The Spectrum for the conjugate invariant subgroups of perpendicular arrays

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1. Introduction.

Perpendicular arrays are not as well-known as orthogonal arrays, but one day they will be! However, in the interim, a definition is a good place to start. A *perpendicular array,* or *PA*, is an $\binom{n}{2}$ by k array A such that each cell is occupied by one of the numbers $1,2,...,n$, and such that if we run our fingers down any two columns of A, we obtain each of the $\binom{n}{2}$ 2-element subsets of $\{1,...,n\}$ exactly once. The number n is called the *order*, and the number k is called the *strength* of the array A. We will abbreviate the phrase "perpendicular array of order n and strength k" to $PA(n,k)$.

A trivial necessary condition for the existence of a $PA(n,k)$ is that $k \leq n$ and n is odd. Whether or not these conditions are (generally) sufficient for the existence of a $PA(n,k)$ is, not too surprisingly, an open problem. However, except for a few cases, the existence problem has been settled for $k = 3$, 4, and 5, by Mullin, Schellenberg, van Rees, and Vanstone [5]. In particular, they show that a $PA(n,3)$ exists for all odd $n \geq 3$; a $PA(n,4)$ exists for all off $n \geq 5$ (except possibly $n = 87$); and a $PA(n,5)$ exists for all odd $n \geq 5$ (except possibly $n \in \{33,39,51,87,219\}$).

If A is a $PA(n,k)$ and α is any permutation in S_k (the symmetric group on $\{1,...,k\}$), we will denote by $A\alpha$ the perpendicular array obtained from A by permuting the columns of A according to α. Two perpendicular arrays are *equal* if they contain the same rows (not necessarily in the same order). Two perpendicular arrays A and B are *conjugate* if there exists $\alpha \in S_k$ such that $A\alpha = B$. If $A\alpha = A$, for some α, we say that A is *invariant* under conjugation by α. The subgroup of S_k consisting of all α such that $A\alpha = A$ is called the *conjugate invariant subgroup* of A.

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Example. A $PA(5,5)$ with conjugate invariant subgroup $<(12345)>$

1	2	4	5	3
3	1	2	4	5
5	3	1	2	4
4	5	3	1	2
2	4	5	3	1
1	4	3	2	5
5	1	4	3	2
2	5	1	4	3
3	2	5	1	4
4	3	1	5	1

In this paper, we investigate the following natural problem. For a subgroup H of S_k ($k \geq 3$), determine the set of all integers n (i.e. the *spectrum)* for which there exists a $PA(n,k)$ having H as its conjugate invariant subgroup. We give the solution of this problem (except for a handful of cases) for $k = 3$, 4, and 5. Apart from being of interest in its own right, the solution of this problem has (at least) two significant applications.

(1) The nesting problem for Steiner triple systems (*STS*).

For which n does there exist an *STS* of order n, (S,T), with the property that one can adjoin one point of S of each triple in T, obtaining a *BIBD* with block-size 4 and $\lambda = 2$? We say that such an *STS* can be *nested*. A necessary condition is $n \equiv 1 \pmod 6$. It turns out that an *STS* which can be nested is equivalent to a $PA(n,4)$ having conjugate invariant subgroup $C_3 = <(123)>$. We show that these designs exist for all $n \equiv 1 \pmod 6$, with 15 possible exceptions. (This result has been obtained independently by Colbourn and Colbourn in [1]).

(2) The Steiner pentagon problem.

In 1966, Alex Rosa proved that the complete graph K_n can be decomposed into edge-disjoint pentagons if and only if $n \equiv 1$ or $5 \pmod{10}$ [6]. Subsequently, the authors [3] obtained such a decomposition with the additional property that every pair of distinct vertices of K_n is joined by a path of length two in exactly one pentagon. (Such a decomposition is called a *Steiner pentagon system,* or *SPS*.) An *SPS* of order n is equivalent to a $PA(n,5)$ having conjugate invariant subgroup $C_5 = <(12345)>$. In [3], the authors show that the spectrum for *SPS* is precisely the set of all positive $n \equiv 1$ or $5 \pmod{10}$, except $n = 15$. We also remark that an *SPS* is equivalent to a quasigroup satisfying the identities $x^2 = x$, $(yx)x = y$, and $x(yx) = y(xy)$, so the spectrum for these quasigroups has also been determined.

These examples provide some indication of the importance of the conjugate invariant subgroup problem for perpendicular arrays. An extensive amount of work has been done on the analogous problem for orthogonal arrays; the interested reader is referred to [2] for a detailed account of progress to date.

2. Preliminaries.

A bit of reflection reveals that a perpendicular array of strength k cannot be invariant under conjugation by a transposition, a product of disjoint transpositions, or any permutation which moves less than $k-1$ symbols. Hence the only possible conjugate invariant subgroups are: $<1>$ and $<(123)>$ for $k = 3$; $<1>$ and $<(ijk)>$ for $k = 4$; and $<1>$ and $<(ijklm)>$ for $k = 5$. In the sequel, we consider the subgroup $<(123)>$ of S_4 and $<(12345)>$ of S_5; the other nontrivial subgroups give rise to the same spectra.

We also observe that a Steiner triple system of order n is equivalent to a $PA(n,3)$ with conjugate invariant subgroup $<(123)>$. This is quite easy to see. Let (S,t) be an STS of order n. Define A, an $\binom{n}{2}$ by 3 array by: for each triple $\{x,y,z\} \in t$, place in A the three rows (x,y,z), (y,z,x) and (z,x,y) (or the three rows (x,z,y), (z,y,x), and (y,x,z)). Then A is a $PA(n,3)$ with conjugate invariant subgroup $<(123)>$. Conversely, suppose A is a $PA(n,3)$ (based on a set S) with $<(123)>$ as its conjugate invariant subgroup. Define a set t of triples of S by: $\{x,y,z\} \in t$ iff (x,y,z), (y,z,x), and (z,x,y) are rows of A. Then (S,t) is an STS of order n.

3. Conjugate invariant subgroups of S_3.

We observed in the last section that a $PA(n,3)$ with conjugate invariant subgroup $<(123)>$ is equivalent to an STS of order n ($n \geq 3$). Hence the spectrum is precisely the set $\{n \geq 3: n \equiv 1 \text{ or } 3 \pmod 6\}$.

The other possible conjugate invariant subgroup is $<1>$. First, we note that there are two distinct $PA(3,3)$'s, and both are invariant under conjugation by $<(123)>$. So, let $n = 2k+1 \geq 5$. Define A to be the $\binom{n}{2}$ by 3 array with rows $(i,i+j,i+2j) \pmod n$, $0 \leq i \leq 2k$, $1 \leq j \leq k$. It can easily be shown that A is a $PA(n,3)$ with conjugate invariant subgroup $<1>$. So, the spectrum for $PA(n,3)$'s with conjugate invariant subgroup $<1>$ is precisely the set $\{n \geq 5: n \text{ odd}\}$.

4. Conjugate invariant subgroups of S_4.

We first consider the case of the conjugate invariant subgroup $<1>$. Suppose that $n > 3$ is odd, $n \notin \{33,39,51,87,219\}$. Then there exists a $PA(n,5)$, say A, by the results of [5]. Let B be the $PA(n,4)$ consisting of the first four columns of A, and let C be the $PA(n,4)$ formed from columns 1, 2, 3, and 5 of A. It is straightforward to see that at least one of B and C has $<1>$ as its conjugate invariant subgroup.

It remains to consider n = 33,39,51,87, and 219. For n = 33,39,51, and 219, there exists a $PA(n,4)$, by [5]. We have observed in the introduction that a $PA(n,4)$ with conjugate invariant subgroup $<(ijk)>$ is equivalent to an *STS* of order n which can be nested, and this requires $n \equiv 1 \pmod 6$. Thus, any $PA(n,4)$ with $n \equiv 3$ or $5 \pmod 6$ has conjugate invariant subgroup $<1>$. So, in particular, the perpendicular arrays $PA(33,4)$, $PA(39,4)$, $PA(51,4)$, and $PA(219,4)$ have conjugate invariant subgroup $<1>$. Hence the spectrum for $PA(n,4)$'s with conjugate invariant subgroup $<1>$ is the set of all odd $n \geq 5$, except (possibly) 87.

The case of the conjugate invariant subgroup $<(123)>$ (or nested Steiner triple systems) is more difficult. We first construct an *STS* which can be nested for all orders $n \equiv 1 \pmod 6$, with 19 possible exceptions.

The technique of proof is the same as was used for an unrelated problem, which we now describe. If a Kirkman triple system (i.e. a resolvable *STS*) of order v contains an *STS* of order $(v-1)/2$ as a subsystem, then $v \equiv 3 \pmod{12}$ (and $(v-1)/2 \equiv 1 \pmod 6$). In [4], Mullin, Stinson and Vanstone investigate the existence of such designs, it is established that, except for 19 possible exceptions, if $n \equiv 1 \pmod 6$, then there exists a Kirkman triple system of order $2n+1$ which contains an *STS* of order n as a subsystem. This is done as follows: 1) *PBD*-closure is established, 2) a prime-power construction, and singular direct and indirect products are given, and 3) enough "small" designs are produced using the constructions of 2) so that 1) can be applied to determine the spectrum (modulo the 19 aforementioned possible exceptions).

The constructions of 1) and 2) depend only on the orders n of the subsystem of the *KTS* of order $2n+1$, and we have noted that $n \equiv 1 \pmod 6$. So if we have constructions for nested *STS* analogous to those of 1) and 2), then 3) can be applied without change. This is what we proceed to do. It is most convenient to describe a nesting of an *STS* (S,T) as a mapping $\alpha: T \to S$ such that (S,B), where $B = \{\{a,b,c,t\alpha\}: t = \{a,b,c\} \in T\}$, is a *BIBD* with block-size 4 and $\lambda = 2$.

The prime power construction: If $n = 6t+1$ is a prime power, let $F = GF(n)$ and let x be primitive in F. Define $B = \{\{x^j, x^{j+2t}, x^{j+4t}\}: 0 \leq j \leq t-1\}$, and $T = \{\{a+i, b+i, c+i\}: \{a,b,c\} \in B$ and $i \in F\}$. Also, let $\alpha: T \rightarrow F$ be defined by $\{a+i, b+i, c+i\}\alpha = i$. Then (F,T) is an STS of order n and α is a nesting.

The singular direct product: Let (V,v) be an STS, and take a fixed (but arbitrary) ordering of the three points in each triple in v. Let (Q,q) be an STS containing a sub-$STS(P,p)$, and write $X = Q \backslash P$. For each $i \in V$, let $(P \cup (X \times \{i\}), q(i))$ be the STS obtained from (Q,q) by replacing each $x \in X$ with (x,i) in each triple of q in which it appears. Let (X,o) be a quasigroup, denote $S = P \cup (X \times V)$, and define T to consist of the following set of triples of S:

1) the triples in P

2) for each $i \in V$, the triples in $q(i) \backslash p$

3) for each triple $\{a,b,c\} \in v$ $(a < b < c)$, the triples $\{(x,a),(y,b),(x \ o \ y, c)\}$, for all $x,y \in X$.

Then (S,T) is an STS, which is called the *singular direct product* of (V,v), (Q,q), (P,p), and (X,o).

Theorem. *Suppose that the STS (V,v) can be nested, and that the STS (Q,q) can be nested in such a way that the restriction of the nesting to P is a nesting of (P,p). If $|X| \neq 6$, then there is a quasigroup (X,o) such that the singular direct product (S,T) can be nested.*

Proof. Let (X,o) be a quasigroup with an orthogonal mate (this requires $|X| \neq 6$). Then (X,o) can be partitioned into transversals t_x, $x \in X$. Let α be a nesting of (Q,q) which induces a nesting of (P,p), and denote by $\alpha(i)$ the corresponding nesting of $(P \cup (X \times \{i\}), q(i))$. Let β be a nesting of (V,v). We now define $\theta: T \rightarrow S$ by

$$\{a,b,c\}\theta = \begin{cases} \{a,b,c\}\alpha, \text{ if } \{a,b,c\} \in p \\ \{a,b,c\}\alpha(i), \text{ if } \{a,b,c\} \in q(i)\backslash p \\ (z,\{i,j,k\}\beta), \text{ if } \{a,b,c\} = \{(x,i),(y,j),(x \ o \ y, k)\} \\ \text{where } \{i < j < k\} \in v \text{ and } (x,y,x \ o \ y) \in t_z. \end{cases}$$

It is straightforward to see that θ is a nesting of (S,T).

The singular indirect product: Before plunging into a description of the singular indirect product, we need a few preliminaries. Let $X^* \subseteq X$. A *quasigroup* (with *hole* X^*) is a partial quasigroup (X, o) in which $x\ o\ y$ is defined if and only if $(x,y) \notin X^* \times X^*$, and $x\ o\ y \notin X^*$ if $x \in X^*$ or $y \in X^*$. Two quasigroups (X, o_1) and (X, o_2) with the same hole X^* are said to be *orthogonal* if the set of ordered pairs resulting from the superposition of the corresponding partial Latin squares is precisely $(X \times X)\backslash(X^* \times X^*)$. Now define $t_a = \{(x,y,x\ o_1\ y): x\ o_2\ y = a\}$, $a \in X$. If $a \in X^*$, then $|t_a| = |X| - |X^*|$ and we call t_a a *short transversal.* If $a \in X\backslash X^*$, then $|t_1| = |X|$ and we call t_a a *long transversal.*

As before, let (V,v) be an *STS*, and order each triple in v. Let (Q,q) be an *STS* containing a sub-*STS* (Q^*,q^*), and let $P \subseteq Q^*$. Define $X = Q\backslash P$ and $X^* = Q^*\backslash P$. Let $q(i)$ (resp. $q^*(i)$) denote the triples obtained from q (resp. q^*) by replacing any $x \in X$ by (x,i). Let $(P \cup (X^* \times V), t)$ be an *STS*. Finally, let (X, o) be a quasigroup with hole X^*, and let $S = P \cup (X \times V)$. Define a set T of triples of S to consist of the following:

1) the triples in t

2) for each $i \in V$, the triples in $q(i)\backslash q^*(i)$

3) for each triple $\{a,b,c\} \in v$ $(a < b < c)$, the triples $\{(x,a),(y,b),(x\ o\ y,c)\}$, for all x,y such that $(x,y) \notin X^* \times X^*$.

Then (S,T) is an *STS*, called the *singular indirect product* of (V,v), (Q,q), (Q^*,q^*), and (X,o) with hole X^*.

Theorem. *Suppose there are nestings of the STS (V,v), $(P \cup (X^* \times V),t)$, and (Q,q) (in which the sub-STS (Q^*,q^*) can be nested). If the quasigroup (X,o) with hole X^* has an orthogonal mate, then the singular indirect product (S,T) can be nested.*

Proof. As described above, we can partition (X, o) into transversals (short and long) t_x, $x \in X$. Let α be a nesting of (Q,q) which induces a nesting of (Q^*,q^*), and denote by $\alpha(i)$ the corresponding nesting of $(P \cup (X \times \{i\}), q(i))$. Also, let β be a nesting of $(P \cup (X^* \times V),t)$, and let γ be a nesting of (V,v). We define $\theta: T \rightarrow S$ by

$$\{a,b,c\}\theta = \begin{cases} \{a,b,c\}\beta, \text{ if } \{a,b,c\} \in t \\ \{a,b,c\}\alpha(i), \text{ if } \{a,b,c\} \in q(i)\backslash q^*(i) \\ (z,\{i,j,k\}\gamma), \text{ if } \{a,b,c\} = \{(x,i),(y,j),(x \circ y,k)\} \\ \text{where } \{i < j < k\} \in v, \text{ and } (x,y,x \circ y) \in t_z. \end{cases}$$

As with the singular direct product, it is straightforward to verify that θ is a nesting of (S,T).

PBD-**closure:** If (X,A) is a *PBD*, and for every block $a \in A$, we have an *STS* $(a,t(a))$ with a nesting θ_a, then $(X,\bigcup_{a \in A} t(a))$ is an *STS*, and θ is a nesting, where $\theta(t) = \theta_a(t)$ where $t \in t(a)$.

As previously mentioned, the above constructions, together with the machinery in [4], are sufficient to construct a nested *STS* for all orders $n \equiv 1 \pmod 6$, with the possible exceptions of $n = 55$, 115, 145, 187, 205, 265, 355, 415, 493, 649, 655, 697, 943, 955, 979, 1003, 1243, 1285, and 1819. In [1], Colbourn and Colbourn have constructed nested *STS* of orders 55, 115, and 145. The methods of [4] then enable one to eliminate 1819 as a possible exception. Hence, the spectrum for $PA(n,4)$'s which are invariant under conjugation by $<(123)>$ is precisely the set of all $n \equiv 1 \pmod 6$ except possibly $n = 187$, 205, 265, 355, 415, 493, 649, 655, 497, 943, 955, 979, 1003, 1243, and 1285.

5. Conjugate invariant subgroups of S_5.

As mentioned in the introduction, the case $<(12345)>$ has been handled in [3]. For completeness we restate it. The spectrum for $PA(n,5)$'s with conjugate invariant subgroup $<(12345)>$ is precisely the set of all $n \equiv 1$ or $5 \pmod{10}$, except $n = 15$. We remark that this removes $n = 51$ from the list of unknown $PA(n,5)$'s in [5].

We now consider the case of the conjugate invariant subgroup $<1>$. Any $PA(n,5)$, where $n \equiv 3,7$, or $9 \pmod{10}$, must have conjugate invariant subgroup $<1>$. By the results of [5] we have $PA(n,5)$'s for all such $n \geq 7$ (except possibly $n \in \{33,39,87,219\}$).

So, we now suppose that $n \equiv 1$ or $5 \pmod{10}$. We handle each case separately. First we consider $n \equiv 5 \pmod{10}$. A straightforward argument shows that any $PA(5,5)$ has a conjugate invariant subgroup of the form $<(ijklm)>$, so 5 is not in the spectrum for $<1>$. On the other hand, since there is a $PA(15,5)$, and since no $PA(15,5)$ can have a conjugate invariant subgroup of the form $<(ijklm)>$, 15 belongs to the spectrum for $<1>$.

So we assume $n \geq 25$. A careful inspection of the $PA(n,5)$'s with conjugate invariant subgroup $<(12345)>$ ($n \equiv 5 \pmod{10}$), constructed in [3] shows that each has at least one $PA(5,5)$ as a subsystem. Let A be one of these $PA(n,5)$'s, let B be a sub-$PA(5,5)$, and let B^* be the $PA(5,5)$ obtained from B by interchanging the first two columns. B^* has conjugate invariant subgroup $<(21345)>$. If we construct A^* by unplugging B from A and replacing it by B^*, we obtain a $PA(n,5)$ which must have $<1>$ as the conjugate invariant subgroup (since A^* cannot be invariant under conjugation by (12345) and (21345)).

Now we assume $n \equiv 1 \pmod{10}$. The $PA(n,5)$'s with conjugate invariant subgroup $<(12345)>$ ($n \equiv 1 \pmod{10}$) constructed in [3] each have either a sub-$PA(5,5)$ or a sub-$PA(11,5)$, except for n = 31, 151, 331, and 751. The cases with a sub-$PA(5,5)$ can be handled as before. The cases with a sub-$PA(11,5)$ can be handled similarly, by replacing the sub-$PA(11,5)$ can be handled similarly, by replacing the sub-$PA(11,5)$ by one with conjugate invariant subgroup $<1>$ (assuming it exists). But, we have a prime construction which produces the desired $PA(11,5)$, and also handles n = 31, 151, 331, and 751. Suppose $n \equiv 1 \pmod{10}$ is a prime, and define an $\binom{n}{2}$ by 5 array A having rows $(i, i+j, i+2j, i+3j, i+4j)$ $\pmod{n}$, for all $0 \leq i < n$, $i \leq j \leq (n-1)/2$. It is clear that A is a $PA(n,5)$ with conjugate invariant subgroup $<1>$.

Combining all of the above, we have a $PA(n,5)$ with conjugate invariant subgroup $<1>$ for all odd $n \geq 7$, except possibly n = 33, 39, 87, or 219. Also, such a PA cannot exist for n = 3, or 5.

6. Summary.

We summarize the results of this paper in the following table.

	subgroup	spectrum	comments
	$\langle 1 \rangle$	all odd $n \geq 5$	
$k = 3$	$\langle (123) \rangle$	all $n \geq 3$, $n \equiv 1$ or 3 (mod 6)	equivalent to a Steiner triple system of order n
	$\langle 1 \rangle$	all odd $n \geq 5$, except possibly $n = 87$	
$k = 4$	$\langle (123) \rangle$	all $n \geq 3$, $n \equiv 1$ (mod 6) except possibly $n = 187$, 205,265,355,415,493,649, 655,697,943,955,979, 1003,1243, or 1285	equivalent to a Steiner triple system of order n which can be nested
	$\langle 1 \rangle$	all odd $n \geq 7$, except possibly $n = 33$, 39, 87, or 219	
$k = 5$	$\langle (12345) \rangle$	all $n \equiv 1$ or 5 (mod 10), except $n = 15$	equivalent to a Steiner pentagon system of order n, and to a quasigroup of order n satisfying the identities $x^2 = x$, $(yx)x = y$, and $x(yx) = y(xy)$

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