CONCERNING THE SPECTRUM OF SKEW ROOM SQUARES

R.C. Mullin[1], D.R. Stinson[2] and W.D. Wallis

Abstract

It is shown that for odd $v \geq 46019$, there exists a skew Room square of side v. Further it is shown that if v is odd, $v \not\equiv 1 \bmod 6$, and $v \geq 17301$, then there exists a skew Room square of side v.

1. *Introduction.*

It is assumed that the reader is familiar with the concepts of pairwise balanced designs (PBD), Room squares, orthogonal arrays, and sets of mutually orthogonal latin squares (cf [5] and [10]). In [7], using Wilson's theorems on PBD closure, it was shown that skew (Room) squares of side v exist for all but a finite number of odd positive v, but no bounds for the number of exceptions could be given conveniently. More recently it was shown in [5] that skew squares exist for all positive integers v of the form $2^{\alpha}t+1, (t,2) = 1$, where $\alpha \notin \{1,2,6,7\}$. These results can be extended to show that for odd positive $v \geq 46019$ there exists a skew square of side v. For convenience we let $SS = \{v : \exists$ a skew Room square of side $v\}$. (The authors conjecture that $SS = \{v : (2,v) = 1, v \geq 7\}$). It is well known [8] that neither 3 nor 5 belongs to SS. We also note that SS is PBD closed [4].

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[1] Supported in part by NRC Grant #A3701.

[2] Present Address: Department of Mathematics, Ohio State University, Columbus, Ohio.

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2. *Squares of side* $v \equiv 1 \bmod 8$.

As noted above, it was shown in [5] that if v is an odd positive integer such that $v \notin SS$, then v is of the form $2^{\alpha}t+1, (t,2) = 1$, where $\alpha \in \{1,2,6,7\}$. Moreover, it was also shown in [5] that if $65 \in SS$, then the condition $\alpha \in \{1,2,6,7\}$ above could be replaced by $\alpha \in \{1,2,7\}$. Since that time Dinitz [3], a student of R.M. Wilson at Ohio State University, has shown the existence of a skew Room square of side 65 by producing the skew strong starter given below. (For definition of starter and adder see [5]. In a strong starter, the adder element for the pair (x_i,y_i) is given by $-(x_i+y_i)$.)

(36,37) (7,9) (43,46) (10,15) (38,44) (1,8) (25,33) (61,5) (35,45) (48,59) (11,23) (39,52) (2,16) (26,41) (62,13) (17,34) (53,6) (12,31)(40,60) (3,24) (27,49) (63,21) (18,42) (54,14) (30,56) (58,20) (4,32) (28,57) (64,29) (19,50) (55,22).

In this paper we show that if $v \geq 9$ and $v \equiv 1 \bmod 8$, then $v \in SS$ except possible for $v = 129$. The results in the current section rely heavily on some of the results of [5], which are cited below. It is well known that the existence of s mutually orthogonal latin squares of side n is equivalent to the existence of an orthogonal array $OA(n,s+2)$. Let $oa(k) = \{n : \exists\ OA(n,k)\}$. For definition of group divisible design, see Wilson [10]. It is well known that the existence of a group divisible design $GDD(k,n)$ is equivalent to the existence of an orthogonal array $oa(n,k)$.

LEMMA 2.1. *For* $n \geq 9403$, $n \in oa(10)$. *(A complete list of values* n *for which the authors could not construct an* $OA(n,10)$ *is given in* [5]*).* □

By a *9-head* of order v we mean a PBD, say D, of order v with block size from SS containing an ideal element ∞ which occurs only in blocks of size 9 in D. Clearly the order v of a 9-head satisfies the congruence $v \equiv 1 \pmod 8$. By the *generalized replication number* of a 9-head of order v we mean the integer $(v-1)/8$.

LEMMA 2.2. *The set* $GR = \{(v-1)/8 : v$ *is the order of a 9-head*$\}$ *is PBD-closed.* □

LEMMA 2.3. $\{7,9,10,17,19,137,144,337\} \subset GR$.

The following easily established results for PBD-closed sets are also used. □

LEMMA 2.4. *Let* K *be a PBD-closed set. Suppose that* $m, m+1, m+2, \ldots, m+\ell \in K$ *and* $a_1, a_2, \ldots, a_\ell \in K$. *If* $t \in oa(m+\ell) \cap K$ *and* $0 \leq a_i \leq t$, *for* $i = 1,2,\ldots,\ell$, *then* $mt + \sum_{i=1}^{\ell} a_i \in K$. □

COROLLARY 2.5. *Let* $t \in oa(10) \cap GR$ *and let* $a \in GR$, $0 \leq a \leq t$. *Then* $9t + a \in GR$. □

LEMMA 2.6. *Let* K *be a PBD-closed set. Suppose that* $m, m+1, \ldots, m+\ell \in K$ *and* $a_1, a_2, \ldots, a_\ell \in K$. *If* $t \in K$ *and* $t \in oa(m+\ell)$, *and* $0 \leq a_i \leq t$ *for* $i = 1,2,\ldots,\ell$ *then* $m(t-1) + \sum_{i=1}^{\ell} (a_i - 1) \in K$.

Proof: Adjoin another element ∞ to the groups of a $GD(m+\ell, t-1)$ continue as in Lemma 2.4. □

COROLLARY 2.7. *Suppose that* $t-1 \in oa(10)$ *and that* $t \in GR$. *If* $a \in GR$, $0 \leq a \leq t$, *then* $9(t-1) + a \in GR$. □

LEMMA 2.8. *Let* K *be a PBD-closed set. If* $s,t \in K$, *and* $s \in oa(t)$ *then* $st \in K$. □

LEMMA 2.9. *Let* K *be a PBD-closed set. If* $s,t \in K$, *and* $s-1 \in oa(t)$ *then* $(s-1)t + 1 \in K$. □

LEMMA 2.10. *Let* K *be a PBD-closed set. Let* $m-1, m, t, t-1$ *and* $a \in K$, $0 \leq a \leq t$. *If* $t \in oa(m)$, *then* $m(t-1) + a \in K$.

Proof: Delete t-a varieties of a block from a GD(m,t). □

COROLLARY 2.11. *Let* $t, t-1$ *and* $a \in GR$, $0 \leq a \leq t$. *If* $t \in oa(10)$, *then* $10(t-1) + a \in GR$. □

LEMMA 2.12. *Let* X *be a PBD with block sizes from* $K = L \cup M$ *where* $L \cap M = \emptyset$. *Suppose* $M \subseteq SS$ *and for every* $\ell \in L$ *there is a 9-head of order* ℓ. *Furthermore, suppose there is an element, say* ∞, *which is only contained in blocks of sizes from* L. *Then there is a 9-head of order* $|X|$ *and hence* $(|X|-1)/8 \in GR$.

Proof: Let B be a block of X such that $|B| \in L$. If $\infty \in B$, then replace the block B by a 9-head whose ideal element is ∞. If $\infty \notin B$, replace B by any PBD with block sizes from SS. The result is a 9-head of order $|X|$. □

COROLLARY 2.13. *Let* $t \in GR$ *and* $s \in SS$. *If* $8t \in oa(s)$ *then* $st \in GR$.

Proof: X is the PBD obtained by adjoining an ideal element ∞ to each group of a GD(s,8t). Then $L = \{8t+1\}$ and $M = \{s\}$ and Lemma 2.12 implies the result. □

LEMMA 2.14. *{234, 252, 421, 463, 883, 1153, 1723, 1873, 2017, 3067, 3319}* $\subseteq$ *GR.*

Proof: The elements of the above set are either r-values of BIBD's with $k = 9$, $\lambda = 1$, or v-values of BIBD's with $\lambda = 1$, $k = 7$ or 9, constructed by Wilson in [9]. □

Let $\overline{GR}$ be the complement of GR with respect to the positive integers. For $a,b \in GR$, $a < b$, define $\delta(a,b)$ to be the length of the longest sequence of consecutive integers in $\overline{GR} \cap \{a,a+1,\ldots,b\}$.

LEMMA 2.15. *Suppose that* $A \in GR \cap oa(10)$, $a,b \in GR$ *and* $a \leq b \leq A$. *Then* $\delta(9A+a,9A+b) \leq \delta(a,b)$.

Proof: If $a \leq r \leq b$ and $r \in GR$, then $9A+r \in GR$ by Corollary 2.5. □

Let $\overline{GR \cap oa(10)}$ be the complement of $GR \cap oa(10)$ with respect of the positive integers. For $A,B \in GR \cap oa(10)$, $A < B$, define $\Delta(A,B)$ to be the length of the longest sequence of consecutive integers in $\overline{GR \cap oa(10)} \cap \{A,A+1,\ldots,B\}$.

THEOREM 2.16. *Suppose* $a,b \in GR$, $a \leq b$ *and* $\delta(a,b) \leq m$. *Suppose that* M *is a positive integer,* $A,B \in GR \cap oa(10)$, $A \leq B$ *and* $\Delta(A,B) \leq M - 1$. *If* $b \leq A$ *and* $9M + a - b \leq m+1$ *then* $\delta(9A+a,9B+b) \leq M$.

Proof: Let $\{A_1,A_2,\ldots,A_n\} \subset GR \cap oa(10)$ where $A = A_1 < A_2 < \ldots < A_n = B$, and $A_{i+1} - A_i \leq M$ for $i = 1,2,\ldots,n-1$. By

Lemma 2.15, $\delta(9A_i+a, 9A_i+b) \le m$ for $i = 1,2,\ldots,n$. Also, $9A_{i+1}+a - (9A_i+b) \le 9M + a - b \le m+1$, for $i = 1,2,\ldots,n-1$. □

LEMMA 2.17. *If one can find* a, A, b *and* B *which satisfy the hypotheses of Theorem 2.16, and if further* $B \ge 9A + a > 9403$ *and* $m \le M$, *then* $\delta(9A+a,x) \le m$ *for any* $x \ge 9A + a$.

Proof: By Theorem 2.16, $\delta(9A+a,9B+b) \le m$. Since $y \in oa(10)$ for all $y \ge 9403$, we have $\Delta(9A+a,9B+b) \le m \le M$. Since $\Delta(A,B) \le M$ and $B \ge 9A+a$, therefore $\Delta(A,9B+b) \le M$. Applying Theorem 2.16 again yields $\delta(9A+a,9(9B+b)+b) \le m$. This process may be repeated indefinitely to obtain $\delta(9A+a,x) \le m$ for any $x > 9A+a$. □

By means of a computer, we have been able to show, using the results of this section, that $\delta(12198, 12244) = 0$, $\delta(38743, 125000) = 0$ and $\Delta(12250, 125000) \le 4$.

THEOREM 2.18. *If* $r \ge 38743$, *then* $r \in GR$.

Proof: We have $\Delta(12250, 125000) \le 4$. Applying Theorem 2.17 with $a = 12198$, $b = 12244$, $A = 12250$, $B = 125000$, yields $\delta(122448,x) = 0$, if $x > 122448$. Since $\delta(38743, 125000) = 0$, therefore $\delta(38743,x) = 0$ if $x > 38743$.

COROLLARY 2.19. *If* $v \equiv 1 \bmod 8$ *and* $v \ge 309945$, *then* $v \in SS$. □

3. *General Constructions for Skew Squares.*

We require other constructions for Room squares to investigate cases of sides $v \equiv 1 \bmod 8$ not handled by the previous theory, as well as other residue classes mod 8.

A survey of results on skew Room squares is given in [5]. The results from that paper required here are updated and listed as lemmata below.

Beaman and Wallis [1] have constructed a skew Room square of side nine. Thus we have

LEMMA 3.1. *For* v *an odd prime power,* $v \neq 3, 5$, $v \in SS$.

LEMMA 3.2. *Suppose there is a skew Room square of side* v_2 *which contains a skew Room subsquare of side* v_3.

i) If $v_2 - v_3 \neq 6$ *and if there is a skew Room square of side* v_1 *then there is a skew Room square of side* $v_1(v_2-v_3) + v_3$ *which contains skew Room subsquares of sides* v_1, v_2 *and* v_3. *This result also holds for* $v_3 = 0$.

ii) (Beaman and Wallis [2]). If $v_3 \neq 0$ *and* $v_2-v_3 \neq 12$, *then there is a skew Room square of side* $5(v_2-v_3) + v_3$ *which contains skew Room squares of sides* v_2 *and* v_3.

The following results are also given in [5].

LEMMA 3.3. *If* v *is odd and* $7 \leq v \leq 53$, *then* $v \in SS$.

LEMMA 3.4. *If* $v \geq 7$ *and* $v \notin SS$, *then* $v = 3n$, $5n$ *or* $75n$ *where* $(n, 3\cdot 5\cdot 7) = 1$.

It is also shown in [6] that $55 \in SS$.

The following was noted in Section 2.

LEMMA 3.5. $v \notin SS$, *then* $v = 2^{\alpha}t+1$, *where* $(2,t) = 1$ *and* $\alpha = 1, 2$ *or* 7.

We establish other lemmata concerning the existence of skew squares below.

LEMMA 3.6. *If* v *is an odd integer and if* $d|v$ *where* $d \in \{7,11,13,17,57\}$ *then* $v \in SS$.

The proof is similar to that of Corollary 5.6 in [5].

LEMMA 3.7. *If* $v \equiv 1 \bmod 10$, *then* $v \in SS$.

Proof: Write v in the form $2^{\alpha}\cdot 5m + 1$ where m is odd. As noted earlier, we need only consider $\alpha = 1,2$ or 7. However $2^{\alpha}\cdot 5 + 1 \subset SS$ for $\alpha = 1,2$ and 7, hence, in virtue of lemmata 3.2(i) and 3.4 we need only consider v of the form $2^{\alpha}\cdot 5m + 1$ where $m = 3n$, $5n$ or $7n$, and $(n,15) = 1$. Hence it is sufficient to show that $2^{\alpha}\cdot 25 + 1$, $2^{\alpha}\cdot 15 + 1$ and $2^{\alpha}\cdot 375 + 1 \in SS$ for $\alpha \in \{1,2,7\}$. However $(2^{\alpha}\cdot 15+1,15) = 1$ and $(2^{\alpha}\cdot 375+1,15) = 1$ for all positive integers α, and $(2^{2}\cdot 25+1,15) = 1$ also. Moreover $2\cdot 25 + 1 = 51 \in SS$ by lemma 3.3 and $2^{7}\cdot 25 + 1 = 33.97 \in SS$ by lemmata 3.2, 3.3 and 3.4.

The same type of argument can be used to establish the following.

LEMMA 3.8. *If* $v \equiv 1 \bmod 12$, *then* $v \in SS$. □

LEMMA 3.9. *If* $v \equiv 1 \bmod 18$, *then* $v \in SS$. □

LEMMA 3.10. *Suppose* $v \equiv 5 \bmod 8$. *If* $\beta|(v-1)$ *where* $\beta \in \{7,11,13,19,31,37\}$, *then* $v \in SS$. □

LEMMA 3.11. *Suppose* $v \equiv 3 \bmod 4$. *If* $\beta|(v-1)$ *where* $\beta \in \{7,11,13,17,23,29,31\}$ *then* $v \in SS$. *Further if* $(v-1,3) = 1$ *and* $19|(v-1)$, *then* $v \in SS$. □

4. *Squares of side* $\equiv 1 \bmod 8$ *revisited.*

From the corollary 2.19 and the results of section 1, we see that if $v \equiv 1 \bmod 8$, and $v \notin SS$, then $v = 256t + 129$ where $0 \le t \le 1210$, or equivalently $v = 128s + 1$, where s is odd and $1 \le s \le 2421$.

LEMMA 4.1. *If* $v = 128p + 1 \in SS$ *for all prime* p *such that* $7 \le p \le 2417$, *then* $v = 128s + 1 \in SS$ *for all odd* s *satisfying* $3 \le s \le 2421$.

Proof: With lemmata 3.7 and 3.8 in mind, the proof is immediate. □

LEMMA 4.2. *If* $v = 128s + 1$ *where* s *is an odd positive integer and* $v \notin SS$, *then either* $s \equiv 1 \bmod 3$ *or* $s \equiv 3 \bmod 5$.

Proof: Immediate. □

THEOREM 4.3. *If* $v \equiv 1 \bmod 8$ *is a positive integer, then either* $v \in SS$ *or* $v = 129$.

Proof: In view of the above we need only consider $v = 128p + 1$ where p is a prime, such that $7 \le p \le 2417$ and $p \equiv 1 \bmod 3$ or $p \equiv 3 \bmod 5$. The cases $p \equiv 1 \bmod 3$ are investigated in table 1 and those cases of $p \equiv 3 \bmod 5$ not treated in table 1 are treated in table 2. (N.B., cases eliminated by 9 heads previously constructed by machine are not listed). □

Table 1.

p	v		construction
19	2433		see section 5, lemma 5.3
37	4737	=	5(953-7) + 7, 953 = 7(137-1) + 1
79	10113	=	25(417-13) + 13, 417 = 13(33-1) + 1
97	12417	=	117(121-15) + 15, 121 = 15(9-1) + 1
181	23169	=	13(1665-1) + 1, 1665 = 45·37
271	34690	=	31·1119, 1119 = 43(27-1) + 1
277	35457	=	709(57-7) + 7, 57 = 7(9-1) + 1
547	70017	=	5(14009-7)+7,14009=7(2009-9)+9,2009=251(9-1)+1.

Table 2.

p	v		construction
23	2945	=	19.155, 155 = 11(15-1) + 1
53	6785	=	23·295, 295 = 21(15-1) + 1
173	22145	=	5(4441-15) + 15, 4441 = 15(297-1) + 1
263	33665	=	71(485-11) + 11, 485 = 11(45-1) + 1
293	37505	=	13·2885, 2885 = 103(29-1) + 1.

5. *The Spectral Problem.*

In this section a bound for the number of odd $v \geq 7$ which are not the sides of skew squares is obtained. An important construction based on Wilson's techniques is given below. If a GDD has e_i groups of size m_i, $1 \leq i \leq n$, then we say it is of group type $e_1\{m_1\} + e_2\{m_2\} + \ldots + e_m\{m_n\}$.

LEMMA 5.1. *There exist group divisible designs with block sizes* $k \in \{7,9\}$ *and group types (i)* $\{8\} + 9\{6\}$ *and (ii)* $\{8\} + 8\{6\}$.

Proof: This follows from remark 3.2 in [10] and the fact that $\{8,9\} \subset oa(7)$.

THEOREM 5.2. *Suppose that* m *and* t *are integers,* $0 \leq t \leq m$, *such that* $m \in oa(10)$ *and* $\{6t+1, 6m+1, 8m+1\} \subset SS$. *Then* $56m + 6t + 1 \in SS$.

Proof: Let D be an oa(m,10) with groups $G_1, G_2, G_3, \ldots, G_{10}$. Applying the Fundamental Construction 3.1 of [10] in conjunction with the above GDD's produces a GDD of type $\{6t\} + \{8m\} + 8\{6m\}$ and block sizes from $\{7,9\}$. Adjoining a new point to these groups produces a PBD with block sizes from $\{6t+1, 8m+1, 6m+1, 7, 9\}$. Since $\{7,9\} \subset SS$ and $\{6t+1, 8m+1, 6m+1\} \subset SS$ by the hypothesis, the result follows since

SS is PBD closed. □

As shown in section 4, if $v \equiv 1 \bmod 8$ and $v \geq 7$, then $v \in SS$ with the possible exceptions of $v = 129, 2433$. This can be improved as below.

LEMMA 5.3. *There exists a skew square of side 2433.*

Proof: Note that $2433 = 56\cdot 43 + 6\cdot 4 + 1$. Further $25, 259, 345 \in SS$, since $25 = 5^2$, $259 = 7\cdot 37$, $345 = 15\cdot 23$. Finally $43 \in oa(10)$. □

In view of the above, we have the following version of Theorem 5.2.

THEOREM 5.4. *If* m *and* t *are integers,* $0 \leq t \leq m$, $m \neq 16$, *and if* $m \in oa(10)$ *and* $\{6m+1, 6t+1\} \subset SS$, *then* $56m + 6t + 1 \in SS$.

Proof: By theorem 4.3 if $m \geq 0$ and $m \neq 16$, then $8m + 1 \in SS$. □

LEMMA 5.5. *If* $t \not\equiv 4 \bmod 5$ *then* $6t + 1 \in SS$.

Proof: Certainly $(6t+1,3) = 1$ and if $t \not\equiv 4 \bmod 5$ then $(6t+1,5) = 1$. □

The following is a well known result of McNeish.

LEMMA 5.6. *If* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ *is the factorization of* n *into prime powers then there exist at least* $\min\{p_i^{\alpha_i}-1 : 1 \leq i \leq k\}$ *pairwise orthogonal latin squares of side* n.

As a result of lemmas 5.5, 5.6 and Theorem 5.4, we obtain the following theorem.

THEOREM 5.7. *Suppose* m *and* t *are non-negative integers such that* $m, t \not\equiv 4 \bmod 5$, $(m,70) = 1$ *and either* $(m,3) = 1$ *or* $9 | m$. *If* $0 \leq t \leq m$, *then* $56m + 6t + 1 \in SS$.

Proof: Note that if $m > 0$ then $m \in oa(10)$ by Lemma 5.6. By Lemma 5.5, $6t + 1$ and $6m + 1 \in SS$. Since m is odd, therefore $m \neq 16$ and Theorem 5.4 may be applied. □

LEMMA 5.8. *Let v be an odd positive integer. If (i) $v \equiv 1 \bmod 6$ and if $v > 46017$ or if (ii) $v \equiv 3 \bmod 6$ and $v \geq 17301$ or if (iii) $v \equiv 5 \bmod 6$ and $v \geq 17303$, then v can be written in the form $v = 56m + 6t + 1$ where $0 \leq t \leq m$, $(m,70) = 1$, $m \not\equiv 4 \bmod 5$, $t \not\equiv 4 \bmod 5$ and either $(m,3) = 1$ or $9|m$.*

Proof: Let $v = 2n + 1$. Then n can be written uniquely in the form $n = 28m_0 + 3t_0$ where m_0 and t_0 are integers such that $0 \leq t_0 \leq 27$. For any integer θ, if $m = m_0 - 3\theta$ and $t = t_0 + 28\theta$, then $n = 28m + 3$. Let $M = \{m':0 \leq m \leq 629,\ m' \not\equiv 4 \bmod 5,\ (m',70) = 1$ and either $(m',3) = 1$ or $9|m'\}$. (The elements of M are listed in table 3.) Now if $m > 0$ and $m \equiv m' \bmod 630$ for some $m' \in M$, then m satisfies the conditions $(70,m) = 1$, $m \not\equiv 4 \bmod 5$ and either $(m,3) = 1$ or $9|m$. Suppose that $m \equiv 0 \bmod 3$ and $m \equiv m' \pmod{630}$, Then $m' \in \{27, 81, 117, 153, 171, 207, 243, 261, 297, 333, 351, 387, 423, 477, 513, 531, 603, 621\} = M_3$. Let M_3^* be the images of M_3 mod 630 in any complete set of consecutive residues. Note that the difference between pairs of elements of M_3^* separated by exactly one member of M_3^* does not exceed 90 that further none of the differences between consecutive numbers of M_3^* is a multiple of 5. Thus any integer $m_0 \equiv 0 \bmod 3$ has the property that there exists a pair of integers $\theta_1(m_0)$ and $\theta_2(m_0)$ such that

i) $m_0 - 3\theta_1 \equiv m_1' \in M_3$,

(congruence mod 630)

ii) $m_0 - 3\theta_2 \equiv m_2' \in M_3$,

iii) $0 \leq \theta_i \leq 29$, $i = 1,2$,

and

iv) $\theta_1 \not\equiv \theta_2 \bmod 5$.

Now suppose that $v = 2n + 1 > 0$ and $n = 28m_0 + 3t_0$ where $0 \leq t \leq 27$, and that $n \equiv 0 \bmod 3$. Then $m_0 \equiv 0 \bmod 3$. Let $\theta_1(m_0)$ and $\theta_2(m_0)$ be chosen as above. Let $t^* = t_0 + 28\theta_1$. If $t^* \not\equiv 4 \bmod 5$, then let $m = m_0 - 3\theta_1$ and $t = t^*$. Otherwise let $m = m_0 - 3\theta_2$ and $t = t_0 + 28\theta_2$.

Let $\theta = (m_0 - m)/3$. Clearly m and t satisfy all conditions of the lemma with the possible exception of the condition $0 \leq t \leq m$. However since $0 \leq t \leq 27 + 28\theta$, if $m_0 \geq 27 + 31\theta$, then we have $0 \leq t \leq m$. However, as noted above, $0 \leq \theta \leq 29$. Thus for $m_0 \geq 926$, we have $0 \leq t \leq m$. However if $n \geq 23009$, then $m_0 \geq 926$, that is, if $v \geq 46019$ and $v \equiv 1 \bmod 6$, then v can be represented as claimed in the enunciation. The results for $v \equiv 3$ and $5 \bmod 6$ can be obtained similarly. □

COROLLARY 5.9. *Let v be an odd positive integer. If (i) $v \equiv 1 \bmod 6$ and if $v > 46017$ or if (ii) $v \equiv 3 \bmod 6$ and $v \geq 17301$ or if (iii) $v \equiv 5 \bmod 6$ and $v \geq 17303$, then there exists a skew Room square of side v.*

Proof: Apply Theorem 5.7. □

Table 3 - Elements of M

1, 11, 13, 17, 23, 27, 31, 37, 41, 43, 47, 53, 61, 67, 71, 73, 81, 83, 97, 101, 103, 107, 113, 117, 121, 127, 131, 137, 143, 151, 153, 157, 163, 167, 171, 173, 181, 187, 191, 193, 197, 207, 211, 221, 223, 227, 233, 243, 247, 251, 253, 257, 261, 263, 271, 277, 281, 283, 293, 297, 307, 311, 313, 317, 323, 331, 333, 337, 341, 347, 351, 353, 361, 367, 373, 377, 383, 387, 391, 397, 401, 403, 407, 414, 423, 431, 433, 437, 443, 451, 457, 461, 463, 467, 473, 477, 481, 487, 491, 493, 503, 513, 517, 521, 523, 527, 531, 533, 541, 547, 551, 557, 563, 571, 577, 583, 587, 593, 601, 603, 607, 611, 613, 617, 621.

6. *Conclusion.*

In view of the above, by means of a computer, the constructions of section 3 and other ad hoc methods it is possible to investigate the remaining possible sides for Room squares. In view of limited space this is not done here. We have been able to show that for odd $v \geq 4537$, $v \in SS$, These results are contained in [6], where a possible computer free proof of the results is outlined.

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University of Waterloo.
University of Newcastle.