On the Existence of Skew Room Frames of Type 2^n

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ABSTRACT

We prove that there exists a skew Room frame of type 2^n for all $n > 67$.

1. Introduction.

We need to begin with some definitions. Let S be a set, and let $\{S_1,\ldots,S_n\}$ be a partition of S. An $\{S_1,\ldots,S_n\}$-*Room frame* is an $|S|$ by $|S|$ array, F, indexed by S, which satisfies the following properties:

1) every cell of F either is empty or contains an unordered pair of symbols of S,

2) the subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are referred to as *holes*),

3) each symbol of $S \backslash S_i$ occurs once in row (or column) s, for any $s \in S_i$,

4) the pairs occurring in F are precisely those $\{s,t\}$, where $(s,t) \in (S \times S) \backslash \bigcup_{1 \leq i \leq n} (S_i \times S_i)$.

We shall say that F is *skew* if, for any pair of cells (s,t) and (t,s), where $(s,t) \in (S \times S) \backslash \bigcup (S_i \times S_i)$, precisely one is empty. The *type* of F is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. We usually use an "exponential" notation to describe types: a type $1^i 2^j 3^k \cdots$ denotes i occurrences of 1, j occurrences of 2, etc.

A Room frame of type 1^n gives rise to a *Room square* of side n by filling in each diagonal cell (s,s) with the pair $\{\infty,s\}$, where ∞ is a new symbol. If the Room frame is skew, then the resulting Room square is also skew.

Room frames have proven to be an important tool in the investigation of many problems related to Room squares (see, for example, [8,9, and 11]). Frames for other types of designs have also been useful. For example, frames for Kirkman triple systems are investigated in [12]. In this paper, we consider only Room frames; hence the term *frame* can be taken to mean "Room frame".

The spectrum for Room squares was determined in 1975 by Mullin and Wallis, who proved the following in [5].

Theorem 1.1. *There exists a Room square of side n if and only if n is an odd positive integer,* $n \neq 3$ *or* 5.

A similar result was shown to hold for skew Room squares a few years later. A short proof of the following is presented in [9].

Theorem 1.2. *There exists a skew Room square of side n if and only if n is an odd positive integer,* $n \neq 3$ *or* 5.

More generally, one can ask the question "when does there exist a (skew) frame of type t^u?" (These frames are called *uniform,* since all the holes have the same size.) The existence of (non-skew) uniform frames was studied in [2], [3], and [10], and the following theorem summarizes the results obtained there.

Theorem 1.3.

1) *There does not exist a frame of type* t^u *if* $u = 2$ *or* 3; *if* $t(u-1)$ *is odd; or if* $t^u = 2^4$ *or* 1^5.
2) *There exists a frame of type* t^4 *if* $4|t$.
3) *There exists a frame of type* t^5 *if* $GCD(t,210) > 1$.
4) *For* $u \geq 6$, *there exists a frame of type* t^u *if and only if* $t(u-1)$ *is even.*

Much less is known about the existence of uniform skew frames with holes of size at least 2. The following "asymptotic" result was proven in [7, Theorem 2.5.3].

Theorem 1.4. *For any* $t \geq 1$, *there exists a constant* $u(t)$ *such that, if* $u > u(t)$, *then there exist a frames of type* t^u *if and only if* $t(u-1)$ *is even.*

The purpose of this paper is to begin the investigation of uniform skew frames. The motivation for our research is that skew Room frames have recently been used to construct nested designs (see, for example, [4]). In this paper, we restrict our attention to skew frames of type 2^n. Our main result is the following.

Theorem 1.5. *There exists a skew Room frame of type* 2^n *for all* $n > 67$.

Of course, there is no skew frame of type 2^n if $n = 2,3$, or 4. The 23 values of n ($6 \leq n \leq 67$) for which the existence of a skew Room frame of type 2^n is unknown are those $n \in X$, where

$$X = \{6,11,15,19,20,22,23,24,26,27,28,30,31,34,35,38,43,46,51,58,59,62,67\}.$$

2. Direct Constructions.

The basic direct construction for frames is the "starter-adder" construction and modifications thereof. Let G be an abelian group, written additively, and let H be a subgroup of G. Denote $g = |G|$, $h = |H|$ and suppose that $g - h$ is even. A *frame starter* in $G \backslash H$ is a set of unordered pairs

$$S = \{\{s_i, t_i\}: 1 \leq i \leq (g-h)/2\}$$

satisfying

1) $\bigcup_{1 \leq i \leq (g-h)/2}(\{s_i\} \cup \{t_i\}) = G \backslash H$, and
2) $\bigcup_{1 \leq i \leq (g-h)/2}(\{\pm(s_i - t_i)\}) = G \backslash H$.

An *adder* for S is an injection $A: S \rightarrow G \backslash H$, such that

$$\bigcup_{1 \leq i \leq (g-h)/2}(\{s_i + a_i\} \cup \{t_i + a_i\}) = G \backslash H,$$

where $a_i = A(s_i, t_i)$, $1 \leq i \leq (g-h)/2$. A is *skew* if, further,

$$\bigcup_{1 \leq i \leq (g-h)/2}(\{a_i\} \cup \{-a_i\}) = G \backslash H.$$

We have the following construction for skew frames.

Theorem 2.1. ([11,Lemma 3.1]). *Suppose there exists a frame starter* S *in* $G \backslash H$, *and a skew adder* A *for* S. *Then there is a skew frame of type* $h^{g/h}$, *where* $g = |G|$ *and* $h = |H|$.

If we are interested in constructing (skew) frames of type 2^n by means of the starter-adder construction, then the subgroup H must be of order 2. As well, the group G must contain a unique element of order 2. Since G is abelian, this restricts the possibilities to groups of the form $G = Z_{2^j} \times K$, where K has odd order m and $j \geq 1$ (whence $H = \{(0,0),(2^{j-1},0)\}$). It is shown in [7, Lemma 2.1.7] that there is no frame starter in $G \backslash H$ when $j = 1$ and $m \equiv 3 \pmod 4$; or when $j = 2$. Hence, we have

Theorem 2.2. *There does not exist a frame starter in* $G\backslash H$ *whenever* $|H| = 2$ *and* $|G|/|H| \equiv 2$ *or* 3 *(mod 4).*

We can construct (skew) frames of type 2^n, $n \equiv 2$ or 3 (mod 4) by means of a modified starter-adder construction, which we now describe. As before, let G be an abelian group and let H be a subgroup of G, where $g = |G|$, $h = |H|$, and suppose that $g - h$ is even. An *intransitive starter* in $G\backslash H$ is defined to be a triple (S,R,C), where

$$S = \bigcup_{1 \le i \le (g-h-2)/2}(\{\{s_i,t_i\}: 1 \le i \le (g-h-2)/2\}) \cup \{\{u\},\{v\}\},$$

$$C = \{\{p,q\}\}, \text{ and}$$

$$R = \{\{p',q'\}\},$$

satisfying

1) $\bigcup_{1 \le i \le (g-h-2)/2}(\{s_i\} \cup \{t_i\}) \cup \{u,v,p,q\} = G\backslash H$, and
2) $\bigcup_{1 \le i \le (g-h-2)/2}(\{\pm(s_i-t_i)\}) \cup \{\pm(p-q)\} \cup \{\pm(p'-q')\} = G\backslash H$, and
3) both $p - q$ and $p' - q'$ have even order in G.

An *adder* for (S,C,R) is an injection $A\colon S \to G\backslash H$, such that

$$\bigcup_{1 \le i \le (g-h-2)/2}(\{s_i+a_i\} \cup \{t_i+a_i\}) \cup \{u+A(u),v+A(v),p',q'\} = G\backslash H,$$

where $a_i = A(s_i,t_i)$, $1 \le i \le (g-h)/2$.

A is *skew* if, further,

1) $\bigcup_{1 \le i \le (g-h-2)/2}(\{a_i\} \cup \{-a_i\}) \cup \{A(u),A(v),-A(u),-A(v)\} = G\backslash H$, and
2) for some $i \ge 1$, $p - q$ has order $2^i m_1$ and $p' - q'$ has order $2^i m_2$, where m_1 and m_2 are odd.

Figure 1
A skew frame of type 2^7

	0	1	2	3	4	5	6	7	8	9	10	11	x	y
0					2 5			11y		4 8	1 3	7 x	9 10	
1	8 x					3 6			0 y		5 9	2 4		10 11
2	3 5	9 x					4 7			1 y		6 10	11 0	
3	7 11	4 6	10 x					5 8			2 y			0 1
4		8 0	5 7	11 x					6 9			3 y	1 2	
5	4 y		9 1	6 8	0 x					7 10				2 3
6		5 y		10 2	7 9	1 x					8 11		3 4	
7			6 y		11 3	8 10	2 x					9 0		4 5
8	10 1			7 y		0 4	9 11	3 x					5 6	
9		11 2			8 y		1 5	10 0	4 x					6 7
10			0 3			9 y		2 6	11 1	5 x			7 8	
11				1 4			10 y		3 7	0 2	6 x			8 9
x		3 10		5 0		7 2		9 4		11 6		1 8		
y	2 9		4 11		6 1		8 3		10 5		0 7			

We have the following result.

Theorem 2.3. (11, Lemmata 3.3 and 3.4]). *If there is an intransitive frame starter and a skew adder in $G\backslash H$, where $g = |G|$ and $h = |H|$, then there is a skew frame of type $h^{g/h}2^1$.*

By constructing intransitive starters with skew adders in $Z_{2n-2}\backslash\{0,n-1\}$ when $n-1 \equiv 1$ or $2 \pmod 4$, we can obtain skew frames of type 2^n, when $n \equiv 2$ or $3 \pmod 4$. As an example of this construction, we present in Figure 1 the skew frame of type 2^7 constructed from the intransitive starter and skew adder in $Z_{12}\backslash\{0,6\}$ which is presented in the Appendix.

The starters and adders presented in the Appendix establish the following.

Lemma 2.4. *There is a skew frame of type* 2^n *for* $n = 7,8,10,12,14,16,$ *and* 18.

3. Recursive Constructions.

In this section, we describe recursive constructions for skew frames. We need to define some design-theoretic terminology (see [1] for any undefined terms).

A *pairwise balanced design* (or PBD) is a pair $(X,\mathbf{A})$, such that $\mathbf{A}$ is a set of subsets (called *blocks*) of X, each of cardinality at least two, such that every unordered pair of *points* (i.e. elements of X) is contained in a unique block in $\mathbf{A}$. If v is a positive integer and K is a set of positive integers, each of which is greater than or equal to 2, then we say that $(X,\mathbf{A})$ is a (v,K)-PBD if $|X| = v$, and $|A| \in K$ for every $A \in \mathbf{A}$.

A *group-divisible design* (or GDD) is a triple $(X,\mathbf{G},\mathbf{A})$, which satisfies the following properties: 1) $\mathbf{G}$ is a partition of X into subsets called *groups;* 2) $\mathbf{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point; and, 3) every pair of points from distinct groups occurs in a unique block.

The group-type, or *type,* of a GDD $(X,\mathbf{G},\mathbf{A})$ is a multiset $\{|G|: G \in \mathbf{G}\}$. We will say that a GDD is a K-GDD if $|A| \in K$ for every $A \in \mathbf{A}$.

A *transversal design* $TD(k,m)$ can be defined to be a $\{k\}$-GDD of type m^k. It is well-known that a $TD(k,m)$ is equivalent to $k-2$ mutually orthogonal Latin squares of order m. For results on the existence of transversal designs we refer to [1].

The following is our main recursive construction for frames, found in [11, Construction 2.2].

Theorem 3.1. *Let* $(X,\mathbf{G},\mathbf{A})$ *be a GDD, and let* $w: X \rightarrow \mathbf{Z}^+ \cup \{0\}$ *(we say that* w *is a weighting).* For every $A \in \mathbf{A}$, suppose there is a skew frame of type $\{w(x): x \in A\}$. Then there is a skew frame of type $\{\sum_{x \in G} w(x): G \in \mathbf{G}\}$.

Define $SF_2 = \{n:$ there exists a skew frame of type $2^n\}$. Then we have the following corollary to Theorem 3.1, which says that the set SF_2 is PBD-closed.

Lemma 3.2. ([6, Theorem 3.2]). *Suppose there is an* (n,SF_2)*-PBD. Then* $n \in SF_2$.

Proof. The hypothesized PBD can be thought of as a GDD in which every group has size 1. Give every point weight 2 and apply Theorem 3.1.

As another useful corollary, to Theorem 3.1, we have the following modification of [9, Lemma 3.1].

Lemma 3.3. *Suppose $m \geq 4$, $m \neq 6$ or 10, and suppose $0 \leq t \leq 3m$. Suppose also that there exist skew frames of types 2^{2m} and 2^t. Then there exists a skew frame of type 2^{8m+t}.*

Proof. Since $m \notin \{2,3,6,10\}$, there exists a $TD(5,m)$ (see [13] and [14]). In four groups of the TD, give every point weight 4, and in the fifth group, assign weights 0,2,4 and 6 so that the weights sum to $2t$. Now, apply Theorem 3.1, employing skew frames of types 4^4, $4^4 2^1$, 4^5, and $4^4 6^1$ (these frames are constructed in [11, Lemma 5.1]). A skew frame of type $(4m)^4(2t)^1$ results. then, fill in the holes with skew of type 2^{2m} and 2^t.

We can inflate the size of each hole in a skew frame by any constant factor t other than 2 or 6, by using a pair of orthogonal Latin squares of order t.

Theorem 3.4. *Suppose there is a skew Room frame of type $t_1^{u_1} t_2^{u_2} \cdots t_j^{u_j}$, and suppose also that $t \neq 2$ or 6. then there exists a skew Room frame of type $(t \cdot t_1)^{u_1} (t \cdot t_2)^{u_2} \cdots (t \cdot t_j)^{u_j}$.*

The following two constructions are both special cases of a construction called the *singular direct product*. They are accomplished by applying Theorem 3.4, and then filling in the holes of the resulting frame.

Lemma 3.5. *Suppose $s = u(v-1)+1$, and let t be a rational number such that $2t$ and $(v-1)/t$ are both integers. Suppose there exist skew frames of type $(2t)^u$ and 2^v, and suppose that $(v-1)/t \neq 2$ or 6. Then there exists a skew Room frame of type 2^s.*

Lemma 3.6. *Suppose $s = u \cdot v$, and let t be a rational number such that $2t$ and v/t are both integers. Suppose there exist skew frames of type $(2t)^u$ and 2^v, and suppose that $v/t \neq 2$ or 6. Then there exists a skew Room frame of type 2^s.*

4. Skew frames of type 2^n, for $n \leq 339$.

In this section, we construct skew frames of type 2^n for all $5 \leq n \leq 339$, except for the 23 exceptions mentioned in the introduction. We begin by looking at skew frames of type 2^n, $n \equiv 1 \pmod 4$. The following result was proved in [6].

Theorem 4.1. *If $n \equiv 1 \pmod 4$, $n \neq 33$, 57, 93, or 133, then there exists a skew frame of type 2^n.*

First, we remove the four exceptions above. These are all applications of the singular direct product construction, Lemma 3.5. First, write $33 = 8(5-1) + 1$, and apply Lemma 3.5 with $t = 1$, using skew frames of types 2^8 and 2^5. Next, write $57 = 7(9-1) + 1$ and take $t = 1/2$, using Skew frames of types 1^7 and 2^9. Next, $93 = 23(5-1) + 1$. Taking $t = 1/2$ and using skew frames of types 1^{23} and 2^5, we get a skew frame of type 2^{93}. Finally, we construct a skew frame of type 2^{133} by writing $133 = 12(12-1) + 1$, using skew frames of type 2^{12}. Hence, we have

Lemma 4.2. *If $n \equiv 1 \pmod 4$, then there exists a skew frame of type 2^n.*

We can construct most of the skew frames in the desired range by the following corollary of Lemma 3.2.

Lemma 4.3. *Suppose there exists a $TD(10,m)$, and let $0 \leq t,u,v \leq m$. Suppose there exist skew frames of types 2^m, 2^t, 2^u, and 2^v. Then there exists a skew frame of type $2^{7m+t+u+v}$.*

Proof. From three groups of the given TD, delete $m - t$, $m - u$, and $m - v$ points, respectively. Considering the groups as blocks, we obtain a $(7m+t+u+v,\{7,8,9,10,m,t,u,v\})$-PBD. Since $\{7,8,9,10,m,t,u,v\}$ is a subset of SF_2, we obtain $7m+t+u+v \in SF_2$, from Lemma 3.2.

First, we present in Table 1 constructions for skew frames of type 2^n, $n \leq 95$, using various recursive constructions. Hence, we have

Lemma 4.4. *If $5 \leq n \leq 95$, then there is a skew frame of type 2^n unless $n \in X$.*

Proof. This is an immediate consequence of Lemmata 2.4 and 4.2, and Table 1.

Table 1
Recursive Constructions for Skew Frames of type 2^n, $32 \leq n \leq 95$

n	construction
$32 = 8{\cdot}4 + 0$	Lemma 3.3
$35 = 7{\cdot}5$	Lemma 3.6 ($t = 1/2$)
$39 = 8{\cdot}4 + 7$	Lemma 3.3
$40 = 8{\cdot}5 + 0$	Lemma 3.3
$42 = 8{\cdot}4 + 10$	Lemma 3.3
$44 = 8{\cdot}4 + 12$	Lemma 3.3
$47 = 8{\cdot}5 + 7$	Lemma 3.3
$48 = 8{\cdot}5 + 8$	Lemma 3.3
$50 = 8{\cdot}5 + 10$	Lemma 3.3
$52 = 8{\cdot}5 + 12$	Lemma 3.3
$54 = 8{\cdot}5 + 14$	Lemma 3.3
$55 = 11{\cdot}5$	Lemma 3.6 ($t = 1/2$)
$56 = 8{\cdot}7 + 0$	Lemma 3.3
$60 = 5{\cdot}12$	Lemma 3.6 ($t = 1$)
$63 = 7{\cdot}9 + 0 + 0 + 0$	Lemma 4.3
$64 = 7{\cdot}9 + 1 + 0 + 0$	Lemma 4.3
$66 = 7{\cdot}9 + 1 + 1 + 1$	Lemma 4.3
$68 = 7{\cdot}9 + 5 + 0 + 0$	Lemma 4.3
$70 = 7{\cdot}9 + 7 + 0 + 0$	Lemma 4.3
$71 = 7{\cdot}9 + 8 + 0 + 0$	Lemma 4.3
$72 = 7{\cdot}9 + 9 + 0 + 0$	Lemma 4.3
$74 = 7{\cdot}9 + 9 + 1 + 1$	Lemma 4.3
$75 = 7{\cdot}9 + 7 + 5 + 0$	Lemma 4.3
$76 = 7{\cdot}9 + 7 + 5 + 1$	Lemma 4.3
$78 = 7{\cdot}9 + 9 + 5 + 1$	Lemma 4.3
$79 = 7{\cdot}9 + 9 + 7 + 0$	Lemma 4.3
$80 = 7{\cdot}9 + 9 + 7 + 1$	Lemma 4.3
$82 = 7{\cdot}0 + 9 + 5 + 5$	Lemma 4.3
$83 = 7{\cdot}9 + 8 + 7 + 5$	Lemma 4.3
$84 = 7{\cdot}9 + 9 + 7 + 5$	Lemma 4.3
$86 = 7{\cdot}9 + 9 + 9 + 5$	Lemma 4.3
$87 = 7{\cdot}9 + 9 + 8 + 7$	Lemma 4.3
$88 = 7{\cdot}9 + 9 + 9 + 7$	Lemma 4.3
$90 = 7{\cdot}9 + 9 + 9 + 9$	Lemma 4.3
$91 = 7{\cdot}13 + 0 + 0 + 0$	Lemma 4.3
$92 = 7{\cdot}13 + 1 + 0 + 0$	Lemma 4.3
$94 = 7{\cdot}13 + 1 + 1 + 1$	Lemma 4.3
$95 = 19{\cdot}5$	Lemma 3.6 ($t = 1/2$)

Now, we have to consider $96 \leq n \leq 339$. It is a bit tedious, but not difficult, to check that all n in this interval can be handled by Lemma 4.3, using m = 13, 17, 25, 29, and 37, with 5 possible exceptions: n = 171, 172, 173, 174, and 179. First, we have 171 = 10(18−1) + 1, and apply Lemma 3.5 with $t = 1$. Next, 172 = 8·20 + 12 and 174 = 8·20 + 14, so Lemma 3.3 can be applied. 173 ≡ 1 mod 4, so the frame exists by Lemma 4.2. Finally, we handle $n = 179$ by observing that the deletion of 4 collinear points from a projective plane of order 13 produces a (179,{10,13,14})-PBD; whence a skew frame of type 2^{179} exists by Lemma 3.2.

Summarizing, we get

Lemma 4.5. *If* $96 \leq n \leq 339$, *then there is a skew frame of type* 2^n.

5. The existence of skew frames of type 2^n, $n \geq 68$.

Given the existence of skew frames of type 2^n for $68 \leq n \leq 339$, it is a simple matter to show that there is a skew frame of type 2^n for all $n \geq 68$.

Theorem 5.1. *There is a skew frame of type* 2^n *for all* $n \geq 68$.

Proof. The proof is by induction on n. In view of Lemmata 4.4 and 4.5, we can assume $n \geq 340$. We can write $n = 8m + t$, where $68 \leq t \leq 75$, uniquely. Then, $m \geq 34$, so $t \leq 3m$. There is a skew frame of type 2^{2m} (by induction), and one of type 2^t. Apply Lemma 3.4 to construct the skew frame of type 2^n.

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Appendix

Starter-adder constructions for Skew frames of type 2^n

$n = 8$	5	6	12	1	2
	10	12	3	13	15
	14	1	6	4	7
	15	3	7	6	10
	4	9	5	9	14
	7	13	14	5	11
	11	2	1	12	3
$n = 12$	10	11	3	13	14
	17	19	1	18	20
	2	5	20	22	1
	3	7	2	5	9
	18	23	8	2	7
	9	15	6	15	21
	21	4	7	4	11
	8	16	11	19	3
	13	22	19	8	17
	20	6	10	6	16
	14	1	9	23	10
$n = 16$	8	9	1	9	10
	15	17	2	17	19
	25	28	3	28	31
	30	2	4	2	6
	7	12	11	18	23
	31	5	22	21	27
	19	26	26	13	20
	27	3	8	3	11
	20	29	27	15	24
	11	21	19	30	8
	13	24	20	1	12
	10	22	15	25	5
	1	14	25	26	7
	4	18	18	22	4
	23	6	23	14	29

$n = 7$	1	3	2	3	5
	4	8	3	7	11
	2	5	8	10	1
	7		1	8	
	11		5	4	
$C =$	9	10			
$R =$				9	2
$n = 10$	1	13	7	8	2
	2	10	5	7	15
	3	17	14	17	13
	4	15	1	5	16
	6	7	15	3	4
	12	14	16	10	12
	5		6	11	
	16		8	6	
$C =$	8	11			
$R =$				14	1
$n = 14$	15	17	14	3	5
	7	10	10	17	20
	20	24	17	11	15
	2	8	4	6	12
	5	12	23	2	9
	1	9	18	19	1
	23	6	2	25	8
	19	3	11	4	14
	14	25	19	7	18
	18	4	6	24	10
	22		1	23	
	11		5	16	
$C =$	16	21			
$R =$				21	22

$n = 18$	19	20	21	6	7
	9	11	5	14	16
	18	22	31	15	19
	26	31	28	20	25
	27	33	4	31	3
	6	13	26	32	5
	7	15	20	27	1
	16	25	27	9	18
	28	4	18	12	22
	2	14	22	24	2
	24	3	23	13	26
	30	10	32	28	8
	8	23	15	23	4
	5	21	24	29	11
	29		1	30	
	12		9	21	
$C =$	1	32			
$R =$				10	33