Kirkman Triple Systems with Maximum Subsystems

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ABSTRACT

A Kirkman Triple System ($KTS(v)$) is a resolvable $(v,3,1)-BIBD$; it is well-known that such a system exists if and only if $v \equiv 3$ modulo 6. In this paper we investigate the following problem: for which $w \equiv 3$ modulo 6 do there exist $KTS(3w)$, $KTS(3w+6)$ and $KTS(3w+12)$, each containing a sub-$KTS(w)$? We are able to solve this problem for all but four values of w.

1. Introduction

Kirkman Triple Systems have been the source of an intense amount of study since they were first introduced well over a century ago by T.P. Kirkman in his famous 'Schoolgirls' problem (see [4] for a historical account and references to this problem). It was not until 1971 that a complete solution to the problem of determining the spectrum for these designs appeared:

Theorem (Ray-Chaudhuri and Wilson, [4]). There exists a Kirkman Triple System $KTS(v)$ if and only if $v \equiv 3$ modulo 6.

We can think of a KTS as being a triple (X,B,P) where X is the set of points, B is the set of blocks and P is a partition of B into subsets (called *parallel classes*), each parallel class forming a partition of X. Then a KTS (X',B',P') is a *subsystem* of (X,B,P) if $X' \subseteq X$, $B' \subseteq B$ and if for each $p' \in P'$ there is a $p \in P$ such that $p' \subseteq p$. That is, each parallel class on the subsystem must be 'inherited' from the mother system. In particular then it is easy to see that if a $KTS(v)$ has a (proper) sub-$KTS(w)$, it must be that $v \geq 3w$. If $v = 3w, 3w+6$ or $3w+12$ then we will call such a subsystem a *maximum subsystem*, since in these cases a $KTS(v)$ could not contain a subsystem with more than w points.

The general problem of determining the existence of Kirkman Triple Systems with subsystems of a given size was the subject of a recent paper by one of the authors, who obtained the following result.

Theorem 1.1 (Stinson, [9]). Given v and w with $v \equiv w \equiv 3$ modulo 6 and $v \geq 4w-9$ there exists a $KTS(v)$ containing a sub-$KTS(w)$, except possibly when $v = 81$ and $w = 15$, or $v = 87$ and $w = 21$.

In view of this result one only need consider KTSs with 'large' subsystems. A particularly interesting sub-problem of this is to determine the spectrum for KTSs with maximum subsystems, and it is with this problem that we are herein concerned. We will prove the following result.

Theorem 1.2. Given any $w \equiv 3$ modulo 6 there exists a $KTS(v)$ containing a sub-$KTS(w)$ where $v = 3w, 3w+6$ or $3w+12$, except possibly when $v = 3w+12$ and $w = 45$, 51, 63 or 87.

Of central importance to our work will be the notion of a group-divisible design. A *group-divisible design* (*GDD*) is a triple (X,G,B) where X is a set of points, G is a partition of X into subsets (called *groups*) and B is a collection of subsets of X (blocks) such that

(i) $|B_i \cap G_j| \leq 1$ for all $B_i \in B$ and $G_j \in G$, and

(ii) each pair of points from distinct groups occurs in exactly one block.

An *incomplete group-divisible design* (*IGDD*) occurs when we assign to each $G_j \in G$ a (possibly empty) subset $H_j \subseteq G_j$ and replace condition (ii) by

(ii)′ a pair of points $x \in G_{j_1}$ and $y \in G_{j_2}$ $(j_1 \neq j_2)$ occurs in exactly one block unless $x \in H_{j_1}$ and $y \in H_{j_2}$, in which case x and y do not occur together in any block.

Note that when all $H_j = \emptyset$ an *IGDD* is just a *GDD*. We will describe *GDD*'s and *IGDD*'s by means of an exponential notation: a K–*GDD* of type $g_1^{t_1} g_2^{t_2} \ldots g_r^{t_r}$ is a *GDD* in which there are t_i groups of size g_i, $i = 1,\ldots,r$ and in which each block has size from the set K; a K–*IGDD* of type $(g_1,h_1)^{t_1}(g_2,h_2)^{t_2} \cdots (g_r,h_r)^{t_r}$ is an *IGDD* with t_i groups of size g_i, each one assigned a 'subgroup' of size h_i in the aforementioned sense, and in which each block has size from the set K. When some $h_i = 0$ we will suppress it; thus a 4–*IGDD* of type $(9,3)^4 6^1$ means a 4–*IGDD* of type $(9,3)^4(6,0)^1$. In some cases it will be convenient to say instead K–*GDD* of type S, where S is the multiset consisting of t_i copies of g_i, or K–*IGDD* of type S, where S is the multiset consisting of t_i copies of the (ordered) pair (g_i,h_i), where $i = 1,\ldots,r$.

The following construction is essentially equivalent to construction 4.4 in [3].

Construction 1.3. Let (X,G,B) be a group-divisible design and let $w:X \rightarrow \mathbb{Z}^+ \cup \{0\}$ and $d:X \rightarrow \mathbb{Z}^+ \cup \{0\}$ be non-negative integer functions on X, where $d(x) \leq w(x)$ for all $x \in X$. Suppose that for each block $b \in B$ there is a K–*IGDD* of type $\{(w(x),d(x)):x \in b\}$ and that for some fixed non-negative integer a there is a K–*GDD* on $a + \sum_{x \in G_j} w(x)$ points having a group of size a and a group of size $\sum_{x \in G_j} d(x)$, for each $G_j \in G$. Then there is a K–*GDD* on $a + \sum_{x \in X} w(x)$ points having a group of size a and a group of size $\sum_{x \in X} d(x)$.

A group-divisible design is called *resolvable* if its block set can be partitioned into subsets (parallel classes), each parallel class forming a partition of the point set. In [8] the authors considered the problem of constructing resolvable 3–*GDD*s and obtained a result which implies the following.

Theorem 1.4. Let g and u be given with $gu \equiv 0$ modulo 3 and $g(u-1) \equiv 0$ modulo 2, $(g,u) \neq (2,3), (2,6), (6,3)$. Then there exists a resolvable 3–*GDD* of type g^u, except possibly when $g \equiv 6$ modulo 12 and $u = 11$, 14, or $g \equiv 2$ or 10 modulo 12 and $u = 6$.

A *frame* is a group-divisible design (X,G,B) whose block set can be partitioned into *holey parallel classes*, i.e. each holey parallel class is a partition of $X - G_j$ for some group $G_j \in G$. The groups in a frame are referred to as *holes*. Frames were first introduced in connection with the study of Room Squares (see e.g. [13], [3]), while frames with more than one block size have been used by one of the authors in connection with the $g^{(k)}(v)$ problem (see [5], [6] and [7]). We are concerned here with a class of frames called *Kirkman frames*, which are frames in which every block has size 3. These designs were studied by Stinson [9], who obtained necessary and sufficient conditions for the existence of Kirkman frames with uniform group sizes:

Theorem 1.5. There exists a Kirkman frame of type g^u if and only if g is even, $u \geq 4$ and $g(u-1) \equiv 0$ modulo 3.

Remark. It is noted in [9] that in a Kirkman frame (X,G,B) there are $\frac{1}{2}|G_j|$ holey parallel classes that partition $X-G_j$, for each $G_j \in G$. In particular then, it is not difficult to see that a Kirkman frame of type g^u is equivalent to a $4-IGDD$ of type $(\frac{3}{2}g,\frac{1}{2}g)^u$. (For a more detailed discussion of this relationship the reader is referred to [10].)

Kirkman frames (and other classes of frames) can be built from group-divisible designs by means of the following simple construction (which, in view of the above remark, is very similar in nature to construction 1.3).

Construction 1.6 ([9], Construction 3.1). Let (X,G,B) be a group-divisible design and $w:X \rightarrow \mathbb{Z}^+ \cup \{0\}$ be a non-negative integer function on X (w is called a *weighting*). Suppose that for each block $b \in B$ there is a Kirkman frame of type $\{w(x):x \in b\}$. Then there is a Kirkman frame of type $\{\sum_{x \in G_j} w(x):G_j \in G\}$.

A *transversal design* $TD(k,n)$ is a group-divisible design of type n^k in which every block has size k. It is well-known that a $TD(k,n)$ is equivalent to a resolvable $TD(k-1,n)$, which in turn is equivalent to $k-2$ mutually orthogonal latin squares ($MOLS$) of order n. Thus a $TD(3,n)$ exists for all $n>0$ and a $TD(4,n)$ exists for all $n>0$ except $n=2,6$. It has been known for some time (see e.g. [12]) that there exist three $MOLS$ of order n (and hence a $TD(5,n)$) for all $n \geq 4$ except $n=6$ and possibly $n=10$, 14. More recently, Todorov [11] has constructed three $MOLS$ of order 14, thus leaving $n=10$ as the only unsettled value.

An *incomplete transversal design* $ITD(k,(n,m))$ is a $k-IGDD$ of type $(n,m)^k$; such a design is equivalent to $k-2$ mutually orthogonal latin squares of order n which are 'missing' a set of $k-2$ mutually orthogonal latin subsquares of order m. It is well-known that an $ITD(3,(n,m))$ exists if and only if $n \geq 2m$, while an $ITD(4,(n,m))$ exists if and only if $n \geq 3m$, with the exception $n=6$, $m=1$ (see Heinrich and Zhu [2]).

Finally, we will use the notation $PBD(K,v)$ to indicate a pairwise balanced design on v points in which each block has size from the set K. Where there is exactly one block of some size $k \in K$ we will indicate this by writing k^*.

2. KTS(v) with sub-KTS(w) where v = 3w or 3w + 6

The case $v=3w$ is trivial: since w is odd there is a resolvable $TD(3,w)$. Now construct a $KTS(w)$ on each group.

Now let $v=3w+6$, $w \neq 3$, 9, 15, 63 or 81. Write $v-w=2w+6=12t$, where $t \geq 4$ and $t \neq 11,14$. From theorem 1.4 there is a resolvable $3-GDD$ of type 6^t; add a group 'at infinity' of size $3t-3$ to yield a $4-GDD$ of type $6^t(3t-3)^1$. Now apply construction 1.6, using weight 2, to build a Kirkman frame of type $12^t(6t-6)^1$ (note that a Kirkman frame of type 2^4 is just a $KTS(9)$ with a point removed). Add three 'ideal' points to this frame and fill in $KTS(15)$ and a $KTS(6t-3)$. We obtain a $KTS(18t-3)$ with a subsystem of order $6t-3=w$, as desired (note that $18t-3=v$).

There remain the cases $KTS(3w+6)$ with sub-$KTS(w)$ where $w = 3, 9, 15, 63$ or 81. The case $w = 3$ is trivial, while the cases $w = 9,15$ are covered by theorem 1.1. For $w = 63$ we start with a $KTS(33)$, adding a group 'at infinity' of size 15 to yield a $4-GDD$ of type $3^{11}15^1$. Apply construction 1.6 with weight 4 to build a Kirkman frame of type $12^{11}60^1$ (a Kirkman frame of type 4^4 exists by theorem 1.5); add three 'ideal' points and fill in $KTS(15)$ and a $KTS(63)$. Finally, for $w = 81$ proceed as follows. Start with a resolvable $3-GDD$ of type 12^7 (theorem 1.4) and add a group 'at infinity' of size 36 to yield a $4-GDD$ of type $12^7 36^1$; apply construction 1.6 with weight 2 to obtain a Kirkman frame of type $24^7 72^1$. Add nine 'ideal' points and on each hole of size 24 (plus the ideal points) construct a $KTS(33)$ 'missing' a sub-$KTS(9)$ (we have already ascertained the existence of a $KTS(33)$ with a sub-$KTS(9)$; now just remove the blocks from the subdesign) and on the hole of size 72 (plus the ideal points) construct a $KTS(81)$.

We have thus shown:

Lemma 2.1. For each $w \equiv 3$ modulo 6 there exists a $KTS(v)$ containing a sub-$KTS(w)$ where $v = 3w$ or $3w+6$.

3. KTS(v) with sub-KTS(w) where v = 3w+12

These designs are by far the most challenging to construct, and it is here that we will require the full power of the ideas presented in the introduction. Let $\mathcal{W} = \{w \equiv 3$ modulo 6: there exists a $KTS(3w+12)$ with a sub-$KTS(w)\}$. Our main tool will be the following.

Lemma 3.1. Suppose that there is a group-divisible design (X,G,B) on s points in which every block has size ≥ 4 and in which there is a group $G_j \in G$ where

(i) $|G_j| = 3,4,5$ or 6, and

(ii) there is a point $y \in G_j$ such that every block containing y has size 5 or 6.

Then $6s-3 \in \mathcal{W}$.

Proof. Apply construction 1.3 to the GDD, where $w(x) = 9$ and $d(x) = 3$ for all points $x \neq y$, while $w(y) = 6$ and $d(y) = 0$. Set $a = 3$. Each block b in the GDD *not* containing y is replaced by a $4-IGDD$ of type $(9,3)^{|b|}$ (see theorem 1.5 and the remark following it), while each block containing y is replaced either by a $4-IGDD$ of type $(9,3)^4 6^1$ or a $4-IGDD$ of type $(9,3)^5 6^1$ (see appendix) depending on whether the block has size 5 or 6. Each group $G_i \neq G_j$ is replaced by a $4-GDD$ of type $3^{2|G_i|+1}(3|G_i|)^1$ (obtained by adding a group 'at infinity' of size $3|G_i|$ to a $KTS(6|G_i|+3)$), while the group G_j is replaced by a $4-GDD$ of type $3^1 6^4$, $3^6 9^2$, $3^9 6^1 12^1$, or $3^{11} 6^1 15^1$ (see appendix), depending on whether $|G_j| = 3,4,5$ or 6.

In this way we obtain a $4-GDD$ on $9s$ points with group sizes 3,6,9 and a group of size $3s-3$. Apply construction 1.6 with weight 2 and add three 'ideal' points, filling in $KTS(9)$, $KTS(15)$, $KTS(21)$ and a $KTS(6s-3)$. This gives a $KTS(18s+3)$ with a sub-$KTS(6s-3)$, i.e. $6s-3 \in \mathcal{W}$. □

Lemma 3.2. Let $s \geq 19$, $s \neq 22$. Then $6s-3 \in \mathcal{W}$.

Proof. We construct a GDD on s points satisfying the hypothesis of lemma 3.1.

$s = 19,20,21$. Take the projective plane of order 4, viewed as a $5-GDD$ of type $1^{16}5^1$, and remove 2, 1, or 0 points from the group of size 5.

$23 \leq s \leq 30$. If $s = 23,24$ or 25 remove 2, 1, or 0 points from a group in the affine plane of order 5. If $s = 26,...,30$ add a group 'at infinity' of size $s-25$ to the affine plane of order 5.

$s \geq 31$ ($s \neq 43,44,45,46$). Choose r from the set $\{3,4,5,6\}$ so that $s \equiv r$ modulo 4, and add a group 'at infinity' of size r to a resolvable $TD(4,\frac{1}{4}(s-r))$.

$s = 43,44,45,46$. There is a resolvable $(40,4,1)-BIBD$ (see [1]); add a group 'at infinity' of size 3,4,5 or 6 to this design. □

It remains to be shown that $6s-3 \in \mathcal{W}$ where $1 \leq s \leq 18$ or $s = 22$. The case $s = 1$ is trivial and the cases $s = 2,3$ or 4 are covered by theorem 1.1. If $s = 5$ or 6 we can use the $4-GDD$'s of type $3^9 6^1 12^1$ or $3^{11} 6^1 15^1$ (appendix), applying construction 1.6 with weight 2 and using three 'ideal' points. Thus we need consider only $7 \leq s \leq 18$ or $s = 22$.

Lemma 3.3. If $s = 7,10,13,16$ or 22 then $6s-3 \in \mathcal{W}$.

Proof. From theorem 1.4 there is a resolvable $3-GDD$ of type $9^{\frac{2s+1}{3}}$; add a group 'at infinity' of size $3s-3$ to yield a $4-GDD$ of type $9^{\frac{2s+1}{3}}(3s-3)^1$. Apply construction 1.6 with weight 2 and use three 'ideal' points. A $KTS(18s+3)$ with a sub-$KTS(6s-3)$ is obtained, as desired. □

Lemma 3.4. If $s = 12$ or 17 then $6s-3 \in \mathcal{W}$.

Proof. Proceed as in lemma 3.3, starting with a resolvable $3-GDD$ of type 15^5 (if $s = 12$) or 15^7 (if $s = 17$) and constructing a $4-GDD$ of type $15^5 30^1$ or $15^7 45^1$. Use construction 1.6 with weight 2, but add nine 'ideal' points. Fill in $KTS(39)$ 'missing' a sub-$KTS(9)$ (a $KTS(39)$ with a sub-$KTS(9)$ is the case $s = 2$; now just remove the blocks from the subdesign) and either a $KTS(69)$ or $KTS(99)$. □

Lemma 3.5. If $s = 14$ or 18 then $6s-3 \in \mathcal{W}$.

Proof $s = 14$. Start with an $ITD(4,(31,10))$ and add three ideal points a,b,c. Let x_1,x_2,x_3,x_4 be a block in the ITD, where x_1 is in the 'missing' subdesign. Now from theorem 1.4 there is a resolvable $3-GDD$ of type 4^6; add a block 'at infinity' of size 10 to build a $PBD(\{4,10^*\},34)$. Note that in this PBD there are blocks of size 4 which do not intersect the block of size 10. Construct a copy of this PBD on each group of the ITD (plus the ideal points) so that (respectively) a,b,c,x_1, a,b,c,x_2, a,b,c,x_3 and a,b,c,x_4 are blocks of size 4. This yields a $PBD(\{4,7^*,40^*\},127)$ (a,b,c,x_1,x_2,x_3,x_4 is the block of size 7) in which the blocks of size 7 and 40 intersect (in x_1); now remove x_1 to obtain a $4-GDD$ of type $3^{27} 6^1 39^1$. Apply construction 1.6 with weight 2, using three 'ideal' points.

$s = 18$. We proceed as above, starting with an $ITD(4,(40,13))$ and adding three ideal points. In the appendix we construct a $PBD(\{4,7^*,13^*\},43)$ in which the blocks of size 7 and 13 intersect and in which there are blocks of size 4 which do not intersect the block of size 13. Building a copy of this design on the groups (plus the ideal points) of the ITD we can generate a $\{4,7\}-GDD$ of type $3^{33} 6^2 51^1$. Apply construction 1.6 with weight 2, using three 'ideal' points. □

We do not know how to do the cases $s = 8,9,11$ and 15. The results in this section imply the following.

Lemma 3.6. Let $w \equiv 3$ modulo 6. Then there exists a $KTS(3w+12)$ with a sub-$KTS(w)$, except possibly where $w = 45,51,63$ and 87.

4. Conclusion

Theorem 1.2 now follows as a consequence of lemmas 2.1 and 3.6.

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Appendix

A 4-IGDD of type $(9,3)^4 6^1$. We construct a $3,4-GDD$ of type 6^5 whose triples fall into twelve holey parallel classes (i.e. three holey parallel classes corresponding to each of four of the groups).

Points: $(\mathbb{Z}_6 \times \{1,2,3,4\}) \cup (\{a\} \times \mathbb{Z}_2) \cup (\{b\} \times \mathbb{Z}_2) \cup \{\infty_1, \infty_2\}$.

Groups: $\{\{0+i,2+i,4+i\} \times \{1,3\} : i=0,1\} \cup \{\{0+i,2+i,4+i\} \times \{2,4\} : i=0,1\} \cup \{(\{a\} \times \mathbb{Z}_2) \cup (\{b\} \times \mathbb{Z}_2) \cup \{\infty_1,\infty_2\}\}$.

Blocks of size 4: develop the following modulo 6

$$0_1 3_1 2_2 5_2,\ 0_3 3_3 2_4 5_4,\ 0_1 1_2 3_3 4_4.$$

Holey parallel classes: develop each of the following two classes modulo 6 (the subscripts on a and b are to be evaluated modulo 2)

$$\begin{array}{cccccc} 0_2 1_4 3_3 & a_0 3_2 4_2 & & 1_1 0_3 3_4 & a_0 4_1 5_1 \\ 3_1 1_2 0_4 & a_1 3_4 4_4 & ; & 2_1 5_2 3_3 & a_1 4_3 5_3 \\ \infty_1 5_1 5_4 & b_0 1_1 2_4 & & \infty_1 1_2 2_3 & b_0 1_3 1_4 \\ \infty_2 5_2 5_3 & b_1 2_2 1_3 & & \infty_2 0_1 5_4 & b_1 3_1 3_2 \end{array}$$

A 4-IGDD of type $(9,3)^5 6^1$. We proceed as above, constructing a $3,4-GDD$ of type 6^6 whose triples fall into fifteen holey parallel classes.

Points: $(\mathbb{Z}_{15} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup \{\infty_1,\infty_2,\infty_3\}$.

Groups: $\{\{0+i,5+i,10+i\} \times \{1,2\} : i=0,1,2,3,4\} \cup \{(\{a\} \times \mathbb{Z}_3) \cup \{\infty_1,\infty_2,\infty_3\}\}$.

Blocks of size 4: develop the block $0_1 1_1 2_2 4_2$ modulo 15.

Holey parallel classes: develop the following class modulo 15 (the subscripts on a are to be evaluated modulo 3)

$$\begin{array}{cc} 1_1 3_1 7_1 & 2_1 14_1 8_2 \\ 11_1 3_2 9_2 & 1_2 2_2 13_2 \\ 4_1 12_2 \infty_1 & 8_1 4_2 \infty_2 \\ 9_1 6_2 \infty_3 & 12_1 11_2 a_1 \\ 6_1 13_1 a_0 & 7_2 14_2 a_2 \end{array}$$

A 4-GDD of type $3^1 6^4$

Points: $(\mathbb{Z}_{12} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3)$.

Groups: $\{\{0+i,2+i,4+i,6+i,8+i,10+i\} \times \{j\} : i=0,1; j=1,2\} \cup \{\{a\} \times \mathbb{Z}_3\}$.

Blocks: develop the following modulo 12 (the subscript on a is to be evaluated modulo 3)

$$0_1 1_1 0_2 3_2,\ 0_1 3_1 4_2 9_2,\ a_0 0_1 7_2 8_2,\ a_0 2_1 7_1 0_2.$$

A 4-GDD of type $3^6 9^2$. We construct a $3,4-GDD$ of type $3^6 9^1$ whose triples fall into nine parallel classes. Points: $(\mathbb{Z}_9 \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup (\{b\} \times \mathbb{Z}_3) \cup \{\infty_1,\infty_2,\infty_3\}$.

Groups: $\{\{0+i,3+i,6+i\} \times \{j\} : i=0,1,2; j=1,2\} \cup \{(\{a\} \times \mathbb{Z}_3) \cup (\{b\} \times \mathbb{Z}_3) \cup \{\infty_1,\infty_2,\infty_3\}\}$.

Blocks of size 4: develop the block $0_1 2_1 3_2 7_2$ modulo 9.

Parallel classes: develop the following class modulo 9 (the subscripts on a and b are to be evaluated modulo 3)

$$\begin{array}{lll}
a_0 0_1 0_2 & b_0 7_1 2_2 & \infty_1 2_1 4_2 \\
a_1 5_2 6_2 & b_1 1_2 8_2 & \infty_2 6_1 3_2 \\
a_2 3_1 4_1 & b_2 1_1 5_1 & \infty_3 8_1 7_2
\end{array}$$

A 4-GDD of type $3^9 6^1 12^1$. We construct a 3,4$-GDD$ of type $3^9 6^1$ whose triples fall into twelve parallel classes.

Points: $(\mathbb{Z}_9 \times \{1,2,3\}) \cup \{\infty_i : 1 \leq i \leq 6\}$.

Groups: $\{\{(0+i)_1,(1+i)_2,(2+i)_3\} : i \in \mathbb{Z}_9\} \cup \{\infty_i : 1 \leq i \leq 6\}$.

Blocks of size 4: develop the block $0_1 0_2 0_3 3_3$ modulo 9; then there are six more quadruples, namely $\infty_1 2_2 4_2 6_2$, $\infty_2 1_2 3_2 8_2$, $\infty_3 0_2 5_2 7_2$, $\infty_4 2_1 4_1 6_1$, $\infty_5 1_1 3_1 8_1$, $\infty_6 0_1 5_1 7_1$.

Parallel classes: develop the following class modulo 9

$$\begin{array}{lll}
0_1 3_1 2_2 & 7_1 5_2 3_3 & \infty_4 0_2 2_3 \\
1_1 4_2 7_2 & \infty_1 4_1 8_3 & \infty_5 3_2 0_3 \\
2_1 6_2 1_3 & \infty_2 6_1 4_3 & \infty_6 8_2 7_3 \\
5_1 1_2 6_3 & \infty_3 8_1 5_3 &
\end{array}$$

Three more parallel classes are

$$\begin{array}{llclllcll}
0_3 1_3 2_3 & \infty_1 3_2 7_2 & & 0_3 4_3 8_3 & \infty_1 0_2 8_2 & & 0_3 7_3 5_3 & \infty_1 1_2 5_2 \\
3_3 4_3 5_3 & \infty_2 6_2 5_2 & & 3_3 7_3 2_3 & \infty_2 7_2 2_2 & & 3_3 1_3 8_3 & \infty_2 0_2 4_2 \\
6_3 7_3 8_3 & \infty_3 4_2 8_2 & ; & 6_3 1_3 5_3 & \infty_3 6_2 1_2 & ; & 6_3 4_3 2_3 & \infty_3 3_2 2_2 \\
0_1 1_1 2_1 & \infty_4 3_1 7_1 & & 3_1 4_1 5_1 & \infty_4 0_1 8_1 & & 6_1 7_1 8_1 & \infty_4 1_1 5_1 \\
0_2 1_2 2_2 & \infty_5 6_1 5_1 & & 3_2 4_2 5_2 & \infty_5 7_1 2_1 & & 6_2 7_2 8_2 & \infty_5 0_1 4_1 \\
 & \infty_6 4_1 8_1 & & & \infty_6 6_1 1_1 & & & \infty_6 3_1 2_1
\end{array}$$

A 4-GDD of type $3^{11} 6^1 15^1$

Start with an $ITD(4,(13,4))$ and add three ideal points a,b,c. Let x_1,x_2,x_3,x_4 be a block in the ITD where x_1 is contained in the 'missing' subdesign. Construct a $(16,4,1)-BIBD$ on each group (plus the ideal points) so that (respectively) a,b,c,x_1, a,b,c,x_2, a,b,c,x_3 and a,b,c,x_4 are blocks of size 4. This yields a $PBD(\{4,7^*,16^*\},55)$ in which the blocks of size 7 and 16 intersect (in x_1). Now remove the point x_1.

A PBD($\{4,7^*,13^*\}$,43). We construct a 3,4$-GDD$ of type $3^8 6^1$ whose triples fall into twelve parallel classes.

Points: $(\mathbb{Z}_{12} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_2) \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$.

Groups: $\{\{0+i,4+i,8+i\} \times \{j\} : i=0,1,2,3; j=1,2\} \cup \{(\{a\} \times \mathbb{Z}_2) \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}\}$.

Blocks of size 4: develop the blocks $0_1 3_1 6_1 9_1$ and $0_2 3_2 6_2 9_2$ modulo 12.

Parallel classes: develop the following class modulo 12 (the subscripts on a are to be evaluated modulo 2)

$$\begin{array}{ll}
0_1 0_2 2_2 & \infty_2 3_1 10_2 \\
4_1 1_2 8_2 & \infty_3 5_1 6_2 \\
7_1 9_1 3_2 & \infty_4 8_1 11_2 \\
10_1 11_1 9_2 & a_0 1_1 6_1 \\
\infty_1 2_1 7_2 & a_1 4_2 5_2
\end{array}$$