ON RESOLVABLE COVERINGS OF PAIRS BY TRIPLES

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Abstract: In this paper, we consider a type of covering design related to nearly Kirkman triple systems (NKTS). If we consider a NKTS to be a resolvable packing of pairs into triples, this suggests that we consider resolvable coverings of pairs by triples. We denote such a design on 6n points by RC(6n). We are able to determine the spectrum of such coverings, with 9 exceptions. We show that RC(6n) exists for all $n \geq 3$, $n \notin \{6, 7, 8, 10, 11, 13, 14, 17, 22\}$. In proving this result, we make essential use of a type of design called a "frame", which can be thought of as a Kirkman triple system with "holes".

1. Introduction

We need to begin with some definitions.

A *pairwise balanced design* (or, PBD) is a pair (X, A), such that A is a set of subsets (called *blocks*) of X, each of cardinality at least two, such that every unordered pair of *points* (i.e. elements of X) is contained in a unique block in A. If v is a positive integer and K is a set of positive integers, each of which is greater than or equal to 2, then we say that (X, A) is a (v, K)-PBD if $|X| = v$, and $|A| \in K$ for every $A \in$ A.

If $K = \{k\}$, then a (v, K)-PBD is referred to as a (v, k, 1)-BIBD (*balanced incomplete block design*). A (v, 3, 1)-BIBD is called a Steiner triple system; these designs exist for all $v \equiv 1$ or 3 modulo 6.

A *group-divisible design* (or, GDD), is a triple (X, G, A), which satisfies the following properties: 1) G is a partition of X into subsets called *groups*; 2) A is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point; and, 3) every pair of points from distinct groups occurs in a unique block.

The group-type, or *type*, of a GDD (X, G, A) is the multiset $\{|G|: G \in$ G$\}$. We usually use an "exponential" notation to describe group-types: a group-type $1^i 2^j 3^k$... denotes i occurrences of 1, j occurrences of 2, etc. We will say that a GDD is a K-GDD if $|A| \in K$ for every $A \in$ A.

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A *parallel class* in a PBD or GDD is a set of blocks that partitions the set of points. If we can partition the blocks into parallel classes, we say that the design is *resolvable*. Clearly, if a parallel class of blocks of size k exists in a design, then k must divide v.

A resolvable (v, 3, 1)-BIBD is called a *Kirkman triple system* and denoted KTS(v). It was conjectured over a century ago that a KTS(v) exists if and only if $v \equiv 3$ modulo 6, but this was proven only in 1971 by Ray-Chaudhuri and Wilson [6]. A related class of GDDs was defined by Kotzig and Rosa [5]: a resolvable 3-GDD of group-type 2^u is referred to as a *nearly Kirkman triple system*, and denoted NKTS(2u). It has been shown that a NKTS(v) exists if and only if $v \equiv 0$ modulo 6, $v \geq 18$ (see [1], [3], [4], and [5]).

One obvious generalization of NKTS is as follows. If we ignore the groups, then the blocks form a maximum collection of blocks (of size three) such that no pair of points occurs in *at most* one block. Such a design is called a *packing* (of pairs into triples). Further, this packing is *resolvable*. The "opposite" of a packing is a *covering* (of pairs by triples), in which we require the minimum collection of blocks such that every pair of points occur in *at least* one block. In the case of the number of points being a multiple of 6, the covering could be resolvable. Such a covering of v points is denoted RC(v). In such a design, the pairs occuring twice would form a one-factor (or, perfect matching) of the set of points. Note that the number of parallel classes in an RC(v) is v / 2.

With only a few small exceptions, we can give a complete solution to the resolvable covering problem.

For small values, we have the following results.

Lemma 1.1 There exist RC(18), RC(24), and RC(30).

Proof: These designs are presented in the Appendix (the covering on 30 points is due to C. Colbourn (private communication)).

Lemma 1.2 There does not exist RC(6) or RC(12).

Proof: It is easy to see that an RC(6) does not exist. Without loss of generality, we can take the first parallel class to be $\{1, 2, 3\}$ $\{4, 5, 6\}$. Up to isomorphism, there is only one possibility for the next parallel class, so we can take it to be $\{1, 2, 4\}$ $\{3, 5, 6\}$. This causes two pairs to be repeated. The third parallel class must also repeat two pairs, so we already have 4 repeated pairs, which is too many.

The non-existence of an RC(12) was shown by an exhaustive computer search.

2. Recursive Constructions

In this section, we describe recursive constructions for resolvable coverings. For the first construction, we use a particular type of design called a frame as an essential tool. Some definitions are required.

If (X, G, A) is a k-GDD and $G \in$ G, then we say that a set $P \subseteq A$ of blocks is a *holey parallel class* with *hole* G provided that P consists of $(|X| - |G|) / k$ disjoint blocks that partition $X \setminus G$. We write h(P) = G to denote that G is the hole of P. If we can partition the set of blocks A into a set P of holey parallel classes, then we say that (X, G, P) is a *frame* with block-size k. We can think of a frame as being a resolvable BIBD with holes, exactly as a GDD is a BIBD with holes.

We will be using frames with block-size 3, which we refer to as *Kirkman frames*. These are studied in [7], in which they are used to prove new results on the existence of subdesigns in Kirkman triple systems. In the case where all the holes have the same size, their existence was completely determined, as recorded in Theorem 2.1.

Theorem 2.1 There exists a Kirkman frame of type t^u if and only if t is even, $u \geq 4$, and $t(u - 1) \equiv 0$ modulo 3.

Kirkman frames are related to the problems of resolvable coverings of pairs by triples by means of the following simple construction.

Theorem 2.2 Suppose there is a Kirkman frame of type $t_1^{u_1} t_2^{u_2} \ldots t_j^{u_j}$, and suppose also that there exist RC(t_i), for $1 \leq i \leq j$. Then there exists a RC(u), where $u = \sum_{1 \leq i \leq j} t_i \cdot u_i$.

Proof: Let (X, G, P) be a Kirkman frame of type $t_1^{u_1} t_2^{u_2} \ldots t_j^{u_j}$. For every hole $G \in$ G, construct an RC(|G|) having point-set G, denoting the parallel classes by P(G, i), $1 \leq i \leq |G| / 2$.

We need the following property of Kirkman frames, which is proved in [7, Theorem 1.2]: given any hole $G \in$ G, there are precisely $|G| / 2$ holey parallel classes with hole G. Hence, we can name the holey parallel classes P'(G, i), $1 \leq i \leq |G| / 2$, for every hole $G \in$ G.

It is now a simple matter to describe an RC(|X|) on point-set X: the parallel classes are $P(G, i) \cup P'(G, i)$, $1 \leq i \leq |G| / 2$, for every $G \in$ G.

In applying this theorem, it will be useful to have constructed some Kirkman frames. We do this now. We use a recursive construction for frames, which is found in [7, Construction 3.1].

Theorem 2.3 Let (X, G, A) be a GDD, and let w: $X \to Z^+ \cup \{0\}$ (we say that w is a *weighting*). For every $A \in A$, suppose there is a Kirkman frame of type $\{w(x):\ x \in A\}$. Then there is a Kirkman frame of type $\{\Sigma_{x \in G}\, w(x):\ G \in G\}$.

We now mention some useful corollaries of this construction.

Corollary 2.4 Suppose there is a GDD, (X, G, A), in which every group has size 3, 4, or 5, and every block has size at least 4. Then there is a RC(6·|X|).

Proof: Apply Theorem 2.3, giving every point weight 6. There is a Kirkman frame of type $6^{|A|}$ for every block $A \in A$, by Theorem 2.1, since every block has size at least 4. Hence, there is a Kirkman frame of type $\{6 \cdot |G|:\ G \in G\}$. Since every hole of this frame has size 18, 24 or 30, we can fill in RC(18), RC(24), or RC(30), as required (Theorem 2.2). This produces an RC(6·|X|).

We obtain the GDDs required by Corollary 2.4 from the following two constructions. These constructions use transversal designs, which are defined as follows: a transversal design TD(k, n) is a {k}-GDD having group-type n^k. (It is well-known that the existence of a TD(k, n) is equivalent to the existence of k - 2 mutually orthogonal Latin squares of order n.)

Lemma 2.5 Suppose there is a TD(6, m), and $4 \le r \le m$. Then there is a GDD of group-type $4^{m-r}5^r$ in which every block has size at least 4.

Proof: Delete all of the points of one group of the TD(6, m), thereby creating a resolvable {5}-GDD of group-type m^5. Deleting m - r points from one of the remaining groups yields a {4, 5}-GDD of type m^4r^1, in which the blocks are partitioned into parallel classes, each of which consists of m - r blocks of size 4 and r blocks of size 5. Now, take one of these parallel classes as the groups of a new GDD. This design is a {4, 5, r, m}-GDD of group-type $4^{m-r}5^r$.

Lemma 2.6 Suppose there is a TD(5 + r, m), and $1 \le r$. Then there is a GDD of group-type $4^{m-r}5^r$ in which every block has size at least 4.

Proof: Let B be a block of the TD, and let $x \notin B$. Delete all points in the group containing x, and from r other groups, delete all points, except those on B. Taking the blocks through x as groups, we obtain a {4, 5, 4 + r, m}-GDD of group-type $4^{m-r}5^r$.

Our other construction for resolvable coverings fills in the groups of a resolvable group-divisible design with block-size 3.

Construction 2.7 Suppose there is a resolvable GDD with block-size 3, of group-type t^u, and suppose there is an RC(t). Then there is an RC(t·u).

3. The Spectrum of Resolvable Coverings

Lemma 3.1 For $n \geq 48$, there exists an RC(6n).

Proof: Write $n = 6m + r$, where m is odd and $4 \leq r \leq m$ (this can be done in a unique way). There is a TD(6, m) by [2]. Apply Corollary 2.4 and Lemma 2.5.

So, we have yet to consider RC(6n) for $6 \leq n \leq 47$. Many of these can also be constructed by filling in the holes of Kirkman frames.

Lemma 3.2 There is an RC(6n) for $32 \leq n \leq 45$.

Proof: Apply Lemma 2.5 and Corollary 2.4 with $m = 7$, $4 \leq r \leq 7$; with $m = 8$, $4 \leq r \leq 7$; and with $m = 9$, $4 \leq r \leq 9$.

Lemma 3.3 There is an RC(6n) for n = 46 and 47.

Proof: Apply Lemma 2.6 and Corollary 2.4 with $m = 11$, $r = 2$ and 3.

Lemma 3.4 There is an RC(54).

Proof: We apply Construction 2.7. There exists a resolvable {3}-GDD having group-type 18^3 (this GDD is equivalent to a TD(4, 18), or a pair of orthogonal Latin squares of order 18) Since an RC(18) exists, then there is an RC(54).

Lemma 3.5 There is an RC(108).

Proof: From Theorem 2.1, there is a Kirkman frame of type 18^6. Since RC(18), exists, therefore RC(108) exists.

We present constructions for the remaining GDDs in tabular form (see Table 1). In all cases, Corollary 2.4 is applied.

Summarizing the results of Lemmata 1.1, 1.2, 3.1 - 3.5, and Table 1, we have our main existence result.

Theorem 3.4 RC(6n) exists for all $n \geq 3$, except possibly for $n \in \{6, 7, 8, 10, 11, 13, 14, 17, 22\}$.

TABLE 1
Construction of Group-divisible Designs

n	group-type of GDD	construction
12	3^4	TD(4, 3)
15	3^5	affine plane of order 4 minus a point
16	4^4	TD(4, 4)
19	$4^4 3^1$	TD(5, 4) minus a point
20	4^5	TD(5, 4)
21	3^7	delete a point from a {4, 7}-PBD on 22 points (constructed by "completing" a Kirkman triple system on 15 points)
23	$5^4 3^1$	TD(5, 5) minus two points from one group
24	$5^4 4^1$	TD(5, 5) minus a point
25	5^5	TD(5, 5)
26	$5^4 3^2$	TD(6, 5) minus two points from each of two groups
27	$5^4 4^1 3^1$	TD(6, 5) minus two points from one group and one point from another group
28	$5^4 4^2$	TD(6, 5) minus one point from each of two groups
29	$5^5 4^1$	TD(6, 5) minus a point
30	5^6	TD(6, 5)
31	$4^7 3^1$	Complete three parallel classes of a resolvable (28, 4, 1)-BIBD.

REFERENCES

1. R.D. Baker and R.M. Wilson, *Nearly Kirkman triple systems*, Utilitas Math. **11** (1977), 289-296.

2. Th. Beth, D. Jungnickel, and H. Lenz, "Design Theory", Biblio - graphisches Institut, Zurich, 1985.

3. A.E. Brouwer, *Two new nearly Kirkman triple systems*, Utilitas Math. **13** (1978), 311-314.

4. C. Huang, E. Mendelsohn, and A. Rosa, *On partially resolvable t-partitions*, Annals Disc. Math. **12** (1983), 169-183.

5. A. Kotzig and A. Rosa, *Nearly Kirkman systems* , "Proc. Fifth S.E. Conf. on Combinatorics, Graph Theory, and Computing", Boca Raton, 1974, 607-614.

6. D.K. Ray-Chaudhuri and R.M. Wilson, *Solution of Kirkman's school-girl problem*, Amer. Math. Soc. Symp. Pure Math. **19** (1971), 187-204.

7. D.R. Stinson, *Frames for Kirkman triple systems* , Discrete Math., (to appear).

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Appendix

A resolvable covering of 18 points.

Points: $(Z_9 \times \{1, 2\})$

Resolution classes: develop the following modulo 9

{(0, 1), (1, 1), (0, 2)}	{(2, 1), (4, 1), (7, 1)}
{(3, 1), (4, 2), (6, 2)}	{(5, 1), (3, 2), (7, 2)}
{(6, 1), (1, 2), (2, 2)}	{(8, 1), (8, 2), (5, 2)}

A resolvable covering of 24 points.

Points: $(Z_{12} \times \{1, 2\})$

Resolution classes: develop the following modulo 12

{(0, 1), (1, 1), (3, 1)}	{(0, 2), (1, 2), (4, 2)}
{(2, 1), (7, 1), (7, 2)}	{(4, 1), (8, 1), (10, 2)}
{(5, 1), (11, 1), (3, 2)}	{(2, 2), (9, 2), (6, 1)}
{(5, 2), (11, 2), (10, 1)}	{(6, 2), (8, 2), (9, 1)}

A resolvable covering of 30 points.

Points: $(Z_{15} \times \{1, 2\})$

Resolution classes: develop the following modulo 15

{(0, 1), (1, 1), (3, 1)}	{(0, 2), (1, 2), (3, 2)}
{(2, 1), (6, 1), (2, 2)}	{(4, 1), (11, 1), (12, 2)}
{(7, 1), (13, 1), (10, 2)}	{(9, 1), (14, 1), (8, 2)}
{(5, 2), (14, 2), (10, 1)}	{(6, 2), (13, 2), (8, 1)}
{(7, 2), (11, 2), (5, 1)}	{(4, 2), (9, 2), (12, 1)}