| = v \pmod{2},$$

yielding condition 1. Now let $x \in H_\ell$ and consider the blocks that contain x. Let

$$T = \{B \backslash \{x\} : x \in B \in \mathcal{B}\},$$

and let $P = X \backslash H_\ell$. Then every pair $y \in H_i, z \in H_j$, where $i \neq j \neq \ell \neq i$ is in a unique block B of $\mathcal{B}$ and hence in a unique triple $B \backslash \{x\}$ in T. Thus $(P, \mathcal{H} \setminus \{H_\ell\}, T)$ is a transverse STS of type $\prod_{i \neq \ell} h_i$.

Lastly, there are $\sum_{1 \leq i < j < k \leq n} h_i h_j h_k$ transverse triples xyz, each in a unique block. Thus, because each block contains four of them, 4 must divide this sum. ∎

Corollary 4.2 *Suppose that a transverse* SQS *of type* $h_1 h_2 \cdots h_n$ *exists, where* $n \geq 4$*. Then* $v = \sum_{i=1}^{n} h_i$ *and the following hold:*

1. $h_1 \equiv h_2 \equiv \cdots \equiv h_n \pmod{2}$*, and*
2. $v \equiv 0 \pmod{2}$.

Proof: Let $(X, \mathcal{H}, \mathcal{B})$ be a transverse SQS with holes $\mathcal{H} = \{H_1, \ldots, H_n\}$, where $|H_i| = h_i$ for $i = 1, 2, \ldots, n$. Now fix i and j where $1 \leq i < j \leq n$. Then, because $n \geq 4$, there exists a k such that $1 \leq k \leq n$ and $k \neq i, j$. By Theorem 4.1 part 1, $h_i + h_k \equiv h_j + h_k \pmod 2$. Therefore $h_i \equiv h_j \pmod 2$ for all i and j. The fact that v is even now follows from Theorem 4.1, part 1. ■

4.2 Uniform designs

The following result about uniform transverse SQS was established by Mills in [10].

Theorem 4.3 *(Mills , 1990) For $u \geq 4$, $u \neq 5$, a transverse* SQS *of type h^u exists only if and only if hu is even and $h(u-1)(u-2) \equiv 0 \pmod 3$.*

Remarks: With reference to the case $u = 5$, Mills [10] notes the non-existence of a transverse SQS of type 2^5 (which was proved by Stanton and Mullin in [11]). The existence of a transverse SQS of type 6^5 is shown by Mills in [9, Lemma 7]. Mills reports the existence of a transverse SQS of type 4^5, but he does not present a construction for it. We give a construction in the Appendix. Hartman and Phelps [3, Section 7] comment on the relevance of this design to the Granville-Hartman bound for embeddings of SQS.

Now we state and prove a theorem on the existence of (uniform) transverse SQS of type h^5.

Theorem 4.4 *There exists a transverse* SQS *of type h^5 for all $h \equiv 0, 4, 6,$ or $8 \pmod{12}$.*

Proof: Apply Theorem 2.4 with $t = 3$, starting with transverse SQS of types 4^5 and 6^5 (these designs are constructed in the Appendix and [9, Lemma 7], respectively). ■

It is an open problem to settle the existence of transverse SQS of type h^5 when $h \equiv 2$ or $10 \pmod{12}$, $h > 2$.

4.3 New constructions

In this section, we give several new constructions for nonuniform transverse SQS.

Theorem 4.5 *There exists a transverse* SQS *of type $m^s((s-2)m)^1$ if and only if $s(s-1)m^2 \equiv 0 \pmod 6$, $(s-1)m \equiv 0 \pmod 2$, and $(m, s) \neq (1, 7)$.*

Proof: Lei [8] showed that a large set of transverse STS of type m^s exist if and only if $s(s-1)m^2 \equiv 0 \pmod 6$, $(s-1)m \equiv 0 \pmod 2$, and $(m, s) \neq (1, 7)$. Apply Theorem 2.2, with $t = 2$.. ■

The next series of theorems modify the *Doubling One-Factor* (or DOF) construction, which was first described by Hanani as a recursive construction for Steiner quadruple systems. This method constructs a one-factorization of K_v on each of two disjoint SQS(v). It then uses a pairing between the one-factors from each SQS(v) to construct a set of quadruples with the property that each triple consisting of one point from one SQS(v) and two points from the other SQS(v) is covered by exactly one of the quadruples. The result is an SQS$(2v)$.

Theorem 4.6 *If there exists a transverse* SQS *of type m^x and one of type n^y, where $x, y \geq 2$ and $m(x-1) = n(y-1)$, then there exists a transverse* SQS *of type $m^x n^y$.*

Proof: Let $(X, \mathcal{H}, \mathcal{B})$ be a transverse SQS of type m^x and let $(X', \mathcal{H}', \mathcal{B}')$ be a transverse SQS of type n^y, where $m(x-1) = n(y-1)$. Then the complete x-partite graph $K_{m,m,\ldots,m}$ (whose parts are the holes in $\mathcal{H}$) is a regular graph of degree $N = m(x-1)$, where mx is even, and, so it has a one-factorization, $\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$. The complete y-partite graph $K_{n,n,\ldots,n}$ (whose parts are the holes in $\mathcal{H}'$) is also a regular graph of degree N, and it has a one-factorization, $\mathcal{F}' = \{F_1', F_2', \ldots, F_N'\}$. Thus we can pair the one-factors of $\mathcal{F}$ and $\mathcal{F}'$, constructing a set of blocks

$$\mathcal{B}'' = \{\{a, b, c', d'\} : ab \in F_j, c'd' \in F_j', 1 \leq j \leq N\}.$$

Now, an admissible triple of points from X is in a unique block in $\mathcal{B}$; an admissible triple of points from X' is in a unique block in $\mathcal{B}'$; and an admissible triple consisting of two points from X and one point from X', or two points from X' and one point from X, is in a unique block in $\mathcal{B}''$. Therefore, we have a transverse SQS of type $m^x n^y$, namely $(X \cup X', \mathcal{H} \cup \mathcal{H}', \mathcal{B} \cup \mathcal{B}' \cup \mathcal{B}'')$. ■

Corollary 4.7 *If there exists a transverse* SQS *of type* m^x *with* $x \geq 2$ *and* $g = m(x-1)$*, then there exists a transverse* SQS *of type* $m^x g^2$.

Proof: A transverse SQS of type g^2 is a trivial design having no blocks. Therefore it follows by Theorem 4.6 that there exists a transverse SQS of type $m^x g^2$. ■

Theorem 4.8 *Suppose* mn *is even, there exists a transverse* SQS *of type* $(mn)^r(s+t)^1$ *and there exists a transverse* SQS *of type* $m^n s^1 t^1$*. Then there exists a transverse* SQS *of type* $m^{rn} s^1 t^1$.

Proof: Let $H_1, H_2, \ldots, H_r$ and X be disjoint sets with $|H_i| = mn$ ($i = 1, 2, \ldots r$) and $|X| = s + t$. Construct a transverse SQS of type $(mn)^r(s+t)^1$, having blocks $\mathcal{B}$, where these $r+1$ sets are the holes.

Next, for each $i = 1, 2, \ldots, r$, partition H_i into subsets $H_{i,1}, H_{i,2}, \ldots, H_{i,n}$ where $|H_{i,j}| = m$ for $j = 1, 2, \ldots, n$. Also, partition $X = S \cup T$, where $|S| = s$ and $|T| = t$. For $1 \leq i \leq r$, construct a transverse SQS of type $m^n s^1 t^1$ on the holes $\{H_{i,1}, H_{i,2}, \ldots, H_{i,n}, S, T\}$, having block set $\mathcal{B}_i$.

Finally, for $1 \leq i < j \leq r$, we construct a set of blocks contained in $H_i \cup H_j$ that correspond to the blocks $\mathcal{B}''$ in the construction given in Theorem 4.6. More precisely, for $1 \leq i \leq r$, construct an n-partite graph $G_i = K_{m,m,\ldots,m}$ whose parts are the holes in H_i, Each G_i is a regular graph of degree $N = (n-1)m$ having an even number of vertices, and hence it has a one-factorization, $\mathcal{F}_i = \{F_i^1, F_i^2, \ldots, F_i^N\}$. Then for each pair i, j with $1 \leq i < j \leq r$, we can pair the one-factors of $\mathcal{F}_i$ and $\mathcal{F}_j$, constructing a set of blocks

$$\mathcal{B}_{i,j} = \{\{a, b, c', d'\} : ab \in F_i^h, c'd' \in F_j^h, 1 \leq h \leq N\}.$$

The desired design has blocks

$$\mathcal{B} \cup \left(\bigcup_{1 \leq i \leq r} \mathcal{B}_i \right) \cup \left(\bigcup_{1 \leq i < j \leq r} \mathcal{B}_{i,j} \right).$$

It can be shown that this design is a transverse SQS of type $m^{rn} s^1 t^1$. ■

We close this section with a nonexistence result.

Theorem 4.9 *There does not exist a transverse* SQS *of type* $1^1 3^5$.

Proof: If a transverse SQS of type $1^1 3^5$ were to exist, then the derived design with respect to the point in the hole of size 1 would be a transverse STS of type 3^5. Adding a "point at infinity", ∞, to the remaining holes (each having size 3), we get a linear space (or pairwise balanced design) on 16 points having five blocks of size four that intersect in the point ∞, and 30 blocks of size three. The 146 non-isomorphic linear spaces of this type were enumerated by Heathcote in [4]. For each of these 146 linear spaces, we deleted the point ∞ and applied a backtracking algorithm to try to extend it to a transverse SQS of type $1^1 3^5$. No extension was possible. ∎

4.4 Small transverse Steiner quadruple systems

In this section, we present two tables that summarize the existence and nonexistence results we have for transverse Steiner Quadruple systems on at most 24 points. Only the types of designs that are admissible according to Theorem 4.1 are listed in the tables. New designs found by the algorithms described in Section 3 are noted in the tables, and the designs appear in the Appendix.

5 Conclusion and open problems

The parameter case $1^{13} 9^1$ in Table 3 is particularly interesting. If such a design were to exist, it would have several interesting properties. The derived designs through the nine points in the group of size 9 would yield nine disjoint Steiner triple systems of order 13 on the remaining 13 points. This would leave 52 triples uncovered, which would therefore be covered by 13 quadruples. It can be shown that that these 13 quadruples would form a projective plane of order 3, i.e., a 2-$(13, 4, 1)$ design. Hence, the problem of constructing a transverse SQS of type $1^{13} 9^1$ is equivalent to taking a projective plane of order 3 (there is a unique one, up to isomorphism) and partitioning all the non-collinear triples of points into nine block-disjoint Steiner triple systems of order 13.

The problem of constructing transverse t-designs with $t > 3$ remains difficult (other than the designs corresponding to orthogonal arrays, of course). Here is one interesting infinite class of transverse 4-designs that we construct by using a result of Etzion. In [2, Corollary 7], Etzion established the existence of a large set of transverse SQS of type g^{2^r} for every $r \geq 2$ and $g \geq 2$. There are $n = (2^r - 3)g$ designs in the large set. Then, applying Theorem 2.2, we have the following result.

Theorem 5.1 *Let $r \geq 2$ and $g \geq 2$, and denote $n = (2^r - 3)g$. Then a transverse 4-$(2^r g + n, 5, 1)$ design of type $g^{2^r} n^1$ exists.*

It would be of interest to find additional constructions for transverse t-designs with $t > 3$ and $\lambda = 1$.

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Table 2: Transverse SQS on $4, 6, \ldots, 18$ points

v	type	existence	remarks
4	1^4	Yes	Theorem 4.3

v	type	existence	remarks
8	1^8	Yes	Theorem 4.3
	2^4	Yes	Theorem 4.3

v	type	existence	remarks
10	1^{10}	Yes	Theorem 4.3
	2^5	No	Remark following Theorem 4.3
	$1^4 3^2$	Yes	Appendix or Corollary 4.7

v	type	existence	remarks
12	3^4	Yes	Theorem 4.3
	$2^4 4^1$	Yes	Appendix or Theorem 4.5
	$1^7 5^1$	No	Theorem 4.5

v	type	existence	remarks
14	1^{14}	Yes	Theorem 4.3
	2^7	Yes	Theorem 4.3

v	type	existence	remarks
16	1^{16}	Yes	Theorem 4.3
	2^8	Yes	Theorem 4.3
	$1^{13} 3^1$	Yes	Appendix
	$1^{10} 3^2$	Yes	Appendix
	$1^7 3^3$	Yes	Appendix
	$1^4 3^4$	Yes	Appendix
	$1^1 3^5$	No	Theorem 4.9
	4^4	Yes	Theorem 4.3
	$1^9 7^1$	Yes	Theorem 4.5

v	type	existence	remarks
18	3^6	Yes	Theorem 4.3
	$2^7 4^1$	Yes	Appendix
	$2^1 4^4$	Yes	Appendix
	$1^{13} 5^1$	?	?

Table 3: Transverse SQS on $20, 22$ and 24 points

v	type	existence	remarks
20	1^{20}	Yes	Theorem 4.3
	$1^{13}7^1$	?	?
	2^{10}	Yes	Theorem 4.3
	4^5	Yes	Appendix
	5^4	Yes	Theorem 4.3
	$2^4 6^2$	Yes	Appendix or Corollary 4.7
	$2^6 8^1$	Yes	Theorem 4.5
	$2^7 6^1$	?	?
	$3^5 5^1$	?	?

v	type	existence	remarks
22	1^{22}	Yes	Theorem 4.3
	$1^6 3^3 7^1$	?	?
	$1^{10} 3^4$	Yes	Theorem 4.6
	$1^{12} 3^1 7^1$	?	?
	$1^{13} 9^1$	?	?
	$1^{16} 3^2$	Yes	Theorem 4.8 ($s = t = 3$, $m = 1$, $n = r = 4$)
	$1^4 3^6$	Yes	Appendix
	$1^8 7^2$	Yes	Appendix or Corollary 4.7
	2^{11}	Yes	Theorem 4.3
	$2^7 8^1$	Yes	Appendix
	$3^5 7^1$	?	?
	$4^4 6^1$	Yes	Appendix

v	type	existence	remarks
24	$1^{12} 5^1 7^1$	?	?
	$1^4 5^4$	?	?
	$1^{13} 11^1$	Yes	Theorem 4.5
	$1^{19} 5^1$	?	?
	$2^7 10^1$	Yes	Theorem 4.5
	$2^{10} 4^1$	Yes	Appendix or Theorem 4.8 ($s = 2$, $t = 4$, $m = 2$, $n = r = 3$)
	$2^4 4^4$	?	?
	$2^6 4^1 8^1$	?	?
	3^8	Yes	Theorem 4.3
	$3^5 9^1$	Yes	Theorem 4.5
	$4^4 8^1$	Yes	Theorem 2.4 or Theorem 4.3
	6^4	Yes	Theorem 4.3

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A Small transverse Steiner quadruple systems

In this appendix, we list the transverse Steiner quadruple systems we found using the algorithms described in Section 3. For each design, we give its type, an automorphism group G, and a set of base blocks. The base blocks, when developed by the automorphism group G, will yield the blocks of the design. The holes are always written as $\{H_1, H_2, \ldots, H_r\}$ where $|H_1| \leq |H_2| \leq \cdots \leq |H_r|$. H_1 consists of the first $|H_1|$ points, H_2 consists of the next $|H_2|$ points, etc.

Type $1^4 3^2$. . . . .

G = $\langle$ (0,1)(2,3), (0,2)(1,3), (0,1,2)(4,5,6)(7,8,9) $\rangle$.

Base blocks: $\{0,1,2,3\}$, $\{0,1,4,9\}$, $\{0,1,5,8\}$, $\{0,1,6,7\}$

Type $2^4 4^1$. . . . .

G = $\langle$ (0,2)(4,6)(1,3)(5,7)(8,9)(10,11), (0,4)(2,6)(1,5)(3,7)(8,10)(9,11), $(0,2,4)(1,3,5)(8,9,10)\rangle$.

Base blocks: $\{0,2,4,11\}$, $\{0,2,5,9\}$, $\{0,3,5,10\}$, $\{1,3,5,11\}$.

Type $1^{13} 3^1$. . . . .

G = $\langle(0)(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)\rangle$.

Base blocks:	$\{0,1,2,3\}$	$\{0,1,4,15\}$	$\{0,1,5,12\}$	$\{0,1,6,9\}$	$\{0,1,7,14\}$
$\{0,1,8,10\}$	$\{0,1,11,13\}$	$\{0,4,5,6\}$	$\{0,4,8,14\}$	$\{0,4,9,12\}$	$\{0,4,10,13\}$
$\{0,7,8,9\}$	$\{0,7,11,15\}$	$\{0,10,11,12\}$	$\{1,2,4,11\}$	$\{1,2,5,9\}$	$\{1,2,6,15\}$
$\{1,2,7,13\}$	$\{1,2,8,12\}$	$\{1,2,10,14\}$	$\{1,4,5,7\}$	$\{1,4,6,13\}$	$\{1,4,9,14\}$
$\{1,4,10,12\}$	$\{1,5,6,11\}$	$\{1,5,8,15\}$	$\{1,5,10,13\}$	$\{1,6,7,8\}$	$\{1,6,12,14\}$
$\{1,7,9,10\}$	$\{1,7,12,15\}$	$\{1,8,9,13\}$	$\{1,8,11,14\}$	$\{1,9,11,12\}$	$\{1,10,11,15\}$
$\{4,5,8,12\}$	$\{4,5,9,13\}$	$\{4,5,11,15\}$	$\{4,7,8,10\}$	$\{4,7,9,15\}$	$\{4,7,12,13\}$
$\{4,8,11,13\}$	$\{4,9,10,11\}$	$\{4,11,12,14\}$	$\{7,8,12,14\}$	$\{7,10,11,14\}$	

Type $1^{10} 3^2$. . . . .

G = $\langle(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)\rangle$.

Base blocks:	$\{0,1,2,3\}$,	$\{0,1,4,15\}$,	$\{0,1,5,12\}$,	$\{0,1,6,9\}$,	$\{0,1,7,14\}$,
$\{0,1,8,10\}$,	$\{0,1,11,13\}$,	$\{0,4,5,6\}$,	$\{0,4,8,14\}$,	$\{0,4,9,12\}$,	$\{0,4,10,13\}$,
$\{0,7,8,9\}$,	$\{0,7,11,15\}$,	$\{1,2,4,11\}$,	$\{1,2,5,8\}$,	$\{1,2,6,15\}$,	$\{1,2,7,10\}$,
$\{1,2,9,14\}$,	$\{1,2,12,13\}$,	$\{1,4,5,9\}$,	$\{1,4,6,13\}$,	$\{1,4,8,12\}$,	$\{1,4,10,14\}$,
$\{1,5,6,11\}$,	$\{1,5,7,13\}$,	$\{1,5,10,15\}$,	$\{1,6,7,8\}$,	$\{1,6,12,14\}$,	$\{1,7,9,11\}$,
$\{1,7,12,15\}$,	$\{1,8,9,15\}$,	$\{1,8,11,14\}$,	$\{1,9,10,13\}$,	$\{4,5,7,11\}$,	$\{4,5,8,13\}$,
$\{4,5,12,15\}$,	$\{4,7,8,10\}$,	$\{4,7,9,14\}$,	$\{4,7,12,13\}$,	$\{4,8,11,15\}$,	$\{4,9,11,13\}$,
$\{7,8,11,13\}$.					

Type $1^7 3^3$. ....

G = $\langle$ (0,1,2)(3,4,5)(7,10,13)(8,11,14)(9,12,15), $(7,8,9)(10,11,12)(13,14,15)\rangle$.

Base blocks: $\{0,1,2,6\}$, $\{0,1,3,15\}$, $\{0,1,4,11\}$, $\{0,1,5,7\}$, $\{0,1,8,14\}$,
$\{0,1,9,12\}$, $\{0,1,10,13\}$, $\{0,3,4,12\}$, $\{0,3,5,13\}$, $\{0,3,6,11\}$, $\{0,3,7,14\}$,
$\{0,3,9,10\}$, $\{0,4,5,8\}$, $\{0,4,6,7\}$, $\{0,4,9,14\}$, $\{0,4,10,15\}$, $\{0,5,6,15\}$,
$\{0,5,9,11\}$, $\{0,5,10,14\}$, $\{0,6,8,10\}$, $\{0,6,9,13\}$, $\{0,6,12,14\}$, $\{0,7,11,15\}$,
$\{0,7,12,13\}$, $\{0,8,11,13\}$, $\{0,8,12,15\}$, $\{3,4,5,6\}$, $\{3,4,8,11\}$, $\{3,4,9,15\}$,
$\{3,4,10,13\}$, $\{3,6,7,15\}$, $\{3,6,8,12\}$, $\{3,6,10,14\}$, $\{3,7,11,13\}$, $\{3,8,10,15\}$,
$\{3,9,11,14\}$, $\{3,9,12,13\}$, $\{6,7,10,13\}$, $\{6,8,11,14\}$, $\{6,9,12,15\}$.

Type $1^4 3^4$. ....

G = $\langle$(0,1)(2,3)(4,7)(10,13)(5,8)(11,14)(6,9)(12,15),
(0,2)(1,3)(4,10)(7,13)(5,11)(8,14)(6,12)(9,15)$\rangle$.

Base blocks: $\{0,1,2,3\}$, $\{0,1,4,15\}$, $\{0,1,5,14\}$, $\{0,1,6,13\}$, $\{0,2,4,12\}$,
$\{0,2,5,11\}$, $\{0,2,7,15\}$, $\{0,2,8,14\}$, $\{0,3,4,9\}$, $\{0,3,5,7\}$, $\{0,3,6,8\}$,
$\{0,4,7,14\}$, $\{0,4,8,10\}$, $\{0,4,11,13\}$, $\{0,5,8,13\}$, $\{0,5,9,12\}$, $\{0,5,10,15\}$,
$\{0,6,7,11\}$, $\{0,6,9,15\}$, $\{0,6,12,14\}$, $\{0,7,10,13\}$, $\{0,8,12,15\}$, $\{0,9,11,14\}$,
$\{4,7,12,15\}$, $\{4,8,11,15\}$, $\{4,8,12,14\}$, $\{4,9,10,15\}$, $\{4,9,12,13\}$, $\{5,8,11,14\}$.

Type $2^7 4^1$. ....

G = $\langle(0,2,4)(1,3,5)(6,8,10)(7,9,11)\rangle$.

Base blocks: $\{0,2,4,17\}$ $\{0,2,5,16\}$ $\{0,2,6,15\}$ $\{0,2,7,14\}$ $\{0,2,8,13\}$
$\{0,2,9,11\}$ $\{0,2,10,12\}$ $\{0,3,5,13\}$ $\{0,3,6,9\}$ $\{0,3,7,10\}$ $\{0,3,8,17\}$
$\{0,3,11,15\}$ $\{0,3,12,14\}$ $\{0,5,6,11\}$ $\{0,5,7,17\}$ $\{0,5,8,10\}$ $\{0,5,9,14\}$
$\{0,5,12,15\}$ $\{0,6,8,14\}$ $\{0,6,10,17\}$ $\{0,6,12,16\}$ $\{0,7,8,15\}$ $\{0,7,11,12\}$
$\{0,7,13,16\}$ $\{0,8,11,16\}$ $\{0,9,10,16\}$ $\{0,9,12,17\}$ $\{0,9,13,15\}$ $\{0,10,13,14\}$
$\{0,11,13,17\}$ $\{1,3,5,16\}$ $\{1,3,6,11\}$ $\{1,3,7,17\}$ $\{1,3,8,15\}$ $\{1,3,9,12\}$
$\{1,3,10,14\}$ $\{1,6,8,12\}$ $\{1,6,9,14\}$ $\{1,6,13,16\}$ $\{1,7,9,16\}$ $\{1,7,11,15\}$
$\{1,7,13,14\}$ $\{1,8,10,16\}$ $\{1,8,13,17\}$ $\{1,9,11,13\}$ $\{1,10,12,17\}$ $\{1,10,13,15\}$
$\{1,11,12,16\}$ $\{6,8,10,15\}$ $\{6,8,11,13\}$ $\{6,9,11,17\}$ $\{6,9,12,15\}$ $\{6,11,12,14\}$
$\{7,9,11,14\}$

Type $2^1 4^4$. ....

G = $\langle(2,3)(4,5)(6,7)(8,9)(10,11)(12,13)(14,15)(16,17),$
$(2,5)(3,4)(6,9)(7,8)(10,13)(11,12)(14,17)(15,16)\rangle$.

Base blocks: $\{0,2,6,17\}$, $\{0,2,7,15\}$, $\{0,2,8,13\}$, $\{0,2,9,10\}$, $\{0,2,11,16\}$,
$\{0,2,12,14\}$, $\{0,6,10,15\}$, $\{0,6,12,16\}$, $\{1,2,6,12\}$, $\{1,2,7,11\}$, $\{1,2,8,14\}$,
$\{1,2,9,16\}$, $\{1,2,10,15\}$, $\{1,2,13,17\}$, $\{1,6,11,17\}$, $\{1,6,13,14\}$, $\{2,6,10,16\}$,
$\{2,6,11,14\}$, $\{2,6,13,15\}$, $\{2,7,10,14\}$, $\{2,7,12,17\}$, $\{2,7,13,16\}$, $\{2,8,10,17\}$,
$\{2,8,11,15\}$, $\{2,8,12,16\}$, $\{2,9,11,17\}$, $\{2,9,12,15\}$, $\{2,9,13,14\}$.

Type 4^5. . . . .

$G = \langle$ (0,4,8,12,16)(1,5,9,13,17)(2,6,10,14,18)(3,7,11,15,19) $\rangle$.

Base blocks: $\{0,4,8,19\}$ $\{0,4,9,18\}$ $\{0,4,10,14\}$ $\{0,4,11,13\}$ $\{0,4,12,17\}$
$\{0,5,9,15\}$ $\{0,5,10,18\}$ $\{0,5,11,19\}$ $\{0,5,13,17\}$ $\{0,6,8,14\}$ $\{0,6,9,13\}$
$\{0,6,11,17\}$ $\{0,6,15,19\}$ $\{0,7,8,17\}$ $\{0,7,10,12\}$ $\{0,7,11,15\}$ $\{0,7,13,18\}$
$\{0,7,14,19\}$ $\{0,9,14,17\}$ $\{0,10,13,19\}$ $\{0,10,15,17\}$ $\{0,11,14,18\}$ $\{1,5,9,14\}$
$\{1,5,15,19\}$ $\{1,6,10,19\}$ $\{1,6,11,18\}$ $\{1,7,9,19\}$ $\{1,7,10,13\}$ $\{1,7,11,14\}$
$\{1,10,14,18\}$ $\{2,6,11,14\}$ $\{2,7,11,19\}$

Type $2^4 6^2$. . . . .

$G = \langle$ (0,1)(3,2)(4,5)(6,7)(8,14)(9,15)(10,16)(11,17)(12,18)(13,19),
(0,3)(1,2)(4,6)(5,7)(8,9,10,11,12,13)(14,15,16,17,18,19) $\rangle$.

Base blocks: $\{0,2,4,7\}$, $\{0,2,5,6\}$, $\{0,2,8,18\}$, $\{0,3,4,6\}$, $\{0,3,5,7\}$,
$\{0,3,8,19\}$, $\{0,4,8,17\}$, $\{0,4,9,18\}$, $\{0,5,8,16\}$, $\{0,5,9,19\}$, $\{0,6,8,15\}$,
$\{0,6,9,16\}$, $\{0,7,8,14\}$, $\{0,7,9,15\}$, $\{4,6,8,19\}$, $\{4,7,8,16\}$.

Type $1^4 3^6$. . . . .

$G = \langle (4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21),$
$(4,7,10)(5,8,11)(6,9,12)(13,16,19)(14,17,20)(15,18,21) \rangle$.

Base blocks: $\{0,1,2,3\}$, $\{0,1,4,21\}$, $\{0,2,4,20\}$, $\{0,3,4,19\}$, $\{0,4,7,18\}$,
$\{0,4,8,17\}$, $\{0,4,9,16\}$, $\{0,13,16,21\}$, $\{1,2,4,19\}$, $\{1,3,4,20\}$, $\{1,4,7,17\}$,
$\{1,4,8,16\}$, $\{1,4,9,18\}$, $\{1,13,16,19\}$, $\{1,13,17,21\}$, $\{1,13,18,20\}$, $\{2,3,4,21\}$,
$\{2,4,7,11\}$, $\{2,4,13,18\}$, $\{2,4,14,17\}$, $\{2,4,15,16\}$, $\{3,4,7,10\}$, $\{3,4,8,12\}$,
$\{3,4,9,11\}$, $\{3,4,13,17\}$, $\{3,4,14,16\}$, $\{3,4,15,18\}$, $\{4,7,12,20\}$, $\{4,7,13,16\}$,
$\{4,7,14,19\}$, $\{4,7,15,21\}$, $\{4,8,13,21\}$, $\{4,8,14,20\}$, $\{4,8,18,19\}$, $\{4,9,13,20\}$,
$\{4,9,14,21\}$, $\{4,9,15,19\}$, $\{4,15,17,20\}$.

Type $1^8 7^2$. ....

G = $\langle$ (0,1)(2,3)(4,5)(6,7), $(0,3)(1,2)(4,7)(5,6)\rangle$.

Base blocks: $\{0,1,2,7\}$, $\{0,1,4,5\}$, $\{0,1,8,21\}$, $\{0,1,9,20\}$, $\{0,1,10,19\}$, $\{0,1,11,18\}$, $\{0,1,12,17\}$, $\{0,1,13,16\}$, $\{0,1,14,15\}$, $\{0,2,4,6\}$, $\{0,2,8,20\}$, $\{0,2,9,21\}$, $\{0,2,10,18\}$, $\{0,2,11,19\}$, $\{0,2,12,16\}$, $\{0,2,13,15\}$, $\{0,2,14,17\}$, $\{0,3,4,7\}$, $\{0,3,8,19\}$, $\{0,3,9,18\}$, $\{0,3,10,21\}$, $\{0,3,11,20\}$, $\{0,3,12,15\}$, $\{0,3,13,17\}$, $\{0,3,14,16\}$, $\{0,4,8,18\}$, $\{0,4,9,17\}$, $\{0,4,10,16\}$, $\{0,4,11,15\}$, $\{0,4,12,21\}$, $\{0,4,13,20\}$, $\{0,4,14,19\}$, $\{0,5,6,7\}$, $\{0,5,8,17\}$, $\{0,5,9,19\}$, $\{0,5,10,15\}$, $\{0,5,11,16\}$, $\{0,5,12,20\}$, $\{0,5,13,21\}$, $\{0,5,14,18\}$, $\{0,6,8,16\}$, $\{0,6,9,15\}$, $\{0,6,10,20\}$, $\{0,6,11,17\}$, $\{0,6,12,19\}$, $\{0,6,13,18\}$, $\{0,6,14,21\}$, $\{0,7,8,15\}$, $\{0,7,9,16\}$, $\{0,7,10,17\}$, $\{0,7,11,21\}$, $\{0,7,12,18\}$, $\{0,7,13,19\}$, $\{0,7,14,20\}$, $\{4,5,8,21\}$, $\{4,5,9,20\}$, $\{4,5,10,19\}$, $\{4,5,11,18\}$, $\{4,5,12,17\}$, $\{4,5,13,16\}$, $\{4,5,14,15\}$, $\{4,6,8,20\}$, $\{4,6,9,21\}$, $\{4,6,10,18\}$, $\{4,6,11,19\}$, $\{4,6,12,16\}$, $\{4,6,13,15\}$, $\{4,6,14,17\}$, $\{4,7,8,19\}$, $\{4,7,9,18\}$, $\{4,7,10,21\}$, $\{4,7,11,20\}$, $\{4,7,12,15\}$, $\{4,7,13,17\}$, $\{4,7,14,16\}$.

Type $2^7 8^1$. ....

G = $\langle(0,2,4,6,8,10,12)(1,3,5,7,9,11,13)(14,15,16,17,18,19,20)\rangle$.

Base blocks: {0,2,4,20} {0,2,5,18} {0,2,6,13} {0,2,7,17} {0,2,8,15} {0,2,9,16} {0,2,10,21} {0,2,11,14} {0,3,4,15} {0,3,5,9} {0,3,6,20} {0,3,7,19} {0,3,8,14} {0,3,10,16} {0,3,11,21} {0,3,13,18} {0,4,8,16} {0,4,9,19} {0,4,13,17} {0,5,6,15} {0,5,7,21} {0,5,8,20} {0,5,11,19} {0,5,13,14} {0,7,9,14} {0,7,11,20} {0,7,13,16} {0,9,11,18} {0,9,13,21} {0,11,13,15} {1,3,5,16} {1,3,9,20} {1,3,11,19} {1,5,9,15}

Type $4^4 6^1$. ....

G = $\langle(0,1,2)(4,5,6)(8,9,10)(12,13,14)(16,17,18)(19,20,21),$
$(0,3)(1,2)(4,7)(5,6)(8,11)(9,10)(12,15)(13,14)\rangle$.

Base blocks: $\{0,4,8,12\}$, $\{0,4,9,21\}$, $\{0,4,13,18\}$, $\{0,5,8,20\}$, $\{0,5,9,18\}$, $\{0,5,10,15\}$, $\{0,5,11,17\}$, $\{0,5,12,16\}$, $\{0,5,13,21\}$, $\{0,5,14,19\}$, $\{0,8,13,17\}$, $\{0,9,12,20\}$, $\{0,9,13,19\}$, $\{0,9,15,17\}$, $\{4,8,13,19\}$, $\{4,9,12,16\}$, $\{4,9,13,17\}$, $\{4,9,14,19\}$.

Type $2^{10}4^1$. ....

$$G = \langle(0,2,4)(6,8,10)(12,14,16)(1,3,5)(7,9,11)(13,15,17),\ (0,2)(6,8)(12,14)(1,3)(7,9)(13,15)\rangle.$$

Base blocks:	{0,2,4,23}	{0,2,5,22}	{0,2,6,15}	{0,2,7,14}	{0,2,10,21}
{0,2,11,20}	{0,2,16,19}	{0,2,17,18}	{0,3,5,23}	{0,3,6,9}	{0,3,7,8}
{0,3,10,20}	{0,3,11,21}	{0,3,12,14}	{0,3,13,15}	{0,3,16,18}	{0,3,17,19}
{0,6,8,14}	{0,6,12,23}	{0,6,13,22}	{0,6,18,21}	{0,6,19,20}	{0,7,9,15}
{0,7,12,21}	{0,7,13,20}	{0,7,18,22}	{0,7,19,23}	{0,8,10,19}	{0,8,11,18}
{0,8,12,15}	{0,8,16,23}	{0,8,17,22}	{0,9,11,19}	{0,9,13,14}	{0,9,16,22}
{0,9,17,23}	{0,12,18,20}	{0,12,19,22}	{0,13,18,23}	{0,13,19,21}	{0,14,16,21}
{0,14,17,20}	{0,15,17,21}	{1,3,5,22}	{1,3,6,14}	{1,3,7,15}	{1,3,10,21}
{1,3,11,20}	{1,3,16,19}	{1,3,17,18}	{1,6,8,15}	{1,6,12,20}	{1,6,13,21}
{1,6,18,23}	{1,6,19,22}	{1,7,9,14}	{1,7,12,22}	{1,7,13,23}	{1,7,18,20}
{1,7,19,21}	{1,8,10,18}	{1,8,11,19}	{1,8,13,14}	{1,8,16,22}	{1,8,17,23}
{1,9,11,18}	{1,9,12,15}	{1,9,16,23}	{1,9,17,22}	{1,12,18,21}	{1,12,19,23}
{1,13,18,22}	{1,13,19,20}	{1,14,16,20}	{1,14,17,21}	{1,15,17,20}	{6,8,10,23}
{6,8,11,22}	{6,8,16,21}	{6,8,17,20}	{6,9,11,23}	{6,9,12,14}	{6,9,13,15}
{6,9,16,20}	{6,9,17,21}	{6,12,18,22}	{6,12,19,21}	{6,13,18,20}	{6,13,19,23}
{6,14,16,19}	{6,14,17,18}	{6,15,17,19}	{7,9,11,22}	{7,9,16,21}	{7,9,17,20}
{7,12,18,23}	{7,12,19,20}	{7,13,18,21}	{7,13,19,22}	{7,14,16,18}	{7,14,17,19}
{7,15,17,18}	{12,14,16,23}	{12,14,17,22}	{12,15,17,23}	{13,15,17,22}	