A Perfect One-factorization of K_{50}

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Abstract We enumerate the perfect one-factorizations of K_{50} which are generated by starters in $\mathbf{Z}_{49}$ fixed by multiplication by 18 and 30. There are precisely 67 non-isomorphic examples.

A *one-factorization* of a complete graph K_{2n} is a partition of the edge-set of K_{2n} into 2n - 1 *one-factors*, each of which contains n edges that partition the vertex set of K_{2n}. A *perfect* one-factorization (P1F) is a one-factorization in which every pair of distinct one-factors forms a Hamiltonian cycle.

P1Fs of K_{2n} are known to exist when n is prime, when 2n - 1 is prime, and when $2n \in \{16, 28, 36, 244, 344\}$ (see [1] and [3]). These were the only known examples of P1Fs. We use a backtracking algorithm to generate starter-induced one-factorizations of a special type, and discover a P1F of K_{50}.

Many of the known constructions for (perfect) one-factorizations use starters. A *starter* in $\mathbf{Z}_{2n+1}$ is a set $S = \{ \{x_1, y_1\}, \{x_2, y_2\}, \ldots \{x_n, y_n\} \}$ such that every non-zero element of $\mathbf{Z}_{2n+1}$ occurs as

(1) an element of some pair of S, and

(2) a difference of some pair of S.

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Define $S^* = S \cup \{0, \infty\}$ and $\infty + g = g + \infty = \infty$ for all $g \in \mathbf{Z}_{2n+1}$. Then, it is easy to see that $F = \{S^* + g : g \in \mathbf{Z}_{2n+1}\}$ is a one-factorization of K_{2n+2}. Further, F contains $\mathbf{Z}_{2n+1}$ in its automorphism group.

In [1], Anderson enumerates all P1Fs of K_n arising from starters, for all even $n \leq 22$. These empirical results suggest that there exists a starter-induced P1F of K_n for all even $n \geq 12$. One might hope that starters would provide new examples of P1F for larger values of n. Unfortunately, the probability that an arbitrary starter in $\mathbf{Z}_{n-1}$ generates a P1F of K_n seems to approach 0 very quickly. Empirical results in [3] indicate that this probability is approximately $1 / 10^{.28n - 2.6}$. Substituting n = 50, we obtain an estimate of $1 / 10^{11.4}$. Clearly, this is not a practical approach in attempting to generate a P1F of K_{50}.

If we are to find a starter-induced P1F of K_{50}, we must restrict our search to a particular class of starters having more structure. In [2], Ihrig studies automorphism groups of starter-induced P1F, and proves the following result.

Theorem ([2, Theorem 4.1]) If F is a P1F induced by a starter in $\mathbf{Z}_{2n-1}$, then the automorphism group of F is a semidirect product of $\mathbf{Z}_{2n-1}$ with H, where H is a subgroup of $\mathrm{Aut}(\mathbf{Z}_{2n-1})$, $|H|$ divides n - 1, and $|H|$ is odd.

Taking 2n - 1 = 49, we see that $|H| = 1$ or 3. A P1F of order 50 could have as its automorphism group the semidirect product of $\mathbf{Z}_{49}$ with $\mathbf{Z}_3$. The possibility $|H| = 3$ would correspond to the existence of a starter in the ring $\mathbf{Z}_{49}$ which is fixed by the multiplicative subgroup {1, 18, 30} (the three cube roots of unity) and which generates a perfect one-factorization of order 50.

We used a backtracking algorithm to enumerate all such starters, and found that there are precisely 938 of them that generate P1Fs. They fall into 67 isomorphism classes, each containing 14 starters. The 14 starters in any such isomorphism class correspond to the 14 cosets of {1, 18, 30} in the multiplicative group of the 42 units in $\mathbf{Z}_{49}$. One such starter is presented below.

1	2	30	11	18	36
4	6	22	33	23	10
42	45	35	27	21	26
12	16	17	39	20	43
32	38	29	13	37	47
8	15	44	9	46	25
19	28	31	7	48	14
40	3	24	41	34	5

References

1. B. A. Anderson, Some perfect 1-factorizations, Proc. of 7th Southeastern Conf. on Comb., Graph Theory and Computing, Utilitas Math., Winnipeg (1976), 79-91.

2. Edwin C. Ihrig, The structure of symmetry groups of perfect one-factorizations of K_{2n}, preprint.

3. E. Seah and D. R. Stinson, A perfect one-factorization for K_{36}, preprint.