HOLEY SOLSSOMs

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ABSTRACT. In this paper we study sets of three orthogonal partitioned incomplete Latin squares, of type 2^n (n odd), which have the property that two of the squares are mutual transposes and the third is symmetric. Such squares have applications to several problems, which are discussed. We prove that such a set of 3 squares exists for all odd $n > 3$, except possibly for $n = 15, 33, 39, 75$, or 87. The result is proved, in part, by means of a PBD-closure result which is of interest in its own right: If P_5 denotes the set of odd prime powers not less than 5, then there is a pairwise balanced design on v points with block sizes in P_5, for all odd $v > 3$ except possibly for $v = 15, 33, 39, 51, 75, 87, 93, 183, 195$, or 219.

1. *Introduction.*

A useful generalization of the idea of a set of mutually orthogonal Latin squares is to allow certain disjoint subsquares to be missing. Such objects are discussed in [1], [2], and [3], for example.

In this paper we consider the situation where the missing subsquares are spanning. Such arrays are called OPILS (as an acronym for orthogonal partitioned incomplete Latin squares), and were studied in [2]. For convenience, we repeat the definition here.

Let $P = \{S_1,\ldots,S_n\}$ be a partition of a set $S (n \geq 2)$. A *partitioned incomplete Latin square,* (or PILS), having partition P, is an $|S|$ by $|S|$ array L, indexed by S, satisfying the following properties:

UTILITAS MATHEMATICA, Vol. 25 (1984), pp. 159-169

(0) a cell of L either contains an element of S or is empty,

(1) the subarrays indexed by $S_i \times S_i$ are empty, for $1 \le i \le n$ (we will refer to these subarrays as *holes*),

(2) the elements occurring in row (or column) s of L are precisely those in $S \backslash S_i$, where $s \in S_i$.

We will say that the type of L is the multiset $\{|S_1|, \ldots |S_n|\}$. We will use the notation $t_1^{u_1} \ldots t_k^{u_k}$ to describe the type of a PILS, where there are precisely u_i S_j's of cardinality t_i, for $1 \le i \le k$.

Suppose L and M are both PILS having partition P. We say that L and M are *orthogonal* if their superposition yields every ordered pair in $S^2 \backslash \bigcup_{i=1}^{n} S_i^2$. Several PILS, each having partition P, are said to be orthogonal if each pair is. We abbreviate the term orthogonal PILS to OPILS.

In this paper we investigate sets of three OPILS of a special kind. (However, the results we prove are new results for sets of three OPILS, even without the extra conditions we impose.)

A *holey SOLSSOM* having partition P will denote a set of three OPILS (having partition P), say A, B, C, where $B = A^T$ and $C = C^T$. (SOLSSOM is an acronym for self-orthogonal latin square with a symmetric orthogonal mate. Such squares are used for the construction of certain tournaments; see [9].)

A holey SOLSSOM of type 2^n is a particularly useful combinational object (applications are given in [4] and [5]). We construct holey SOLSSOMs of type 2^n for n odd. Such arrays can be constructed for $n > 3$ an odd prime power (Section 2), and such an array does not exist for $n = 3$. In Section 3 we prove a PBD-closure result which reduces the list of possible exceptions to $n \in \{15,33,39,51,75,87,93,183,195,219\}$. In Section 4, we produce holey SOLSSOMs of type 2^n for $n = 51, 93, 183, 195$, and 219. Thus the spectrum is determined, except for $n = 15, 33, 39, 75$, and 87.

2. *Direct constructions.*

The following result is proved in [4].

LEMMA 2.1. *If* $q \equiv 1 \bmod 4$ *is a prime power, then there exists a holey SOLSSOM of type* 2^q.

We now give a construction for the remaining odd prime powers, except 3. It is trivial to observe that a holey SOLSSOM of type 2^3 cannot exist (see [2], for example).

Our construction is accomplished by difference methods. We will take two copies of the Galois field $G = GF(q)(q \equiv 3 \bmod 4)$, say $G \times \{1\}$ and $G \times \{2\}$. Choose $c \neq 0$ so that $c^2 - 1$ is a quadratic non-residue in G (this can be done whenever $q > 3$). Denote by Q the $(q-1)/2$ quadratic residues in G, and let B denote the set of $4(q-1)$ quintuples:

$$\left.\begin{array}{ccccc}
y_0 & -y_1 & ay_1 & -ay_1 & 0_0 \\
-y_1 & y_0 & -ay_1 & ay_1 & 0_0 \\
-y_0 & y_1 & ay_0 & -ay_0 & 0_0 \\
y_1 & -y_0 & -ay_0 & ay_0 & 0_0 \\
ay_1 & -ay_1 & y_0 & -y_1 & 0_1 \\
-ay_1 & ay_1 & -y_1 & y_0 & 0_1 \\
ay_0 & -ay_0 & -y_0 & y_1 & 0_1 \\
-ay_0 & ay_0 & y_1 & -y_0 & 0_1
\end{array}\right\} y \in Q$$

It is not difficult to see that the differences obtained from any two columns contain every value x_{ij} $(x \neq 0)$, once. (The difference $u_i - v_j$ is defined to be $(u-v)_{ij}$.) Hence, if we develop this set B through G, and use any two columns to coördinatize, we get a set of three OPILS of type 2^q (having partition $\{x \times \{1,2\}: x \in G\}$). If, however, we use the first two columns to coördinatize, we get a holey SOLSSOM of type 2^q. This is easily seen as follows. The quintuples of B have the property that $(a,b,c,d,e) \in B$ if and only if $(b,a,d,c,e) \in B$. This property remains true for the set of $4q(q-1)$ quintuples obtained by developing through G. So, if C, D, and E are the three squares obtained from columns 3, 4, and 5 (respectively), we get $E(a,b) = E(b,a)$, so E is symmetric, and $C(a,b) = D(b,a)$, so $C = D^T$. Thus we have

LEMMA 2.2. *If* $q \equiv 3 \bmod 4$ *is a prime power exceeding* 3, *then there is a holey SOLSSOM of type* 2^q.

In Figure 1 below, we present the holey SOLSSOM of type 2^7 obtained by this construction (A is orthogonal to A^T and B, and B is symmetric).

Figure 1

A holey SOLSSOM of type 2^7

	0_0	1_0	2_0	3_0	4_0	5_0	6_0	0_1	1_1	2_1	3_1	4_1	5_1	6_1
0_0		6_1	5_1	4_0	3_1	2_0	1_0		5_0	3_0	2_1	6_0	1_1	4_1
1_0	2_0		0_1	6_1	5_0	4_1	3_0	5_1		6_0	4_0	3_1	0_0	2_1
2_0	4_0	3_0		1_1	0_1	6_0	5_1	3_1	6_1		0_0	5_0	4_1	1_0
3_0	6_1	5_0	4_0		2_1	1_1	0_0	2_0	4_1	0_1		1_0	6_0	5_1
4_0	1_0	0_1	6_0	5_0		3_1	2_1	6_1	3_0	5_1	1_1		2_0	0_0
5_0	3_1	2_0	1_1	0_0	6_0		4_1	1_0	0_1	4_0	6_1	2_1		3_0
6_0	5_1	4_1	3_0	2_1	1_0	0_0		4_0	2_0	1_1	5_0	0_1	3_1	
0_1		3_1	6_1	1_0	5_1	4_0	2_0		2_1	4_1	6_0	1_1	3_0	5_0
1_1	3_0		4_1	0_1	2_0	6_1	5_0	6_0		3_1	5_1	0_0	2_1	4_0
2_1	6_0	4_0		5_1	1_1	3_0	0_1	5_0	0_0		4_1	6_1	1_0	3_1
3_1	1_1	0_0	5_0		6_1	2_1	4_0	4_1	6_0	1_0		5_1	0_1	2_0
4_1	5_0	2_1	1_0	6_0		0_1	3_1	3_0	5_1	0_0	2_0		6_1	1_1
5_1	4_1	6_0	3_1	2_0	0_0		1_1	2_1	4_0	6_1	1_0	3_0		0_1
6_1	2_1	5_1	0_0	4_1	3_0	1_0		1_1	3_1	5_0	0_1	2_0	4_0	

Figure 1 (continued)

	0_0	1_0	2_0	3_0	4_0	5_0	6_0	0_1	1_1	2_1	3_1	4_1	5_1	6_1
0_0		4_0	1_0	5_0	2_0	6_0	3_0		4_1	1_1	5_1	2_1	6_1	3_1
1_0	4_0		5_0	2_0	6_0	3_0	0_0	4_1		5_1	2_1	6_1	3_1	0_1
2_0	1_0	5_0		6_0	3_0	0_0	4_0	1_1	5_1		6_1	3_1	0_1	4_1
3_0	5_0	2_0	6_0		0_0	4_0	1_0	5_1	2_1	6_1		0_1	4_1	1_1
4_0	2_0	6_0	3_0	0_0		1_0	5_0	2_1	6_1	3_1	0_1		1_1	5_1
5_0	6_0	3_0	0_0	4_0	1_0		2_0	6_1	3_1	0_1	4_1	1_1		2_1
6_0	3_0	0_0	4_0	1_0	5_0	2_0		3_1	0_1	4_1	1_1	5_1	2_1	
0_1		4_1	1_1	5_1	2_1	6_1	3_1		4_0	1_0	5_0	2_0	6_0	3_0
1_1	4_1		5_1	2_1	6_1	3_1	0_1	4_0		5_0	2_0	6_0	3_0	0_0
2_1	1_1	5_1		6_1	3_1	0_1	4_1	1_0	5_0		6_0	3_0	0_0	4_0
3_1	5_1	2_1	6_1		0_1	4_1	1_1	5_0	2_0	6_0		0_0	4_0	1_0
4_1	2_1	6_1	3_1	0_1		1_1	5_1	2_0	6_0	3_0	0_0		1_0	5_0
5_1	6_1	3_1	0_1	4_1	1_1		2_1	6_0	3_0	0_0	4_0	1_0		2_0
6_1	3_1	0_1	4_1	1_1	5_1	2_1		3_0	0_0	4_0	1_0	5_0	2_0	

3. *A PBD-closure result.*

In this section we use the notions of pairwise balanced design (PBD), group-divisible design (GDD), transversal design (TD), and PBD-closure. For definitions, we refer to Wilson [10].

We state the following simple result without proof.

LEMMA 3.1. *The set* $S = \{n$: *there exists a holey SOLSSOM of type* $2^n\}$ *is PBD-closed.*

Let us denote by P_5 the set of odd prime powers greater than or equal to 5. We know that $P_5 \subseteq S$ and S is PBD-closed, so $\mathbb{B}(P_5) \subseteq S$. Thus, in this section we study the set $\mathbb{B}(P_5)$, which denotes the PBD-closure of P_5.

We note that Wilson has shown that $\mathbb{B}(P_3) = \{v \geq 3 : v \text{ is odd}\}$, but $\mathbb{B}(P_5)$ has not previously been studied. In this section we show that $\mathbb{B}(P_5) \supseteq \{v \geq 5 : v \text{ is odd}\} \backslash \{15,33,39,51,75,87,93,183,219\}$.

The following result is an easy extension of the result of MacNeish [6].

LEMMA 3.2. *Let* S *be a PBD-closed set. If* $q_1,\ldots,q_k$ *are all prime powers in* S, *then* $\prod_{i=1}^{k} q_i \in S$.

COROLLARY 3.3. *If* v *is odd,* $v \geq 5$ *and* $v \not\equiv 3$ *or* 15 *mod* 18, *then* $v \in \mathbb{B}(P_5)$.

LEMMA 3.4. *If* $v \equiv 1$ *mod* 4, $v \geq 5$, *and* $v \neq 33, 93$, *then* $v \in \mathbb{B}(P_5)$.

Proof. It is shown in [7] that $\mathbb{B}(5,9,13,17) \supseteq \{v \geq 5 : v \equiv 1 \text{ mod } 4\} \backslash \{29,33,49,57,93,129,133\}$. Now, $\{29,49,133\} \subseteq \mathbb{B}(P_5)$ by Corollary 3.3. Wang [9, p. 61] has shown that $129 \in \mathbb{B}(5,29) \subseteq \mathbb{B}(P_5)$. Also, there is a TD(7,8), so $57 \in \mathbb{B}(7,9) \subseteq \mathbb{B}(P_5)$. □

LEMMA 3.5. *Suppose there is a* TD(17,m), *and* $0 \leq s,t \leq m$. *If* S *is a PBD-closed set and* $\{5,17,m,m+4s,m+4t\} \subseteq S$, *then* $17m + 4(s+t) \in S$.

Proof. First, we note the existence of 3 GDDs. There is trivially a GDD with group-type 1^{17} and one block of size 17. We can obtain a GDD with group type $1^{16}5^1$ and blocks of size 5 from the projective plane of order 4. From the affine plane of order 5, we obtain a GDD with group-type $1^{15}5^2$ and blocks of size 5.

We now apply Wilson's fundamental construction for GDDs [10]. Give all points of a TD(17,m) weight one, except for s points in one group and t points in another, which get weight 5. The result is a PBD with blocks of size $5, 17, m, m+4s$, and $m+4t$. □

LEMMA 3.6. *Suppose* $m \equiv 3 \bmod 4$, $m \in \mathbb{B}(P_5)$, *and there exists a* $TD(17,m)$. *If* $v \equiv 3 \bmod 4$ *and* $17m + 8 \le v \le 25m - 8$, *then* $v \in \mathbb{B}(P_5)$.

Proof. From Corollary 3.3 it follows that, given three integers $m + 4a$, $m + 4a + 4$, $m + 4a + 8$ $(a \ge 0)$, at least two of them belong to $\mathbb{B}(P_5)$. Thus any integer a satisfying $2 \le a \le 2m - 2$ can be written as $a = s + t$ where $0 \le s,t \le m$, $\{4m+s, 4m+t\} \subseteq \mathbb{B}(P_5)$. □

LEMMA 3.7. *If* $m \equiv 3 \bmod 4$ *is a positive integer, then at least one of* $m, m+4, m+8, \ldots, m+44$ *is relatively prime to* $60060 = 2\cdot3\cdot5\cdot7\cdot11\cdot13$.

Proof. This is a matter of checking residue classes mod 60060. □

LEMMA 3.8. *If* v *is odd and* $v \ge 331$, *then* $v \in \mathbb{B}(P_5)$.

Proof. First, we note that if $(m,60060) = 1$, then there is a $TD(17,m)$. Let $v \ge 331$ be odd. If $v \equiv 1 \bmod 4$, we apply Lemma 3.4, so assume $v \equiv 3 \bmod 4$. We wish to find $m \equiv 3 \bmod 4$ so that a $TD(17,m)$ exists and $17m + 8 \le v \le 25m - 8$. We can use $m = 19,23,27,31,43,47,59,67,71,79, 83,87$, and 103. This handles $331 \le v \le 2567$. For larger v, we can use the fact that the gap between suitable values of m is no more than 44 (Lemma 3.7), and $17(m+44) + 8 \le 25m - 8$ for $m \ge 103$. □

COROLLARY 3.9. *If* $v \ge 5$ *is odd and* $v \notin \mathbb{B}(P_5)$, *then* $v \in \{15,33,39, 51,75,87,93,111,123,147,159,183,195,219,231,255,267,291,303,327\}$.

Proof. Corollary 3.3, and Lemmata 3.4 and 3.8. □

LEMMA 3.10. $\mathbb{B}(P_5) \supseteq \{111,231,291\}$.

Proof. From TDs of depth 5 we obtain $111 \in \mathbb{B}(5,23)$, $231 \in \mathbb{B}(5,57) \subseteq \mathbb{B}(5,7,9)$, and $291 \in \mathbb{B}(5,59)$. □

We can handle several other values by means of indirect products. These utilize incomplete transversal designs $T(k,n) - T(k,m)$, which are defined in [1]. If K is a set of positive integers, then a (v,K)-PBD will denote a PBD on v points with block-sizes in K.

LEMMA 3.11. ([8, Theorem 2.15]) *Let* u,v,w, *and* a *be positive integers,* $a \le w$, *and let* K *be a set of positive integers. If there exist a* (v,K)-PBD *which contains a block (or subdesign) of size* w, *and a* $TD(u,v-a)-TD(u,w-a)$, *and* $u(w-a) + a \in K$, *then* $u(v-a)+a \in \mathbb{B}(K)$.

In Table 1 we make several applications of Lemma 3.10, where $K = \mathbb{B}(P_5)$.

Table 1

equation $u(v-a) + a$	PBD	w	incomplete TD	$u(w-a) + a$
$123 = 7(21-4) + 4$	PG(2,4)	5	TD(7,17) - TD(7,1)	11
$147 = 5(35-7) + 7$	TD(5,7)	7	TD(5,28) - TD(5,0)	7
$159 = 5(35-4) + 4$	TD(5,7)	5	TD(5,31) - TD(5,1)	9
$255 = 5(55-5) + 5$	TD(5,11)	5	TD(5,50) - TD(5,0)	5
$267 = 5(55-2) + 2$	TD(5,11)	5	TD(5,53) - TD(5,3)	17
$303 = 5(63-3) + 3$	TD(7,9)	7	TD(5,48) - TD(5,4)	23

The incomplete TD(5,53) - TD(5,3) is obtained by using Wilson's construction ([11]) with the equation $53 = 7.7 + 1 + 3$. A TD(5,48) - TD(5,4) is constructed as a direct product $48 = 12.4$.

We can handle one more value, 327, by Lemma 3.5. We write $327 = 17.19 + 4(1+0)$. Since there is a TD(17,19), and $\{19,23,27\} \subseteq \mathbb{B}(P_5)$, we have $327 \in \mathbb{B}(P_5)$.

Summarizing Corollary 3.9, Lemma 3.10, and the above discussion, we obtain

THEOREM 3.12. $\mathbb{B}(P_5) \supseteq \{v \ge 5 : v \textit{ odd}\}\backslash\{15,33,39,51,75,87,93,183,195,219\}$.

4. *A few more holey SOLSSOMs.*

We have two recursive constructions for holey SOLSSOMs : a singular direct product, and a GDD construction. Both these constructions use holey SOLSSOMs of type 1^n. A SOLSSOM of odd order n is equivalent to a holey SOLSSOM of type 1^n. SOLSSOMs can also exist for even orders n, but there exists no holey SOLSSOM of type 1^n for n even. SOLSSOMs are known to exist for all but a few orders; we refer the reader to [4]. (Any orders we use here exist by [4].)

We do not state our constructions in their most general form, but in a form which is sufficient for our needs.

LEMMA 4.1. *(Singular direct product.) Suppose there exists a holey SOLSSOM of type* 1^u, *a holey SOLSSOM of type* 2^v, *and a SOLSSOM of order* $2(v-1)$. *Then there exists a holey SOLSSOM of type* $2^{u(v-1)+1}$.

Proof. The proof is similar to [4, Construction 2.3]. □

COROLLARY 4.2. *There exist holey SOLSSOMs of type* 2^n *for* $n = 51, 93, 183$.

Proof. $51 = 5(11-1) + 1$, $93 = 23(5-1) + 1$, and $183 = 7(27-1) + 1$. □

LEMMA 4.3. *Suppose there is a* GDD (X,G,A) *which satisfies the two properties:*

1) *for every block* $A \in A$, *there is a holey SOLSSOM of type* $1^{|A|}$
2) *for every group* $G \in G$, *there is a holey SOLSSOM of type* 2^n, *where* $n = (|G| + 2)/2$.

Then there is a holey SOLSSOM of type 2^v, *where* $v = (|X| + 2)/2$. □

Proof. This is essentially the usual PBD construction for Latin squares. □

We can eliminate the values 195 and 219 by this construction.

COROLLARY 4.4. *There exist holey SOLSSOMs of types* 2^{195} *and* 2^{219}.

Proof. It is proved in [7, Lemma 6.12] that if there exists a TD(10,m) and $0 \le s,t \le m$, then there is a GDD with group-type $(4s)^1(4t)^1(8m)^1(4m)^7$ with blocks of size 5 and 9. The value 195 is killed with $m = 9$, $s = t = 8$; the value 219 by $m = 11$, $s = t = 5$. □

Summarizing, we obtain our main result.

THEOREM 4.5. *There exist a holey SOLSSOM of type* 2^n *for all odd* $n > 3$ *with the possible exceptions* $n = 15,33,39,75$, *or* 87.

Proof. Lemma 3.1, Theorem 3.12, and Corollaries 4.2 and 4.4. □

In closing we conjecture that holey SOLSSOMs of type 2^n exist for all even $n \ge 6$. At present, however, there are no holey SOLSSOMs of type 2^n known for even n.

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Received October 3, 1983.