On the existence of Kirkman triple systems containing Kirkman subsystems

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Abstract. The obvious necessary conditions for the existence of a Kirkman triple system of order v containing a Kirkman subsystem of order w are $v \equiv w \equiv 3$ modulo 6, $v \geq 3w$. We show that these conditions are sufficient.

1. Introduction

A *pairwise balanced design* (or, PBD) is a pair $(X, \mathcal{A})$, such that X is a set of elements (called *points*) and $\mathcal{A}$ is a set of subsets of X (called *blocks*), such that every unordered pair of points is contained in a unique block of $\mathcal{A}$. If v is a positive integer and K is a set of positive integers, then we say that $(X, \mathcal{A})$ is a (v, K)-PBD if $|X| = v$, and $|A| \in K$ for every $A \in \mathcal{A}$. The integer v is called the *order* of the PBD.

Using this notation, we can define a *Steiner triple system* of order v, which we denote STS(v), to be a $(v, \{3\})$-PBD. It is of course well-known that an STS(v) exists if and only if $v \equiv 1$ or 3 modulo 6.

Let $(X, \mathcal{A})$ be a PBD. If a set of points $Y \subseteq X$ has the property that, for any $A \in \mathcal{A}$, either $|Y \cap A| \leq 1$ or $A \subseteq Y$, then we say that Y is a *subdesign* or *flat* of the PBD. The *order* of the subdesign is $|Y|$. The subdesign Y is *proper* if $Y \neq X$. If Y is a subdesign, then we can delete all blocks $A \subseteq Y$ and replace them by a single block, Y, and the result is a PBD. Also, any block or point of a PBD is itself a subdesign.

The problem of constructing Steiner triple systems containing subsystems was studied by Doyen and Wilson in [2]. The obvious necessary conditions for the existence of an STS(v) containing an STS(w) as a subsystem are $v \geq 2w + 1$, $v \equiv 1$ or 3 modulo 6, $w \equiv 1$ or 3 modulo 6. In [2], it is shown that these necessary conditions are sufficient.

A *parallel class* in a PBD is a set of blocks that form a partition of the point set. A PBD is *resolvable* if the block set can be partitioned into parallel classes. A *Kirkman triple system* of order v, or KTS(v), is defined to be a resolvable STS(v). In [14], Ray-Chaudhuri and Wilson showed that there exists a KTS(v) if and only if $v \equiv 3$ modulo 6.

In this paper, we are interested in KTS(v) which contain KTS(w) as subsystems. We say that a KTS(w) is a subsystem of a KTS(v) only if the parallel classes of the KTS(w) are induced by the parallel classes of the KTS(v). We shall describe the subsystem as a sub-KTS(w). The obvious necessary conditions for the existence of a KTS(v) containing a sub-KTS(w) is $v \geq 3w$, $v \equiv w \equiv 3$ modulo 6. This problem has been studied in several recent papers, and the following results have been proved.

Theorem 1.1 [15, 12]. *For all* $v \equiv w \equiv 3$ *modulo 6,* $v \geq 4w - 9$*, there exists a KTS*(v) *containing a sub-KTS*(w).

Theorem 1.2 [11]. *For all* $w \equiv 3$ *modulo 6,* $v = 3w, 3w+6$ *or* $3w+12$*, there exists a KTS*(v) *containing a sub-KTS*(w)*, except possibly when* $w = 45, 51, 63,$ *or* 87 *and* $v = 3w + 12$.

Theorem 1.3 [12]. *Suppose* $v \equiv w \equiv 3$ *modulo 6,* $v \geq 3w$*, and* $v - w = 12s + 6$ *or* $12s + 12$*, where* $s \in \{0, 1, 2, 3, 4, 5, 6, 7, 20, 24, 25, 28, 29, 30, 31, 36, 40, 44, 45, 52, 59, 60, 63, 64, 65\}$ *or* $s \geq 68$*. Then there exists a KTS*(v) *containing a sub-KTS*(w).

There are precisely 884 ordered pairs (v, w), where $v \equiv w \equiv 3$ modulo 6 and $v \geq 3w$, which are not covered by any of the three theorems above. These are listed in the Appendix. In this paper, we eliminate all of these possible exceptions.

2. Constructions for Kirkman triple systems containing subsystems

We need to define several types of designs. First, we define a useful generalization of a PBD called a group-divisible design. A *group-divisible design* (or, GDD), is a triple $(X, \mathcal{G}, \mathcal{A})$, which satisfies the following properties:

(1) $\mathcal{G}$ is a partition of X into subsets called *groups*;
(2) $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point;
(3) every pair of points from distinct groups occurs in a unique block.

The *group-type* (or type) of a GDD $(X, \mathcal{G}, \mathcal{A})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. We usually use an "exponential" notation to describe group-types: a group-type $1^i 2^j 3^k \ldots$ denotes i occurrences of 1, j occurences of 2, etc. As with PBDs, we will say that a GDD is a K-GDD if $|A| \in K$ for every $A \in \mathcal{A}$. As well, we say that a GDD is resolvable if the blocks can be partitioned into parallel classes.

Now, we define the idea of a GDD with a hole. Informally, an *incomplete* GDD, or IGDD, is a GDD from which a sub-GDD is missing (this is the "hole"). We give a formal definition. An IGDD is a quadruple $(X, Y, \mathcal{G}, \mathcal{A})$ which satisfies the following properties:

(1) X is a set of *points*, and $Y \subseteq X$;
(2) $\mathcal{G}$ is a partition of X into *groups*;
(3) $\mathcal{A}$ is a set of *blocks*, each of which intersects each group in at most one point;
(4) no block contains two members of Y;
(5) every pair of points $\{x, y\}$ from distinct groups, such that at least one of x, y is in $X \backslash Y$, occurs in a unique block of $\mathcal{A}$.

We say that an IGDD $(X, Y, \mathcal{G}, \mathcal{A})$ is a K-IGDD if $|A| \in K$ for every block $A \in \mathcal{A}$. The *type* of the IGDD is defined to be the multiset of ordered pairs

$\{(|G|, |G \cap Y|): G \in \mathcal{G}\}$. As with GDDs, we shall use an exponential notation to describe types. Note that if $Y = \emptyset$, then the IGDD is a GDD.

We have already defined PBDs with subdesigns. If we allow the subdesign to be missing (i.e., a hole), we have an incomplete PBD, as follows. An *incomplete* PBD (or IPBD) is a triple $(X, Y, \mathcal{A})$, where X is a set of points, $Y \subseteq X$, and $\mathcal{A}$ is a set of blocks which satisfies the following properties:

(1) for any $A \in \mathcal{A}$, $|A \cap Y| \leq 1$;

(2) any two points x, y, not both in Y, occur in a unique block.

Hence, Y is the hole. Note that $(X, Y, \mathcal{A})$ is an IPBD if and only if $(X, \mathcal{A} \cup \{Y\})$ is a PBD. We say that $(X, Y, \mathcal{A})$ is a $(v, w; K)$-IPBD if $|X| = v$, $|Y| = w$, and $|A| \in K$ for every $A \in \mathcal{A}$.

We also employ a more general type of incomplete PBD. We are interested in the situation when we have two subdesigns, of given sizes, which intersect in a third subdesign of a given size. However, as usual, the subdesigns need not be present, i.e. we allow holes. We will refer to these designs as $\diamond$-IPBDs, in order to suggest the structure of the holes. We give a formal definition. An *incomplete* $\diamond$-PBD is a tuple $(X, Y_1, Y_2, \mathcal{A})$, where $Y_1 \subseteq X$, $Y_2 \subseteq X$, and $\mathcal{A}$ is a set of blocks such that every pair of points $\{x, y\}$ occurs in a unique block, unless $\{x, y\} \subseteq Y_1$ or $\{x, y\} \subseteq Y_2$, in which case the pair occurs in no block. We say that the $\diamond$-IPBD is a $(v; w_1, w_2; w_3; K)$-$\diamond$-IPBD if $|X| = v$, $|Y_1| = w_1$, $|Y_2| = w_2$, $|Y_1 \cap Y_2| = w_3$, and $|A| \in K$ for every $A \in \mathcal{A}$.

Our main application of $\diamond$-IPBDs involves using them to fill in the groups of IGDDs. This construction was presented in [17].

Construction 2.1 Filling in groups Let K be a set of positive integers, and let $ba0$. Suppose that the following designs exist:

(1) a K-IGDD of type $\{(t_1, u_1), (t_2, u_2), \ldots, (t_n, u_n)\}$;

(2) a $(t_i + b; u_i + a, b; a; K)$-$\diamond$-IPBD, for $1 \leq i \leq n - 1$; and

(3) a $(t_n + b; u_n + a)$-IPBD.

Then there exists a $(t + b, u + a; K)$-IPBD, where $t = \sum t_i$ and $u = \sum u_i$.

In this paper, we also make extensive use of Kirkman triple systems with holes of various types. We refer to these as *incomplete* KTS, or IKTS. If we have a KTS(v) containing a sub-KTS(w), we can remove the subsystem, leaving a hole. We shall denote the resulting incomplete system by (v, w)-IKTS. (Of course, if we have a (v, w)-IKTS, where $v \equiv w \equiv 3$ modulo 6, then we can fill a KTS(w) into the hole, constructing a KTS(v) containing a sub-KTS(w).) Next, suppose we have a KTS(v) which contains sub-KTS(w_1) and sub-KTS(w_2) which intersect in a sub-KTS(w_3). If we remove these subsystems, we obtain an incomplete system which we denote by $(v; w_1, w_2; w_3)$-$\diamond$-IKTS.

We also make extensive use of an object which can be thought of as a resolvable GDD having a *spanning* set of holes. A *K-frame* is a K-GDD $(X, \mathcal{H}, \mathcal{A})$, in which the set of blocks $\mathcal{A}$ can be partitioned into *holey parallel classes*, each of which is a partition of X/H, for some $H \in \mathcal{H}$. The members of $\mathcal{H}$ are called *holes*. We refer to a $\{3\}$-frame as a *Kirkman frame*.

As before, the *type* of a frame $(X, \mathcal{H}, \mathcal{A})$ is defined to be the multiset $\{|H|: H \in \mathcal{H}\}$. It can be shown (see [15] and [9]) that for each hole H of a $\{k\}$-frame, there are $|H|/(k-1)$ holey parallel classes which partition $X\backslash H$. It is also interesting to note that a $\{k\}$-frame of type $\{t_1, t_2, \ldots, t_n\}$ is equivalent to a $\{k+1\}$-IGDD of type $\{(kt_1/(k-1), t_1/(k-1)), (kt_2/(k-1), t_2/(k-1)), \ldots, (kt_n/(k-1), t_n/(k-1))\}$.

If we fill in the holes of Kirkman frames, we can construct KTS with sub-KTS, as follows.

Construction 2.2 Filling In Holes Let $a \geq 0$, and suppose that there exists a Kirkman frame of type $\{t_1, t_2, \ldots, t_n\}$; a KTS$(t_i + a)$ containing a sub-KTS(a), for $1 \leq i \leq n-1$; and a KTS$(t_n + a)$. Then there exists a KTS$(t + a)$, where $t = \sum t_i$, containing a sub-KTS$(t_n + a)$.

We also use incomplete (Kirkman) frames, which bear the same relationship to Kirkman frames as IGDDs do to GDDs. An *incomplete K-frame* is a K-IGDD $(X, Y, \mathcal{H}, \mathcal{A})$ in which the set of blocks $\mathcal{A}$ can be partitioned into *holey parallel classes*, each of which is a partition of X/H, for some $H \in \mathcal{H}$, or a partition of $X\backslash(H \cup Y)$, for some $H \in \mathcal{H}$. It can be shown that for each hole H, there are $|H \cap Y|/2$ holey parallel classes which partition $X\backslash(H \cup Y)$, and $|H\backslash Y|/2$ holey parallel classes which partition $X\backslash H$.

We also construct KTS containing sub-KTS by filling in the holes of incomplete Kirkman frames with $\diamond$-IKTS.

Construction 2.3 Generalized Filling In Holes Let $b \geq a \geq 0$. Suppose that the following designs exist:

(1) an incomplete Kirkman frame of type $\{(t_1, u_1), (t_2, u_2), \ldots, (t_n, u_n)\}$;
(2) a $(t_i + b;\ u_i + a,\ b;\ a)$-$\diamond$-IKTS, for $1 \leq i \leq n-1$; and
(3) a $(t_n + b;\ u_n + a)$-IKTS.

Then there exists a $(t + b,\ u + a)$-IKTS, where $t = \sum t_i$ and $u = \sum u_i$.

We also observe that if we fill in all but one group of a Kirkman frame (Construction 2.2), we obtain an IKTS, and if we fill in all but two groups, we obtain a $\diamond$-IKTS.

It will be necessary to build families of IGDDs. Our basic construction for IGDDs is a recursive one. We refer to it as the "Fundamental IGDD Construction" (see [8] and [11]).

Construction 2.4 Fundamental IGDD Construction Suppose $(X, Y, \mathcal{G}, \mathcal{A})$ is an IGDD, and let $t, s: X \rightarrow \mathbf{Z}^+ \cup \{0\}$ be functions such that $t(x) \leq s(x)$, for every $x \in X$. For every block $A \in \mathcal{A}$, suppose that we have a K-IGDD of type $\{(s(x), t(x)): x \in A\}$. Suppose also that we have a K-IGDD of type

$\{(\sum_{x\in G\cap Y} s(x), \sum_{x\in G\cap Y} t(x)) : G \in \mathcal{G}\}$. Then there exists a K-IGDD of type $\{(\sum_{x\in G} s(x), \sum_{x\in G} t(x)) : G \in \mathcal{G}\}$.

We use a similar IGDD construction to build frames and incomplete frames (see [15]).

Construction 2.5 Fundamental Frame Construction Suppose $(X, Y, \mathcal{G}, \mathcal{A})$ is an IGDD, and let $s: X \to \mathbf{Z}^+ \cup \{0\}$ be a function. For every block $A \in \mathcal{A}$, suppose that we have a Kirkman frame of type $\{(s(x): x \in A\}$. Then there exists an incomplete Kirkman frame of type $\{(\sum_{x\in G} s(x), \sum_{x\in G\cap Y} s(x)): G \in \mathcal{G}\}$.

As applications of the above, we mention a family of constructions which are called the product constructions. These utilize (incomplete) transversal designs, which we now define. A *transversal design* TD(k, n) is a $\{k\}$-GDD of type n^k. It is well-known that a TD(k, n) is equivalent to $k-2$ mutually orthogonal Latin squares (MOLS) of order n. We also define a TD(k, n) - TD(k, m) (an *incomplete* transversal design) to be a $\{k\}$-IGDD of group-type $(n, m)^k$.

The most general product construction is referred to as the *Generalized Singular Indirect Product*, or GSIP.

Construction 2.6 Generalized Singular Indirect Product Suppose u, t, v, w, a, and b are non-negative integers such that $0 \leq b - a \leq v - w$, $a \leq w \leq v$. Suppose that the following designs exist:

(1) a Kirkman frame of type t^u;
(2) a TD$(u, (v-b)/t)$–TD$(u, (w-a)/t)$;
(3) a $(v; w, b; a)$-$\diamond$-IKTS; and
(4) a (b, a)-IKTS.

Then there exists a $(u(v-b)+b, u(w-a)+a)$-IKTS.

Proof: Start with the given incomplete TD, give every point weight t, and apply the Fundamental Frame Construction. We get an incomplete Kirkman frame of type $(v-b, w-a)^u$. Now fill in the holes. ∎

When $b = a$, we obtain the *Singular Indirect Product*, or SIP.

Construction 2.7 Singular Indirect Product Suppose u, t, v, w, and a are non-negative integers such that $0 \leq a \leq w \leq v$. Suppose that the following designs exist:

(1) a Kirkman frame of type t^u;
(2) a TD$(u, (v-a)/t)$–TD$(u, (w-a)/t)$; and
(3) a $(v; w)$-IKTS.

Then there exists a $(u(v-a)+a, u(w-a)+a)$-IKTS.

When $b = a = w$, we obtain the *Singular Direct Product*, or SDP.

Construction 2.8 Singular Direct Product Suppose u, t, v and w are non-negative integers such that $w \leq v$. Suppose that the following designs exist:

(1) a Kirkman frame of type t^u;

(2) a $TD(u,(v-w)/t)$;
(3) a (v,w)-IKTS; and
(4) a $KTS(w)$.

Then, there is a $(u(v-w)+w,w)$-IKTS and a $(u(v-w)+w,v)$-IKTS.

3. Applications of the constructions

We now describe two recursive constructions for producing Kirkman triple systems containing Kirkman subsystems, which will eliminate all but 31 of the 884 exceptions. These constructions will make use of Kirkman frames constructed in [15].

Lemma 3.1. *There exists a Kirkman frame of type t^u if and only if t is even, $u \geq 4$, and $t(u-1) \equiv 0 \bmod 3$.*

Proof: See [15, Theorem 4.5]. ∎

We shall use certain incomplete TDs.

Lemma 3.2. *For all positive integers v and w such that $v \geq 3w$ and $(v,w) \neq (6,1)$, there is a $TD(4,v)-TD(4,w)$.*

Proof: See [4]. ∎

We shall also require some particular classes of ◇-IKTS.

Lemma 3.3. *For all $m \geq 0$, there exists a $(18m+9;6m+3,9;3)$-◇-IKTS.*

Proof: For $m=0$, the design exists trivially. For $m>0$, start with a $TD(4,6m+3)-TD(4,3)$ (Lemma 3.2). Delete all the points in one group. Then, on two of the remaining groups, fill in $(6m+3,3)$-IKTS. ∎

Lemma 3.4. *For all $m \geq 2$, there exists a $(18m+15;\ 6m+3,15;\ 3)$-◇-IKTS.*

Proof: For $m=2$, use a Kirkman frame of type 12^4, filling in two holes with $(15,3)$-IKTS. For $m \geq 3$, start with a resolvable $\{3\}$-GDD of type 6^{m+1} (see [10] and [1]). Adjoin infinite points to the parallel classes of this GDD, to construct a $\{4\}$-GDD of type $6^{m+1}(3m)^1$. Give every point weight 2 and apply the Fundamental Frame Construction. This produces a frame of type $12^{m+1}(6m)^1$. Now fill in all but one hole of size 12 with $(15,3)$-IKTS. ∎

Construction 3.1 Suppose there exists a $TD(6,m)$, $0 \leq t \leq m$, and $0 \leq u \leq m$. Let $a \geq 0$. Suppose there exist $KTS(6m+a)$ and $KTS(6m+6t+a)$, each containing a $KTS(a)$, and a $KTS(6m+6u+a)$. Then there exists a $KTS(36m+6t+6u+a)$ containing a sub-$KTS(6m+6u+a)$.

Proof: Start with a $TD(6,m)$, and give the points in the first four groups weight 6; give t of the points in the 5th group weight 12, and give the remaining points in the 5th group weight 6; and give u of the points in the 6th group weight 12, and give the remaining points in the 6th group weight 6. In order to apply the Fundamental Frame Construction, we need Kirkman frames of types 6^6, $6^5 12^1$ and $6^4 12^2$, which are obtained as follows. A frame of type 6^6 exists by Lemma 3.1. We get a frame of type $6^5 12^1$ by applying the Fundamental Construction to a $\{4\}$-GDD of type $3^5 6^1$, giving every point weight 2 (this GDD is produced by adjoining infinite points to 6 parallel classes of a KTS(15)). Similarly, we get a frame of type $6^4 12^2$ by applying the Fundamental Construction to a $\{4\}$-GDD of type $3^4 6^2$, giving every point weight 2 (this GDD is exhibited in the Appendix of [12]). Hence, we build a Kirkman frame of type $(6m)^4(6m+6t)^1(6m+6u)^1$ (for any $0 \le t \le m$, $0 \le u \le m$). We now add on a infinite points, and fill in KTS containing sub-KTS(a), and the KTS($6m+6u+a$). ∎

Construction 3.2 Suppose there is a TD($6, m$), $0 \le t \le m$, $0 \le u \le m$, and $a = 3$ or 6. Then there exists a KTS($72m + 18t + 12u + 2a + 3$) containing a sub-KTS($24m + 6t + 3$).

Proof: Start with a TD($6, m$), delete $m - t$ points from the 5th group, and delete $m - u$ points from the 6th group. Then, give the points in the first five groups weights (9, 3), and give the points in the 6th group weights (6, 0). In order to apply the Fundamental IGDD construction, we need $\{4\}$-IGDDs of types $(9,3)^4$, $(9,3)^5$, $(9,3)^4 6^1$, and $(9,3)^5 6^1$. The first two IGDDs are equivalent to frames of types 6^4 and 6^5 respectively (see the remark preceding Construction 2.2). The last two IGDDs are presented in the Appendix of [11]. Then, we obtain a $\{4\}$-IGDD of type $(9m, 3m)^4(9t, 3t)^1(6u)^1$. Next, assign every point weight 2, and apply the Fundamental Frame Construction. This produces an incomplete frame of type $(18m, 6m)^4(18t, 6t)^1(12u)^1$. Next, we will fill ◇-IKTS into the holes of the frame, using Construction 2.3. We adjoin a total of $2a + 3$ points, 3 of which are incorporated into the sub-KTS.

If $a = 3$, then we fill in $(18m+9; 6m+3, 9; 3)$-◇-IKTS, $(18t+9; 6t+3, 9; 3)$-◇-IKTS, and $(12u + 9, 3)$-IKTS. These exist from Lemma 3.3.

If $a = 6, t > 1$, then we fill in $(18m + 15; 6m + 3, 15; 3)$-◇-IKTS, $(18t + 15; 6t + 3, 15; 3)$-◇-IKTS, and $(12u + 15, 3)$-IKTS. These exist from Lemma 3.4.

If $a = 6, t = 1, u \neq 1, 2$, then we instead use $(18m + 15; 6m + 3, 15; 3)$-◇-IKTS, $(12u + 15; 3, 15; 3)$-◇-IKTS, and $(18t + 15, 9)$-IKTS.

If $a = 6, t = 1$, and $u = 1$ or 2, we proceed slightly differently. We start with a TD($5, m$) and delete $m - u - 1$ points from the 5th group. Give all points weight (9, 3), except for u points in the fifth group, which get weights (6, 0). Proceeding as before, we obtain an incomplete Kirkman frame of type $(18m, 6m)^4(12u + 18, 6)^1$. Now, fill in $(18m + 15; 6m + 3, 15; 3)$-◇-IKTS, and $(12u + 33, 9)$-IKTS.

This covers all cases, so the proof is complete. ∎

By computer, we established that Constructions 3.1 and 3.2 eliminate all but 31 ordered pairs (v, w), which are presented in Table 1. Appropriate applications of Constructions 3.1 and 3.2 for the remaining 853 ordered pairs are given in the research report [13].

Table 1
The 31 exceptions remaining after application of Constructions 3.1 and 3.2

(141,39)	(147,45)	(153,45)	(159,45)	(165,45)
(165,51)	(171,51)	(177,51)	(183,51)	(189,51)
(189,57)	(195,57)	(201,57)	(201,63)	(207,63)
(255,69)	(261,69)	(261,75)	(267,75)	(273,75)
(279,75)	(261,81)	(267,81)	(273,81)	(279,81)
(285,81)	(273,87)	(279,87)	(285,87)	(291,87)
(297,93)				

15 of the exceptions in Table 1 can be eliminated by the product constructions. We list these in Table 2. In all applications, $t = 2$ and $u = 4$, so we are using Kirkman frames of type 2^4. The requisite incomplete TDs exist by Lemma 3.2.

At this point, 16 ordered pairs remain as possible exceptions. We eliminate most of these using a well-known PBD construction. First, we define the set $K_{1,3} = \{k \equiv 1 \bmod 3,\ k \geq 4\}$.

Lemma 3.5. *Suppose there is a $(v, w; K_{1,3})$-IPBD. Then there is a $(2v+1, 2w+1)$-IKTS.*

Proof: The IPBD gives rise to a GDD of type $w^1 1^{v-w}$. Give every point weight 2, and apply the Fundamental Frame Construction. This produces a frame of type $(2w)^1 2^{v-w}$. Now fill KTS(3) into the holes of size 2. ∎

Lemma 3.6. *Suppose there is a resolvable $\{4\}$-GDD of type t^u, where $t \equiv 0 \bmod 3$, and let $0 \leq s \leq t(u-1)/3$. Then there is a $(6tu+6s+9, 2tu+3)$-IKTS.*

Proof: Adjoin infinite points to s of the parallel classes of the GDD. This produces a $\{4,5\}$-GDD of type $t^u s^1$ in which every block of size 5 hits the group of size s. Assign weights (3, 1) to every point of the original GDD, and assign weights (3, 0) to the s infinite points. Apply the Fundamental IGDD construction, using $\{4\}$-IGDDs of types $(3,1)^4(3,0)^1$ and $(3,1)^4$. (These arise from deleting a block from $\{4\}$-GDDs of types 3^4 and 3^5, respectively.) We construct in this way a $\{4\}$-IGDD of type $(3t,t)^u(3s)^1$.

Next, we fill in the groups of this IGDD with ◇-IPBDs using Construction 2.1. Set $a = 1$ and $b = 4$. We use $(3t+4; t+1, 4; 1; K_{1,3})$-◇-IPBDs, which are constructed by adjoining $t+1$ infinite points to the parallel classes of a KTS$(2t+3)$. For the last group, we use a block of size $3s+4$. This gives us a $(3tu+3s+4, tu+1; K_{1,3})$-IPBD. Now, apply Lemma 3.5. ∎

Table 2

v	w	construction		ingredients	remarks
153	45	SDP	153 = 4(45 − 9) + 9	(45,9)-IKTS TD(4,18)	
165	45	SIP	165 = 4(45 − 5) + 5 45 = 4(15 − 5) + 5	 TD(4,20) − TD(4,5)	(45,15)-IKTS
171	51	SIP	171 = 4(45 − 3) + 3 51 = 4(15 − 3) + 3	 TD(4,21) − TD(4,6)	(45,15)-IKTS
177	51	SDP	177 = 4(51 − 9) + 9	(51,9) − *IKTS* TD(4,21)	
183	51	GSIP	183 = 4(57 − 15) + 15 51 = 4(15 − 3) + 3	(57; 15,15; 3)-◇-IKTS TD(4,21) − TD(4,6)	a {4}-GDD of type $6^4 3^1$ gives rise to a frame of type $12^4 6^1$, and so to the required ◇-IKTS.
201	57	SDP	201 = 4(57 − 9) + 9	(57,9)-IKTS TD(4,24)	
207	63	SDP	207 = 4(63 − 15) + 15	(63,15)-IKTS TD(4,24)	
261	69	SIP	261 = 4(69 − 5) + 5 69 = 4(21 − 5) + 5	 TD(4,32) − TD(4,8)	(69,21)-IKTS
267	75	SIP	267 = 4(69 − 3) + 3 75 = 4(21 − 3) + 3	 TD(4,33) − *TD*(4,9)	(69,21)-IKTS
273	75	SDP	273 = 4(75 − 9) + 9	(75,9)-IKTS TD(4,33)	
279	75	GSIP	279 = 4(81 − 15) + 15 75 = 4(21 − 3) + 3	(81; 15,21; 3)-◇-IKTS TD(4,33) − TD(4,9)	a {4}-GDD of type $6^5 9^1$ gives rise to a frame of type $12^5 18^1$, and so to the required ◇-IKTS.
261	81	SDP	261 = 4(81 − 21) + 21	(81,21)-IKTS TD(4,30)	
273	81	SIP	273 = 4(69 − 1) + 1 81 = 4(21 − 1) + 1	(69,21)-IKTS TD(4,34) − TD(4,10)	
279	81	SDP	279 = 4(81 − 15) + 15	(81,15)-IKTS TD(4,33)	
285	87	SDP	285 = 4(87 − 21) + 21	(87,21)-IKTS TD(4,33)	

In a similar fashion, we have

Lemma 3.7. *Suppose there is a resolvable* $\{4\}$*-GDD of type* t^u*, where* $t \equiv 1 \bmod 3$*, and let* $0 \le s \le t(u-1)/3$*. Then there is a* $(6tu + 6s + 3, 2tu + 1)$*-IKTS.*

Proof: As before, construct a $\{4\}$-IGDD of type $(3t, t)^u (3s)^1$. Then, set $a = 0$ and $b = 1$. We fill in $(3t+1, t; K_{1,3})$-IPBDs (which are constructed by adjoining t infinite points to the parallel classes of a KTS$(2t+1)$) and a block of size $3s+1$ (if $s > 0$). We get a $(3tu + 3s + 1, tu; K_{1,3})$-IPBD. Now, apply Lemma 3.5. ∎

A slightly different application of the same idea gives us

Lemma 3.8. *Suppose there is a* $\{4\}$*-frame of type* t^u*, where* $t \equiv 0 \bmod 3$*, and* $0 \leq s \leq t/3$*. Also, suppose there is a* $(3t+3s+4, t+1; K_{1,3})$*-IPBD. Then there exists a* $(6tu+6s+9, 2tu+3)$*-IKTS.*

Proof: Adjoin s points to a hole of the frame, constructing a $\{4,5\}$-IGDD of type $t^{u-1}(t+s, s)^1$. Assign weights (3, 1) to every point of the original GDD, and assign weights (3, 0) to the s infinite points. Apply the Fundamental IGDD Construction, resulting in a $\{4\}$-IGDD of type $(3t,t)^{u-1}(3t+3s,3t)^1$. Then, set $a = 1$ and $b = 4$. We use $(3t+4; t+1, 4; 1; K_{1,3})$-$\diamond$-IPBDs, which are constructed as in Lemma 3.6. For the last group, we use the $(3t+3s+4, t+1; K_{1,3})$-IPBD. This gives us a $(3tu+3s+4, tu+1; K_{1,3})$-IPBD. Apply Lemma 3.5. ▮

We list several applications of these constructions in Table 3.

Table 3

v	w	lemma		ingredients	remarks
141	39	3.5		$(70, 19; \{4\})$-IPBD	see [7]
147	45	3.6	$t=3, u=8, s=2$	$(73, 22; \{4,7,10\})$-IPBD	see Appendix
159	45	3.5		$(79, 22; \{4\})$-IPBD	see [6]
165	51	3.6	$t=3, u=8, s=2$	resolvable $\{4\}$-GDD of type 3^8	see [5]
189	51	3.6	$t=3, u=8, s=6$	resolvable $\{4\}$-GDD of type 3^8	see [5]
189	57	3.7	$t=4, u=7, s=3$	resolvable $\{4\}$-GDD of type 4^7	see [3]
195	57	3.7	$t=4, u=7, s=4$	resolvable $\{4\}$-GDD of type 4^7	see [3]
201	63	3.8	$t=6, u=5, s=2$	$\{4\}$-frame of type 6^5 TD(4,7)	see [16]
261	75	3.6	$t=9, u=4, s=6$	resolvable TD(4,9)	
267	81	3.7	$t=4, u=10, s=4$	resolvable $\{4\}$-GDD of type 4^{10}	see [3]
285	81	3.7	$t=4, u=10, s=7$	resolvable $\{4\}$-GDD of type 4^{10}	see [3]
297	93	3.8	$t=9, u=5, s=3$	$\{4\}$-frame of type 9^5 TD(4,10)	see [16]

We can eliminate the four remaining exceptions by ad hoc means.

Lemma 3.9. *There is a KTS(279) containing a sub-KTS(87).*

Proof: Start with a $\{4\}$-GDD of type $6^4 9^1$ (this is obtained by adjoining infinite points to the 9 parallel classes of a resolvable $\{3\}$-GDD of type 6^4, which is constructed in [10]). Give every point weight 8, and apply the frame construction. This produces a frame of type $48^4 72^1$. Now adjoin 15 infinite points, filling in (63, 15)-IKTS and KTS(87). ▮

Lemma 3.10. *There is a KTS(255) containing a sub-KTS(69).*

Proof: Start with a TD(5,4). Give every point weight 12, except for one point which gets weight 6, and apply the frame construction. We fill in frames of type 12^5 and $12^4 6^1$ (this latter frame is obtained by applying the frame construction to a $\{4\}$-GDD of type $6^4 3^1$, giving the points weight 2). This produces a frame of type $48^4 42^1$. Now adjoin 21 infinite points, filling in $(69,21)$-IKTS, $(63,21)$-IKTS, and KTS(69). ▮

Lemma 3.11. *There is a KTS(273) containing a sub-KTS(87).*

Proof: A $\{4\}$-IGDD of type $(9,3)^4 6^1$ is given in the Appendix of [11]. The design is presented as a $\{3,4\}$-GDD of type 6^4 whose blocks of size three can be partitioned into twelve holey parallel classes. Three of these holey parallel classes correspond to each of the first four groups. It is easy to verify that the blocks of size four can be partitioned into two holey parallel classes having the the fifth group as their hole. Hence, we can construct from this design a $\{4,5\}$-IGDD of type $(9,3)^4(8,2)^1$. Now, apply Construction 2.5, giving every point weight 6. We obtain a Kirkman frame of type $(54,18)^4(48,12)^1$. Then apply Construction 2.3 with $b = 9$ and $a = 3$, filling in $(63; 21, 9; 3)$-$\diamond$-IKTS (which is obtained by applying Lemma 3.5 to a $(31,10;\{4\})$-IPBD) and a $(57,15)$-IKTS. ▮

Lemma 3.12. *There is a KTS(291) containing a sub-KTS(87).*

Proof: Remove a point from the hole of the $(73,22;\{4,7,10\})$-IPBD given in the appendix, producing a $\{4,7\}$-GDD of type $3^{14}9^1 21^1$. Give every point weight four and apply the Fundamental Frame Construction. Then fill in KTS(15), a KTS(39) and a KTS(87). ▮

As a result of all the above constructions, we have our main result.

Theorem. *There is a KTS(v) containing a sub-KTS(w) if and only if* $v \equiv w \equiv 3$ *modulo 6 and* $v \geq 3w$.

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Appendix

The 884 exceptions remaining after Theorems 1.1, 1.2 and 1.3

w	values of v									
39 :	141									
45 :	147	153	159	165						
51 :	165	171	177	183	189					
57 :	189	195	201	207	213					
63 :	201	207	213	219	225	231	237			
69 :	225	231	237	243	249	255	261			
75 :	243	249	255	261	267	273	279	285		
81 :	261	267	273	279	285	291	297	303	309	
87 :	273	279	285	291	297	303	309	315	321	327
93 :	297	303	309	315	321	327	333	351	357	
99 :	315	321	327	333	339	357	363	369	375	381
105 :	333	339	345	363	369	375	381	387	393	
111 :	351	369	375	381	387	393	399	429		
117 :	375	381	387	393	399	405	435	441	447	453
123 :	387	393	399	405	411	441	447	453	459	
129 :	405	411	417	447	453	459	465			
135 :	423	453	459	465	471	525				
141 :	459	465	471	477	531	537	543	549		
147 :	465	471	477	483	537	543	549	555	561	567	573

153 : 477 483 489 543 549 555 561 567 573 579 585

159 : 495 549 555 561 567 573 579 585 591 609 615 621

165 : 555 561 567 573 579 585 591 597 615 621 627 633 639 645

171 : 561 567 573 579 585 591 597 603 621 627 633 639 645 651 669

177 : 567 573 579 585 591 597 603 609 627 633 639 645 651 657 675
681 687 693

183 : 573 579 585 591 597 603 609 615 633 639 645 651 657 663 681
687 693 699 705 711

189 : 585 591 597 603 609 615 621 639 645 651 657 663 669 687 693
699 705 711 717

195 : 603 609 615 621 627 645 651 657 663 669 675 693 699 705 711
717 723 753 759 765

201 : 621 627 633 651 657 663 669 675 681 699 705 711 717 723 729
759 765 771 777 783 789

207 : 639 657 663 669 675 681 687 705 711 717 723 729 735 765 771
777 783 789 795 801 807 813

213 : 663 669 675 681 687 693 711 717 723 729 735 741 771 777 783
789 795 801 807 813 819 825 831 837

219 : 675 681 687 693 699 717 723 729 735 741 747 777 783 789 795
801 807 813 819 825 831 837 843 861

225 : 693 699 705 723 729 735 741 747 753 783 789 795 801 807 813
819 825 831 837 843 849 867 873 879 885

231 : 711 729 735 741 747 753 759 789 795 801 807 813 819 825 831
837 843 849 855 873 879 885 891 897 903 909

237 : 735 741 747 753 759 765 795 801 807 813 819 825 831 837 843
849 855 861 879 885 891 897 903 909 915 921 927 933

243 : 747 753 759 765 771 801 807 813 819 825 831 837 843 849 855
861 867 885 891 897 903 909 915 921 927 933 939 945 951

249 : 765 771 777 807 813 819 825 831 837 843 849 855 861 867 873
891 897 903 909 915 921 927 933 939 945 951 957

255 : 783 813 819 825 831 837 843 849 855 861 867 873 879 897 903
909 915 921 927 933 939 945 951 957 963 993 999 1005

261 : 819 825 831 837 843 849 855 861 867 873 879 885 903 909 915
921 927 933 939 945 951 957 963 969 999 1005 1011 1017

267 : 825 831 837 843 849 855 861 867 873 879 885 891 909 915 921
927 933 939 945 951 957 963 969 975 1005 1011 1017 1023

273 : 837 843 849 855 861 867 873 879 885 891 897 915 921 927 933
939 945 951 957 963 969 975 981 1011 1017 1023 1029 1071 1077

279 : 855 861 867 873 879 885 891 897 903 921 927 933 939 945 951
957 963 969 975 981 987 1017 1023 1029 1035 1077 1083 1089 1095

285 : 873 879 885 891 897 903 909 927 933 939 945 951 957 963 969
975 981 987 993 1023 1029 1035 1041 1083 1089 1095 1101

291 : 891 897 903 909 915 933 939 945 951 957 963 969 975 981 987
993 999 1029 1035 1041 1047 1089 1095 1101 1107

297 : 909 915 921 939 945 951 957 963 969 975 981 987 993 999 1005
1035 1041 1047 1053 1095 1101 1107 1113

303 : 927 945 951 957 963 969 975 981 987 993 999 1005 1011 1041
1047 1053 1059 1101 1107 1113 1119
309 : 951 957 963 969 975 981 987 993 999 1005 1011 1017 1047 1053
1059 1065 1107 1113 1119 1125
315 : 963 969 975 981 987 993 999 1005 1011 1017 1023 1053 1059 1065
1071 1113 1119 1125 1131
321 : 981 987 993 999 1005 1011 1017 1023 1029 1059 1065 1071 1077 1119
1125 1131 1137
327 : 999 1005 1011 1017 1023 1029 1035 1065 1071 1077 1083 1125 1131 1137
1143
333 : 1017 1023 1029 1035 1041 1071 1077 1083 1089 1131 1137 1143 1149
339 : 1035 1041 1047 1077 1083 1089 1095 1137 1143 1149 1155
345 : 1053 1083 1089 1095 1101 1143 1149 1155 1161
351 : 1089 1095 1101 1107 1149 1155 1161 1167
357 : 1095 1101 1107 1113 1155 1161 1167 1173
363 : 1107 1113 1119 1161 1167 1173 1179
369 : 1125 1167 1173 1179 1185
375 : 1173 1179 1185 1191
381 : 1179 1185 1191 1197
387 : 1185 1191 1197 1203
393 : 1197 1203 1209
399 : 1215

A $(73, 22; \{4, 7, 10\})$-IPBD

Points: $Z_{42} \cup \{a, b_0, b_1, c_0, c_1, d_0, d_1, e_0, e_1\} \cup \{\infty_i : 0 \le i \le 20\} \cup \{\infty\}$.

Blocks: Develop the following modulo 42, where subscripts on ∞ are developed modulo 21 and subscripts on letters are developed modulo 2.

$0, 6, 12, 18, 24, 30, 36$	$\infty, a, b_0, b_1, c_0, c_1, d_0, d_1, e_0, e_1$
$\infty, 0, 14, 28$	$\infty_0, a, 11, 22$
$\infty_0, b_0, 2, 29$	$\infty_0, c_0, 3, 34$
$\infty_0, d_0, 4, 17$	$\infty_0, e_0, 6, 41$
$\infty_0, 0, 22, 26$	$\infty_0, 28, 31, 36$
$\infty_0, 30, 39, 40$	$\infty_0, 12, 14, 37$