An Assortment of New Howell Designs

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Abstract. We enumerate the (non-isomorphic) one-factorizations and sets of orthogonal one-factorizations (i.e. Howell designs) for several graphs on 10, 12 and 14 vertices. Among our results are the following. From the twelve 6-regular graphs on 12 vertices having transitive automorphism groups, we found that there are precisely 24 non-isomorphic $H(6, 12)$, and precisely one $H_3(6, 12)$. From the ten 7-regular graphs on 12 vertices having transitive automorphism groups, we found that there are precisely 1393 non-isomorphic $H(7, 12)$, and precisely five $H_3(7, 12)$. We also determined that there are exactly three $H^*(7, 12)$ designs, and exactly three skew $H(8, 10)$ designs. Finally, we found an example of an $H^{**}(13, 14)$, which was the smallest case of an $H^{**}(2n-1, 2n)$ which was not previously known to exist.

1. Introduction

Let Gr be an r-regular graph on n vertices. A *one-factorization* of Gr is a partition of the edge-set of Gr into r *one-factors*, each of which contains $n/2$ edges that partition the vertex set of Gr. Two one-factorizations F and G of Gr are *orthogonal* if any two edges of the graph which belong to the same one-factor of G belong to different one-factors of F (and vice-versa).

A *Howell Design* $H(s, t)$ is a square array of side s having the following properties:

(1) each cell of the array is either empty or contains a two-subset of a t-set,

(2) each element of the t-set occurs in exactly one cell of each row and each column, and

(3) any two-subset occurs in at most one cell of the array.

It is easy to see that two orthogonal one-factorizations of Gr, an r-regular graph on n vertices, give rise to an $H(r, n)$; and, conversely, the existence of an $H(r, n)$ implies the existence of a pair of orthogonal one-factorizations of some r-regular graph on n vertices, Gr, which we call the *underlying graph* of the Howell Design.

This idea generalizes to higher dimensions, as well. We can define an i-*dimensional* Howell design $H_i(r, n)$ to be an i-dimensional array which satisfies property (1), such that each two-dimensional projection is an $H(r, n)$. We refer to an $H_3(r, n)$ as a Howell *cube*. An $H_i(r, n)$ is equivalent to i *mutually* orthogonal one-factorizations of the underlying graph.

Howell designs were introduced in [9] and have been extensively studied since then. The existence of Howell designs has been completely determined in [1] and [17]. Note that a trivial necessary condition for the existence of an $H(r, n)$ is that $r + 1 \leq n \leq 2r$.

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Theorem 1.1. *Let r and n be positive integers, where n is even, and $r+1 \leq n \leq 2r$. Then there exists an $H(r,n)$ if and only if $(r,n) \neq (2,4), (3,4), (5,6)$, or $(5,8)$.*

On the other hand, if we ask which graphs are the underlying graphs of Howell designs, then not very much is known. We summarize a few of the known results. Denote $NOF(Gr)$ = the number of non-isomorphic one-factorizations of a graph Gr, and $NH_i(Gr)$ = the number of non-isomorphic $H_i(r,n)$ with underlying graph Gr. Also, let $n(Gr)$ denote the largest number of mutually orthogonal one-factorizations of Gr (i.e. $n(Gr) = \max\{i : NH_i(Gr) > 0\}$).

First, we observe that the cases where Gr is a complete bipartite graph $K_{m,m}$ have been extensively studied in the guise of orthogonal Latin squares. Hence, we have $n(K_{m,m}) \geq 2$ if and only if $m \neq 2$ or 6 (see [5]). The cases where Gr is a complete graph (K_m, with m even) correspond to Room designs, or equivalently, orthogonal symmetric Latin squares. Hence, for example, $n(K_m) \geq 3$ if and only if $m \geq 7$ is odd (see [6]). Other bounds are given in [7]. Also, we note that the underlying graph of an $H(m, m+2)$ (m even) is $K_{m+2} - f$, where f is a one-factor. Thus, we have $n(K_{m+2} - f) \geq 2$ for all even $m \geq 4$. Very few other general results are known.

Even less is known concerning *upper* bounds for $n(Gr)$. It is not difficult to see that, for any r-regular graph on n vertices, $n(Gr) \leq r-1$. This bound can be attained with equality in the case of graphs $K_{m,m}$ when m is a prime power. However, it seems unlikely that this bound can be met with equality for r-regular graphs where $r > n/2$. There is a *conjectured* bound, namely $n(Gr) \leq (n-2)/2$. Note that this conjectured bound is stronger if $r > n/2$, (if $r < n/2$, then $n(Gr) \leq 1$, anyway). The two bounds agree if $r = n/2$. This conjectured bound has been verified for $n \leq 10$ (see [13], [2], and [16]). Also, we note that there are infinitely many such graphs where $n(Gr) = (n-2)/2$ (see [16]).

The non-isomorphic one-factorizations and (i-dimensional) Howell designs have been enumerated (for all i) for all graphs on at most 10 vertices (see [13], [2], and [16]). It is not feasible to continue this enumeration to all graphs Gr on 12 vertices, for two reasons. If Gr is r-regular with r close to 12, the numbers $N_i(Gr)$ will be astronomical, and present techniques would not yield any results in a reasonable amount of time. If Gr is 6- or 7-regular, we can determine the numbers $N_i(Gr)$; the problem here is that there are too many graphs to test them all. In the remaining sections, we discuss the enumeration of one-factorizations and Howell designs for several interesting graphs on 12 and 14 vertices.

2. 6-regular graphs on 12 vertices

The case of 6-regular graphs on 12 vertices is particularly interesting, due to the non-existence of a pair of orthogonal Latin squares of order 6 (i.e. $n(K_{6,6}) = 1$). In [9], Hung and Mendelsohn presented the first example of an $H(6, 12)$. More recently, Brickell found a Howell cube $H_3(6, 12)$ for which the underlying graph is the icosahedron with antipodal points joined (see [4]). It is also worth mentioning that the automorphism group of this cube is the same as the automorphism group of the icosahedron (this group is isomorphic to $Z_2 \times A_5$).

In the hope of finding further examples, we investigated the 6-regular graphs on 12 vertices having a transitive automorphism group. There are precisely 12 such graphs (see [3]); we present a listing of the edges of the complements of these graphs in Table 1. From these 12 graphs, we found that there are precisely 24 non-isomorphic $H(6, 12)$, and precisely one $H_3(6, 12)$ (the Brickell cube). There are no examples of an $H_4(6, 12)$ in this class of graphs. A summary of our results is given in Table 2.

3. 7-regular graphs on 12 vertices

As in Section 2, we looked at the graphs having transitive automorphism groups. For 7-regular graphs on 12 vertices, there are 10 such graphs ([3]). We list the edges in the complements of these graphs in Table 3. From these 10 graphs, we found many more Howell designs: 1393 non-isomorphic $H(7, 12)$, and five non-isomorphic $H_3(7, 12)$. The enumeration is summarized in Table 4. An example of an $H_3(7, 12)$ was not previously known; we present one of the five in Table 5 (the underlying graph is graph #1 in Table 3).

We also investigated two other 7-regular graphs on 12 vertices, namely, the graphs which correspond to the so-called *-*designs*. An $H^*(r, n)$ can be defined as an $H(r, n)$ whose underlying graph has an independent set of $n - r$ vertices (which is the maximum possible size). For $r = 7$, $n = 12$, such a graph has the form $K_5^c + Q$, where Q is either a 7-cycle or the disjoint union of a 3-cycle and a 4-cycle (^{c} denotes complement and $+$ denotes join). In the first case, there are no $H^*(7, 12)$; in the second case, there are three non-isomorphic $H^*(7, 12)$, which are presented in Table 6. These are thus the smallest examples of $H^*(n, 2n-2)$ for n odd, since there are no Howell designs $H(3, 4)$ or $H(5, 8)$ (previously, the smallest example in this class was an $H^*(13, 24)$, constructed in [15]).

4. $H^{**}(13, 14)$

Another special class of Howell designs are called **-*designs*. An $H^{**}(r, n)$ is defined to be an $H(r, n)$ which satisfies the following two properties:

(1) there exists an $(r - n/2) \times (r - n/2)$ subarray of the Howell design which consists of empty cells,

(2) there exists a one-factor of the underlying graph which forms a transversal of the $n/2$ rows and columns which do not meet the empty subarray of (1).

These may seem somewhat unusual properties to ask for, but it turns out that there is a powerful recursive construction for **-designs, which was instrumental in the proof of Theorem 1.1 (see [17]).

There has recently been some interest in $H^{**}(2m-1, 2m)$ (i.e. Room squares which are **-designs). Note that we can define an $H^{**}(2m-1, 2m)$ by requiring only that property (1) holds; property (2) then follows as a consequence. Such a design has several equivalent formulations, which are described in [18]: one of these is a partitioned balanced tournament design $PBTD(n)$, and another is a pair of almost disjoint $H(m, 2m)$. We elaborate on the second formulation. Two $H(m, 2m)$, say D_1 and D_2 (on the same symbol set), having underlying graphs G_1 and G_2, respectively, are said to be *almost disjoint* if the following properties hold:

(1) $G_1 \cap G_2 = f$, where f is a one-factor,
(2) $G_1 \cup G_2 - f = K_{2m}$, the complete graph on $2m$ vertices,
(3) the edges of f occur in a row (or column) of D_1, and in a row (or column) of D_2.

$H^{**}(2m-1, 2m)$ do not exist for $m = 2, 3$, or 4 (see [18]). For $m \geq 5$, such a design is known to exist for all but a few values of m ([11], and private communication from S. Vanstone and E. Lamken). The smallest unknown case was $m = 7$. We were able to construct two non-isomorphic examples of $H^{**}(13, 14)$, which we present in Table 7 as sets of almost disjoint $H(7, 14)$.

These were found as follows. The $H(7, 14)$ labelled D_1 was constructed by E. Lamken (private communication). Call the underlying graph G_1, and let f denote the one-factor occurring in the last column of D_1. First, we enumerated all one-factorizations of the graph $G_2 = (K_{14} - G_1) \cup f$ which contained f as a one-factor. There were precisely 5272 non-isomorphic one-factorizations F of this type. For each such F, we determined all possible one-factorizations G of G_2 orthogonal to F, such that G also contained f as a one-factor. For only two of these 5272 one-factorizations F could we find such a G orthogonal to F. These are exhibited in Table 7.

5. Skew $H(8, 10)$ designs

The last types of Howell designs we investigated are complementary and skew $H(r, r+2)$. These special types of Howell designs are introduced in [10]. Several constructions are given, and it is shown that a pair of complementary $H(r, r+2)$ exist for all even $r \geq 4$. Much less is known regarding skew $H(r, r+2)$. In [10], a skew $H(4, 6)$, is constructed, and it is reported that there does not exist a skew $H(6, 8)$. The first unsettled case was that of a skew

$H(8,10)$. We have done an enumeration of skew $H(8,10)$, and we found that there are exactly three non-isomorphic examples.

We define a skew $H(r, r+2)$ (of course, r must be even). An $H(r, r+2)$, say H, is said to be *skew* if there exist two symbols a, b where $\{a, b\}$ is not an edge of the underlying graph, such that the following properties are satisfied:

(1) Denote the n cells of H which contain a by T_a, and denote the n cells of H which contain b by T_b. Then $T_a \cup T_b$ consists of the r cells on the diagonal of H (say D), and r other cells which form a transversal of cells (say D') of H, such that D' is symmetric with respect to D (i.e. a cell $(i,j) \in D'$ if and only if cell $(j,i) \in D'$).

(2) Given any cell $(i,j) \notin D \cup D'$, precisely one of cell (i,j) and cell (j,i) is empty.

In [16], the authors enumerated all non-isomorphic $H(8,10)$; there are 18220 such Howell designs. It was therefore a straightforward test to see which of these designs could be written down in such a way that it forms a skew $H(8,10)$. This was done as follows. For any given $H(8,10)$, there are five possibilities for the pair $\{a, b\}$. For each possibility, the cells in $T_a \cup T_b$ form four 4-cycles (no matter how the Howell design is written down). For each 4-cycle, there are essentially two inequivalent ways of permuting the rows / columns containing the 4-cycle. There are thus only $2^5 = 32$ row / column permutations that must be considered (for each possible $\{a, b\}$).

As a result of these tests, we found precisely three non-isomorphic skew $H(8,10)$, which we record in Table 8.

Table 1. 5-regular graphs on 12 vertices having transitive automorphism groups

Graph No.	Edges
1	1 - 2, 3, 4, 5, 6; 2 - 3, 4, 5, 6; 3 - 4, 7, 8; 4 - 7, 8; 5 - 6, 9, 10; 6 - 9, 10; 7 - 8, 11, 12; 8 - 11, 12; 9 - 10, 11, 12; 10 - 11, 12; 11 - 12.
2	1 - 2, 3, 4, 5, 6; 2 - 3, 4, 7, 8; 3 - 4, 9, 10; 4 - 11, 12; 5 - 6, 7, 9, 11; 6 - 8, 10, 12; 7 - 8, 9, 11; 8 - 10, 12; 9 - 10, 11; 10 - 12; 11 - 12.
3	1 - 2, 3, 4, 5, 6; 2 - 3, 4, 7, 8; 3 - 4, 9, 10; 4 - 11, 12; 5 - 7, 8, 9, 11; 6 - 7, 8, 10, 12; 7 - 9, 11; 8 - 10, 12; 9 - 11, 12; 10 - 11, 12.
4	1 - 2, 3, 4, 5, 6; 2 - 3, 4, 7, 8; 3 - 4, 9, 10; 4 - 11, 12; 5 - 7, 8, 9, 11; 6 - 7, 9, 10, 12; 7 - 10, 12; 8 - 9, 11, 12; 9 - 11; 10 - 11, 12.
5	1 - 2, 3, 4, 5, 6; 2 - 3, 4, 7, 8; 3 - 5, 7, 9; 4 - 5, 7, 10; 5 - 7, 11; 6 - 8, 9, 10, 11; 7 - 12; 8 - 9, 10, 12; 9 - 11, 12; 10 - 11, 12; 11 - 12.
6	1 - 2, 3, 4, 5, 6; 2 - 3, 4, 7, 8; 3 - 5, 7, 9; 4 - 5, 7, 10; 5 - 7, 11; 6 - 8, 9, 10, 12; 7 - 12; 8 - 9, 11, 12; 9 - 10, 11; 10 - 11, 12; 11 - 12.
7	1 - 2, 3, 4, 5, 6; 2 - 3, 4, 7, 8; 3 - 5, 7, 9; 4 - 6, 8, 10; 5 - 6, 9, 11; 6 - 10, 11; 7 - 8, 9, 12; 8 - 10, 12; 9 - 11, 12; 10 - 11, 12; 11 - 12.
8	1 - 2, 3, 4, 5, 6; 2 - 3, 4, 7, 8; 3 - 5, 9, 10; 4 - 7, 9, 10; 5 - 9, 11, 12; 6 - 8, 9, 11, 12; 7 - 10, 11, 12; 8 - 10, 11, 12; 9 - 11; 10 - 12.
9	1 - 2, 3, 4, 5, 6; 2 - 3, 4, 7, 8; 3 - 5, 9, 10; 4 - 7, 11, 12; 5 - 9, 11, 12; 6 - 8, 10, 11, 12; 7 - 9, 10, 11; 8 - 9, 10, 12; 9 - 12; 10 - 11.
10	1 - 2, 3, 4, 5, 6; 2 - 3, 7, 8, 9; 3 - 10, 11, 12; 4 -5, 7, 8, 10; 5 - 9, 11, 12; 6 - 7, 8, 11, 12; 7 - 9, 11; 8 - 10, 12; 9 - 10, 12; 10 - 11.
11	1 - 2, 3, 4, 5, 6; 2 - 3, 7, 8, 9; 3 - 10, 11, 12; 4 -7, 8, 9, 10; 5 - 7, 8, 10, 11; 6 - 7, 10, 11, 12; 7 - 12; 8 - 11, 12; 9 - 10, 11, 12.
12	1 - 2, 3, 4, 5, 6; 2 - 7, 8, 9, 10; 3 - 7, 8, 9, 11; 4 -7, 8, 10, 11; 5 - 7, 9, 10, 11; 6 - 8, 9, 10, 11; 7 - 12; 8 - 12; 9 - 12; 10 - 12; 11 - 12.

Table 2. Howell designs from 6-regular graphs on 12 vertices having transitive automorphism groups.

Graph No.	|Aut(Gr)|	DPM(Gr)	NOF(Gr)	NH(Gr)	NH_3(Gr)
1	768	368	190	0	0
2	144	348	469	3	0
3	48	344	1248	8	0
4	24	342	2018	0	0
5	96	392	1451	0	0
6	12	386	6932	1	0
7	120	368	733	4	1
8	12	354	4976	0	0
9	24	344	2216	5	0
10	48	344	1021	0	0
11	24	336	1983	3	0
12	1440	376	132	0	0

Notation: DPM(Gr) denotes the number of distinct one-factors of Gr.

Table 3. 4-regular graphs on 12 vertices having transitive automorphism groups

Graph No.	Edges
1	1 - 2, 3, 4, 5; 2 - 3, 4, 6; 3 - 4, 7; 4 - 8; 5 - 6, 9, 10; 6 - 9, 10; 7 - 8, 11, 12; 8 - 11, 12; 9 - 10, 11; 10 - 12; 11 - 12.
2	1 - 2, 3, 4, 5; 2 - 3, 4, 6; 3 - 5, 7; 4 - 6, 8; 5 - 7, 9; 6 - 8, 10; 7 - 9, 11; 8 - 10, 12; 9 - 11, 12; 10 - 11, 12; 11 - 12.
3	1 - 2, 3, 4, 5; 2 - 3, 6, 7; 3 - 8, 9; 4 - 5, 6, 10; 5 - 8, 11; 6 - 7, 10; 7 - 9, 12; 8 - 9, 11; 9 - 12; 10 - 11, 12; 11 - 12.
4	1 - 2, 3, 4, 5; 2 - 3, 6, 7; 3 - 8, 9; 4 - 6, 8, 10; 5 - 7, 9, 10; 6 - 8, 11; 7 - 9, 11; 8 - 12; 9 - 12; 10 - 11, 12; 11 - 12.
5	1 - 2, 3, 4, 5; 2 - 3, 6, 7; 3 - 8, 9; 4 - 6, 8, 10; 5 - 7, 9, 11; 6 - 8, 11; 7 - 9, 12; 8 - 12; 9 - 10; 10 - 11, 12; 11 - 12.
6	1 - 2, 3, 4, 5; 2 - 3, 6, 7; 3 - 8, 9; 4 - 6, 10, 11; 5 - 8, 10, 12; 6 - 11, 12; 7 - 9, 10, 12; 8 - 11, 12; 9 - 10, 11.
7	1 - 2, 3, 4, 5; 2 - 6, 7, 8; 3 - 6, 7, 8; 4 - 6, 9, 10; 5 - 6, 9, 10; 7 - 11, 12; 8 - 11, 12; 9 - 11, 12; 10 - 11, 12.
8	1 - 2, 3, 4, 5; 2 - 6, 7, 8; 3 - 6, 7, 9; 4 - 6, 7, 10; 5 - 8, 9, 10; 6 - 11; 7 - 12; 8 - 11, 12; 9 - 11, 12; 10 - 11, 12.
9	1 - 2, 3, 4, 5; 2 - 6, 7, 8; 3 - 6, 7, 9; 4 - 6, 8, 10; 5 - 7, 9, 10; 6 - 11; 7 - 12; 8 - 11, 12; 9 - 11, 12; 10 - 11, 12.
10	1 - 2, 3, 4, 5; 2 - 6, 7, 8; 3 - 6, 9, 10; 4 - 7, 9, 11; 5 - 8, 10, 12; 6 - 11, 12; 7 - 10, 12; 8 - 9, 11; 9 - 12; 10 - 11.

Table 4. Howell designs from 7-regular graphs on 12 vertices having transitive automorphism groups.

Graph No.	|Aut(Gr)|	DPM(Gr)	NOF(Gr)	NH(Gr)	NH_3(Gr)
1	48	825	127222	84	1
2	24	837	270875	235	3
3	48	827	130176	103	0
4	48	824	130141	166	0
5	24	821	245138	189	0
6	24	808	218138	130	0
7	768	827	9145	47	0
8	144	820	43060	72	1
9	24	818	237042	264	0
10	48	804	110656	103	0

Table 5. A Howell cube $H_3(7, 12)$.

1 2	6 11	5 10	7 12	3 4	8 9	
5 11	1 3	6 7	2 9	8 12		4 10
6 8	2 5	1 4		7 11	3 10	9 12
10 12	7 9	2 8	1 5		4 6	3 11
	8 10	3 9	4 11	1 6	5 12	2 7
4 9		11 12	3 8	2 10	1 7	5 6
3 7	4 12		6 10	5 9	2 11	1 8

1 2		6 11	3 4	8 9	5 10	7 12
8 12	1 3		6 7	4 10	2 9	5 11
3 10	7 11	1 4	9 12	2 5	6 8	
7 9	2 8	10 12	1 5		3 11	4 6
4 11	5 12	2 7	8 10	1 6		3 9
5 6	4 9	3 8		11 12	1 7	2 10
	6 10	5 9	2 11	3 7	4 12	1 8

1 2	4 9	10 12		3 7	6 8	5 11
7 9	1 3	6 11	8 10	2 5	4 12	
	2 8	1 4	6 7	11 12	5 10	3 9
4 11	6 10	3 8	1 5		2 9	7 12
8 12	7 11	5 9	3 4	1 6		2 10
3 10	5 12		2 11	8 9	1 7	4 6
5 6		2 7	9 12	4 10	3 11	1 8

Table 6. Three Howell designs $H^*(7, 12)$.

1 2	7 11		6 10	3 4	8 12	5 9
7 12	1 3	8 10	4 9	2 5	6 11	
8 9		1 4	11 12	7 10	3 5	2 6
4 11	2 8	6 12	1 5		9 10	3 7
	5 12	7 9	3 8	1 6	2 4	10 11
5 10	6 9	2 3		8 11	1 7	4 12
3 6	4 10	5 11	2 7	9 12		1 8

1 2		6 10	3 4	5 9	8 12	7 11
	1 3	8 11	7 12	4 10	6 9	2 5
5 12	7 9	1 4	10 11	2 8		3 6
3 8	6 12	2 7	1 5		4 11	9 10
7 10	5 11		8 9	1 6	2 3	4 12
4 9	8 10	3 5	2 6	11 12	1 7	
6 11	2 4	9 12		3 7	5 10	1 8

1 2		8 12	3 4	5 9	6 10	7 11
	1 3	7 9	6 12	4 10	8 11	2 5
6 9	8 10	1 4	2 7	11 12	3 5	
3 8	7 12	6 11	1 5		2 4	9 10
7 10	5 11	2 3	8 9	1 6		4 12
5 12	4 9		10 11	2 8	1 7	3 6
4 11	2 6	5 10		3 7	9 12	1 8

Table 7. Two sets of almost disjoint Howell designs H(7, 14)

Set 1: $\{D_1, D_2\}$.

D_1

a 3	$\underline{a}$ $\underline{3}$	$\underline{2}$ $\underline{4}$	2 4	$\underline{1}$ $\underline{5}$	1 5	6 $\underline{6}$
$\underline{a}$ 2	a $\underline{2}$	$\underline{1}$ $\underline{3}$	1 3	$\underline{4}$ $\underline{6}$	4 6	5 $\underline{5}$
$\underline{1}$ $\underline{2}$	1 2	a $\underline{5}$	$\underline{a}$ 5	3 6	$\underline{3}$ $\underline{6}$	4 $\underline{4}$
$\underline{3}$ $\underline{4}$	3 4	$\underline{a}$ $\underline{6}$	a 6	2 5	$\underline{2}$ $\underline{5}$	1 $\underline{1}$
4 5	$\underline{4}$ $\underline{5}$	2 6	$\underline{2}$ $\underline{6}$	a 1	$\underline{a}$ $\underline{1}$	3 $\underline{3}$
1 6	$\underline{1}$ $\underline{6}$	3 5	$\underline{3}$ $\underline{5}$	$\underline{a}$ 4	a $\underline{4}$	2 $\underline{2}$
$\underline{5}$ $\underline{6}$	5 6	1 4	$\underline{1}$ $\underline{4}$	$\underline{2}$ $\underline{3}$	2 3	a $\underline{a}$

D_2

$\underline{a}$ $\underline{4}$	3 $\underline{1}$	5 $\underline{3}$	1 $\underline{5}$	a 2	4 $\underline{2}$	6 $\underline{6}$
1 $\underline{2}$	2 $\underline{4}$	4 $\underline{6}$	$\underline{a}$ 3	6 $\underline{1}$	a $\underline{3}$	5 $\underline{5}$
3 $\underline{6}$	6 $\underline{5}$	a $\underline{1}$	2 $\underline{3}$	5 $\underline{2}$	$\underline{a}$ 1	4 $\underline{4}$
6 $\underline{3}$	a 4	3 $\underline{2}$	5 $\underline{4}$	$\underline{a}$ $\underline{5}$	2 $\underline{6}$	1 $\underline{1}$
a 5	$\underline{a}$ $\underline{2}$	2 $\underline{5}$	4 $\underline{1}$	1 $\underline{6}$	6 $\underline{4}$	3 $\underline{3}$
4 $\underline{5}$	1 $\underline{3}$	$\underline{a}$ 6	a $\underline{6}$	3 $\underline{4}$	5 $\underline{1}$	2 $\underline{2}$
2 $\underline{1}$	5 $\underline{6}$	1 $\underline{4}$	6 $\underline{2}$	4 $\underline{3}$	$\underline{5}$ $\underline{3}$	a $\underline{a}$

Set 2: $\{D_1, D_3\}$.

D_3

a 4	$\underline{a}$ $\underline{4}$	2 $\underline{3}$	1 $\underline{5}$	5 $\underline{2}$	3 $\underline{1}$	6 $\underline{6}$
6 $\underline{3}$	3 $\underline{2}$	a $\underline{6}$	4 $\underline{1}$	2 $\underline{4}$	$\underline{a}$ 1	5 $\underline{5}$
1 $\underline{2}$	2 $\underline{5}$	5 $\underline{1}$	3 $\underline{6}$	$\underline{a}$ 6	a $\underline{3}$	4 $\underline{4}$
3 $\underline{4}$	a 5	6 $\underline{5}$	$\underline{a}$ $\underline{2}$	4 $\underline{3}$	2 $\underline{6}$	1 $\underline{1}$
$\underline{a}$ $\underline{5}$	6 $\underline{1}$	4 $\underline{2}$	a 2	1 $\underline{6}$	5 $\underline{4}$	3 $\underline{3}$
5 $\underline{6}$	1 $\underline{3}$	$\underline{a}$ 3	6 $\underline{4}$	a $\underline{1}$	4 $\underline{5}$	2 $\underline{2}$
2 $\underline{1}$	4 $\underline{6}$	1 $\underline{4}$	5 $\underline{3}$	3 $\underline{5}$	6 $\underline{2}$	a $\underline{a}$

Table 8. Three skew H(8, 10) designs.

a = 5, b = 6

6 10		1 9	4 5	2 8		3 7	
8 9	2 5		7 10	3 6	1 4		
	1 7	4 6			2 10	5 9	3 8
1 5		2 7	6 8		3 9	4 10	
	6 9	8 10	1 3	5 7			2 4
2 3				4 9	6 7	1 8	5 10
	4 8	3 5		1 10		2 6	7 9
4 7	3 10		2 9		5 8		1 6

a = 5, b = 6

2 6				3 10	4 7	5 9	1 8
3 8	6 10	2 7	4 5	1 9			
4 9		3 5		2 8	1 10		6 7
7 10	2 5	8 9	3 6			1 4	
			1 7	4 6	5 8	2 10	3 9
	1 3		8 10	5 7	6 9		2 4
1 5	7 9	4 10			2 3	6 8	
	4 8	1 6	2 9			3 7	5 10

a = 7, b = 8

8 10		6 9		5 7	2 4	1 3	
4 9	2 7		1 6		5 10		3 8
	1 9	2 8	3 10		6 7	4 5	
2 5			4 7	1 10	3 9	6 8	
1 7	3 5	4 10		8 9			2 6
		3 7		4 6	1 8	2 10	5 9
	6 10		5 8	2 3		7 9	1 4
3 6	4 8	1 5	2 9				7 10

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