SNAPPY CONSTRUCTIONS FOR TRIPLE SYSTEMS

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1. A *triple system* $T(\lambda,v)$ is a way of selecting unordered triples from a v-set so that every pair of elements appear together in λ triples. They are the first non-trivial case of balanced incomplete block designs. The best known systems are the ones with $\lambda = 1$, which are called *Steiner triple systems* because they are the Steiner systems with block-size three, or $S_1(2,3,v)$ systems. (In classical notation a $T(\lambda,v)$ is an $S_\lambda(2,3,v)$ system.)

Elementary counting arguments show that if a $T(\lambda,v)$ exists then there are integers r and b such that

$$\lambda(v-1) = 2r, \quad vr = 3b$$

(each element belongs to precisely r triples, and b is the total number of triples in the system). These conditions can be expressed in terms of the primacy of λ to 6:

$$(\lambda,6) = 1 \text{ implies } v \equiv 1 \text{ or } 3 \pmod 6;$$
$$(\lambda,6) = 2 \text{ implies } v \equiv 0 \text{ or } 1 \pmod 3;$$
$$(\lambda,6) = 3 \text{ implies } v \equiv 1 \pmod 2;$$

while $(\lambda,6) = 6$ imposes no restriction. Another obvious necessary condition is that $v \neq 2$. These conditions are together sufficient. Clearly if $(\lambda,6) = d$, so that $\lambda = sd$ for some integer s, one could form a $T(\lambda,v)$ by taking s copies of a $T(d,v)$. So to prove sufficiency it is enough to show that the following systems exist.

$$T(1,v) \text{ for all } v \equiv 1,3 \pmod 6;$$
$$T(2,v) \text{ for all } v \equiv 0,4 \pmod 6;$$
$$T(3,v) \text{ for all } v \equiv 5 \pmod 6;$$
$$T(6,v) \text{ for all } v \equiv 2 \pmod 6,\ v \neq 2.$$

2. The sufficiency of the condition $v \equiv 1$ or $3 \pmod 6$ for Steiner triple systems was proven independently by several hands in the nineteenth century [3,5]. Bose [1] settled the case $\lambda = 2$, and the others were done by Hanani [2]. However, as Street [8] points out,

> *Most proofs of sufficiency are awkward; Stanton and Goulden's recent recursive proof ... is an elegant exception ... .*

But even the existence proof for Steiner triple systems given by Stanton and Goulden [2] is lengthy and recursive. For example, to construct a $T(1,31)$ you first need a $T(1,15)$, which requires a $T(1,7)$.

But Lindner, in [4] and in his lectures, has observed that easy direct constructions of $T(1,v)$ have been available since 1958, by combining the result of [1] and [6]. They are simpler for practical use and quite suitable for teaching. Inspired by his work we have sought equally easy direct constructions for all triple systems.

The results appear below, starting with the construction (in Section 3) of $T(1,v)$'s.

A word about the presentation is in order. We use Latin squares, as most readers will be familiar with these beasts; for teaching purposes, if Latin squares are not mentioned before triple systems, then the approach presented is a convenient way to introduce those interesting and important arrays. But the whole construction can be done without overtly using Latin squares; instead of mentioning the array L, one can give a suitable formula for ℓ_{xy}. We omit verifications after the first, as they all follow the same pattern.

3. A Latin square of side (or order) s is an $s \times s$ array with all entries from the set $S = \{1,2,\ldots,s\}$, such that each row and each column is a permutation of S. A set of positions in a Latin square is called a *transversal* if it contains one representative of each row and one representative of each column, and if the positions contain between them each member of the set precisely once. A *transversal* square is a Latin square with a transversal. Given a transversal square of side s, one could first permute the columns so that the transversal positions formed the main diagonal, then permute the names of the elements so that the diagonal becomes $(1,2,\ldots,s)$ in order. We say a transversal square in this form is *standardised*.

Let L be a *symmetric* transversal square of order $2n+1$. Such squares are easy to construct: one example is $L = (\ell_{ij})$, where

$$\ell_{ij} \equiv (n+1)(i+j) \pmod{2n+1}.$$

We define a $T(1,6n+3)$ based on three sets of symbols $\{x^1\}$, $\{x^2\}$ and $\{x^3\}$, where $1 \le x \le 2n+1$. The triples are

$$\{x^1,x^2,x^3\} \quad : \quad 1 \le x \le 2n+1$$

$$\left.\begin{array}{l}\{x^1,y^1,\ell^2_{xy}\}\\ \{x^2,y^2,\ell^3_{xy}\}\\ \{x^3,y^3,\ell^1_{xy}\}\end{array}\right\} \quad : \quad 1 \le x < y \le 2n+1$$

To verify that this is a Steiner triple system, observe that two members of the same set occur together exactly once in one of the blocks of form $\{x^i,y^i,\ell^{i+1}_{xy}\}$; two members of different sets which have the same x-value occur together exactly once: for example, if $x \ne z$, consider x^1 and z^2; there is exactly one column y of L such that $\ell_{xy} = z$, this y satisfies $y \ne x$, and x^1 and z^2 meet together in the triple $\{x^1,y^1,z^2\}$ and nowhere else.

Although it is easy to see that no pair of elements occur together more than once, there is really no need to verify this. For we have constructed $(2n+1) + 3n(2n+1) = 6n^2 + 5n + 1$ triples. Between them they contain $18n^2 + 15n + 3$ unordered pairs. We have verified that each unordered pair of $6n + 3$ elements has occurred at least one: that makes $(6n+3)(6n+2)/2 = 18n^2 + 15n + 3$ pairs. So there is no room for a pair to occur twice. This argument will apply to all our constructions.

Next, let M be a Latin square of side $2n$ which is symmetric and has diagonal $(1,2,\ldots,n,1,2,\ldots,n)$. Such a square always exists: for example $M = (m_{ij})$ where

$$\left.\begin{array}{ll} m_{ij} \equiv \tfrac{1}{2}(i+j) & \text{when } i+j \text{ is even} \\ m_{ij} \equiv \tfrac{1}{2}(i+j+1) & \text{when } i+j \text{ is odd} \end{array}\right\} \pmod{2n}$$

We define a $T(1,6n+1)$ based on $\{x^1\}$, $\{x^2\}$ and $\{x^3\}$ for $1 \le x \le 2n$ and an object ∞, by the triples

$$\{x^1,x^2,x^3\}, \quad 1 \le x \le n$$

$$\left.\begin{array}{l} \{\infty,x^1,(x-n)^2\} \\ \{\infty,x^2,(x-n)^3\} \\ \{\infty,x^3,(x-n)^1\} \end{array}\right\} : \; n+1 \le x \le 2n,$$

$$\left.\begin{array}{l} \{x^1,y^1,m^2_{xt}\} \\ \{x^2,y^2,m^3_{xy}\} \\ \{x^3,y^3,m^1_{xy}\} \end{array}\right\} : \; 1 \le x < y \le 2n.$$

4. For the case $\lambda = 2$ we need to know that there is a transversal square of every order n (except order 2, which is impossible). But this is easy. One simple construction is as follows. If n is odd, define $L_n(\ell_{ij})$ by $\ell_{ij} = 2i-j \pmod n$. If n is even, $n > 2$, define T_n to be the Latin square of order n derived from L_{n-1} by replacing the $(1,2),(2,3),\ldots,(n-2,n-1)$ and $(n-1,1)$ elements by n and then appending a last row and column which make the square Latin: the last column is $(n-1,1,2,\ldots,n-2,n)$ and the last row $(n-2,n-1,1,\ldots,n-3,n)$. T_n is also a transversal square. So transversal squares exist for all orders except 2. (We could also have used the transversal squares of Section 3, instead of L_n, transporting the entries just after the main diagonal in the same way. But the above examples are even simpler to write down.)

There exist a $T(2,6)$ and a $T(2,7)$. Suitable sets of triples are:

$T(2,6)$: 123,124,135,146,156,235,245,256,345,346

$T(2,7)$: 123,124,135,136,157,157,235,247,256,257,345,347,367,456.

(Or one could duplicate the $T(1,7)$ of Section 3.)

Now suppose $n \ne 2$. Let $A = (a_{ij})$ be a transversal square of order n. A $T(2,3n)$ on the symbols $\{x^1\}$, $\{x^2\}$, $\{x^3\}$, $1 \le x \le n$, is formed by the triples.

$$\{x^1,x^2,x^3\} \quad : \; 1 \le x \le n, \text{ twice each}$$

$$\left.\begin{array}{l} \{x^1,y^1,a^2_{xy}\} \\ \{x^2,y^2,a^3_{xy}\} \\ \{x^3,y^3,a^1_{xy}\} \end{array}\right\} : \; 1 \le x \le n,\ 1 \le y \le n,\ x \ne y$$

A $T(2,3n+1)$ on the same symbols together with symbol ∞ is formed by the triples

$$\left.\begin{array}{l}\{x^1,x^2,x^3\}\\ \{x^1,x^2,\infty\}\\ \{x^2,x^3,\infty\}\\ \{x^3,x^1,\infty\}\end{array}\right\} : 1 \le x \le n,$$

$$\left.\begin{array}{l}\{x^1,y^1,a^2_{xy}\}\\ \{x^2,y^2,a^3_{xy}\}\\ \{x^3,y^3,a^1_{xy}\}\end{array}\right\} : 1 \le x \le n,\ 1 \le y \le n,\ x \ne y .$$

5. The case of $T(3,v)$ for $v \equiv 5 \pmod 6$ is the easiest of the constructions. One simply takes all the triples

$$\{y, x+y, 2x+y\} : 0 \le x < \tfrac{1}{2}v,\ 0 \le y < v$$

where all the additions are reduced modulo v.

6. Suppose $A = (a_{ij})$ is a transversal square of order n. The following triples form a $T(6,3n+2)$ based on $\{x^1\}$, $\{x^2\}$, $\{x^3\}$, $1 \le x \le n$, $\{\infty^1,\infty^2\}$:

$$\left.\begin{array}{l}\{x^1,y^1,a^2_{xy}\}\\ \{x^2,y^2,a^3_{xy}\}\\ \{x^3,y^3,a^1_{xy}\}\end{array}\right\} : 1 \le x \le n,\ 1 \le y \le n,\ x \ne y, \text{ each triple taken three times}$$

$$\left.\begin{array}{l}\{\infty^1,x^1,x^2\},\{\infty^2,x^1,x^2\}\\ \{\infty^1,x^2,x^3\},\{\infty^2,x^2,x^3\}\\ \{\infty^1,x^3,x^1\},\{\infty^2,x^3,x^1\}\end{array}\right\} : 2 \le x \le n, \text{ each triple taken three times}$$

$$\left.\begin{array}{l}\{\infty^1,\infty^2,1^1\},\{\infty^1,\infty^2,1^2\},\{\infty^1,\infty^2,1^3\},\\ \{\infty^1,1^1,1^2\},\{\infty^1,1^2,1^3\},\{\infty^1,1^3,1^1\},\\ \{\infty^2,1^1,1^2\},\{\infty^2,1^2,1^3\},\{\infty^2,1^3,1^1\},\\ \{1^1,1^2,1^3\}.\end{array}\right\} : \text{ twice each.}$$

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