SOME CONSTRUCTIONS FOR FRAMES, ROOM SQUARES, AND SUBSQUARES

D.R. Stinson

Abstract

Several constructions are given for frames, Room squares, and subsquares. Among the results obtained are the following:

(1) There is a skew Room square of side 69,

(2) There are skew frames of type $4^4 2^1$ and $4^4 6^1$,

(3) For all $s \equiv 3$ modulo 8, $s > 3$, there is a Room square of side $3s + 2$ with a subsquare of side s.

A *Room square* of side s is a square array R of side s, which satisfies the following:

(1) each cell of R either is empty or contains an unordered pair of elements (called *symbols*) chosen from a set S of size $s + 1$,

(2) each symbol occurs precisely once in each row and each column,

(3) every unordered pair of symbols occurs in a unique cell of R.

Suppose R is a Room square of side s, on symbol set S. A square t by t subarray of R is said to be a *Room subsquare* of side t provided it is itself a Room square (of side T). We shall refer to a Room subsquare simply as a *subsquare*.

A Room square R, on symbol set S, is said to be *standardized* with respect to the symbol $\infty \in S$, provided the rows and columns of R have been permuted (if necessary) so that ∞ occurs in the cells of R on the main diagonal. Given a standardized Room square, it is natural to index the rows and columns of R so that $\{\infty,x\}$ occurs in cell (x,x) of R, for every $x \in S$, $x \neq \infty$.

A standardized Room square R (of side s) is said to be a *skew* Room square (of side s) provided that, for any pair of cells (i,j) and (j,i), where $i \neq j$, precisely one is empty.

A subsquare of a skew Room square R is said to be a *skew* subsquare

ARS COMBINATORIA, VOL. 12 (1981), pp. 229-267

provided it is located symmetrically with respect to the main diagonal of R.

Let S be a set, and let $\{S_1,\ldots,S_n\}$ be a partition of S. An $\{S_1,\ldots,S_n\}$-frame is an $|S|$ by $|S|$ array, F, indexed by S, which satisfies the properties:

(1) every cell either is empty or contains an unordered pair of symbols of S,

(2) the subarrays S_i^2 are empty, for $1 \le i \le n$ (these subarrays are refereed to as *holes*),

(3) each symbol of $S \backslash S_i$ occurs precisely once in row (or column) s, where $s \in S_i$,

(4) the pairs occurring in F are precisely those $\{s,t\}$ where $(s,t) \in S^2 \backslash \bigcup_{i=1}^{n} S_i^2$.

F is *skew* if, for any pair of cells (s,t) and (t,s), where $(s,t) \in S^2 \backslash \bigcup_{i=1}^{n} S_i^2$, precisely one is empty.

The *type* of an $\{S_1,\ldots,S_n\}$-frame F will be the multiset $\{|S_1|,\ldots,|S_n|\}$. We will say that F has type $t_1^{u_1} \ldots t_k^{u_k}$ provided there are u_i S_j's of cardinality t_i, for $1 \le i \le k$.

If a Room square of side s is standardized, with respect to ∞, say, and then the contents of the cells containing ∞ are deleted, a frame of type 1^s is constructed. Conversely, one can produce a Room square of side s from a frame of type 1^s. Also, a skew Room square of side s is equivalent to a skew frame of type 1^s.

More generally, a Room square of side s containing a subsquare of side t gives rise to a frame of type $1^{s-t}t^1$. If t is odd, $t \ne 3$ or 5, then these two arrays are equivalent. However, there do exist frames of type $1^{s-t}t^1$ with $t = 3$ or 5, whereas no Room square has a subsquare of side 3 or 5. (See Theorem 1.1). We will refer to a frame of type $1^{s-t}t^1$ as an *incomplete* Room square (of side s) missing a subsquare of side t.

We have the following existence results.

THEOREM 1.1 (Mullin and Wallis [12]). *There exists a Room square of side* s *if and only if* s *is an odd positve integer other than* 3 *or* 5.

THEOREM 1.2 (Stinson [15]). *There exists a skew Room square of side* s *if and only if* s *is an odd positive integer other than* 3 *or* 5.

THEOREM 1.3 (Stinson [14]). *If* $s \geq max\{t+644, 6t+9\}$, s, t *odd positive intgers, then there is a frame of type* $1^{s-t}t^1$.

THEOREM 1.4 (Dinitz and Stinson [5]). *Let* t *and* u *be positive integers. If any of the following conditions hold, then there exists a frame of type* t^u:

(1) $u \geq 6$ *and* $t(u-1)$ *is even.*

(2) $u = 5$ *and* $gcd(t,210) \neq 1$

(3) $u = 4$ *and* $t \equiv 0$ *modulo* 4.

Notice that Theorem 1.3 says nothing if $s < 6t$. One of the purposes of this paper is to establish the existence of Room squares (of side s) with subsquares (of side t) where t is "large" compared to s. In this situation one must have $s \geq 3t + 2$ (see section 5); we establish that equality can be attained infinitely often.

We require several definitions concerning designs. A *group-divisible design* (GDD) is a triple (X,G,A), where X is a finite set (of *points*), G is a partition of X into subsets called *groups*, and A is a set of subsets of X (called *blocks*), such that (1) every unordered pair of points x_1, x_2, not contained in a group, is contained in a unique block, (2) a group and a block contain at most one common point.

A *pairwise balanced design* (PBD) is a pair (X,A), where X is a finite set of points, and A is a set of blocks, such that every pair of points is contained in a unique block.

Let K be a set of positive integers. (X,A) is said to be a (v,K)-PBD if $v = |X|$, and $A \in A$ implies $|A| \in K$. K is said to be *PBD-closed* provided $v \in K$ where ever there exists a (v,K)-PBD.

A subset of blocks $P \subseteq A$ is a *parallel class* if P partitions X. A PBD is *resolvable* if A can be partitioned into parallel classes.

A *Latin square* (of *order* s) based on symbol set S, where $|S| = x$, is an s by s array L of the symbols of S, such that each symbol occurs precisely once in each row and each column. Two Latin squares, L and M of order s, based on symbol sets S and T respectively, are said to be *orthogonal* provided their superposition yields every ordered pair in $S \times T$ exactly once. Several Latin squares are *mutually orthogonal* if each pair is. We refer to a set of mutually orthogonal Latin squares as (a set of) MOLS. A pair of orthogonal Latin squares will be called a pair of OLS.

The following is a well-known result concerning MOLS.

LEMMA 1.5 *Suppose* $n \geq 2$ *has prime power factorization* $n = \pi p_i^{\alpha_i}$. *Then there exist* k *MOLS of order* n *if* $n \geq min\{p_i^{\alpha_i} - 1\}$.

Let L be a Latin square of order s, on symbol set S. A t-by-t subarray L' of L is said to be a *subsquare* (of L) provided it is a Latin square of order t in its own right (on some symbol set $S' \subseteq S$). Similarly, if L and M are a pair of OLS of order s, we say that t-by-t subarrays L' of L and M' of M are *sub-OLS* (of order t) if L' and M' are respectively subsquares of order t, and their superposition (within the superposition of L and M) yields a pair of OLS of order t.

Suppose one removes a pair of sub-OLS (of order t) from a pair of OLS (of order s). The resulting arrays are called a pair of *incomplete* OLS (of order s) *missing* a pair of sub-OLS (of order t). If $t \neq 2$ or 6, then the incomplete OLS may be "completed" by inserting any pair of OLS of order t on the relevant symbol sets. (It is well-known that a pair of OLS exist for all positive integral orders except 2 and 6). However if $t = 2$ or 6, the incomplete OLS may still exist (and, of course, they cannot be completed.)

We need to define one more array related to a pair of OLS, which resembles a frame in some ways. Let $\{S_1, \ldots, S_n\}$ be a partition of S. A *partitioned pair* of *incomplete* OLS, having partition $\{S_1, \ldots, S_n\}$ consists of two S by S arrays, L and M, indexed by S, whose cells either are empty or contain a symbol from S, such that

(1) The subarrays of L,M indexed by S_i^2 are empty, $1 \le i \le n$,

(2) row or column s of L or M contains the symbols $S \backslash S_i$ where $s \in S_i$,

(3) the ordered pairs which occur in $\{(L(s,t), M(s,t))\}$ are precisely those in $S^2 \backslash \bigcup_{i=1}^{n} S_i^2$.

The *type* of the partition $\{S_1,\ldots,S_n\}$, as for frames, will denote the multiset $\{|S_1|,\ldots,|S_n|\}$.

Finally, we need to define a special type of GDD associated with sets of MOLS. A transversal design TD (m,n) is a $GDD(X,G,A)$ in which $|X| = mn$, G consists of m groups, each of cardinaltiy n, and A consists of n^2 blocks each of size m. It is well-known that the existence of a $TD(m,n)$ is equivalent to the existence of $m - 2$ MOLS of order n.

A $TD(m,n)$ is said to be *resolvable* if its block set can be partitioned into parallel classes, and is denoted $RTD(m,n)$. The existence of an $RTD(m-1,n)$ is equivalent to the existence of a $TD(m,n)$.

In this paper we establish several new results concerning frames and Room squares. The necessary theory is developed in sections 2,3 and 4, and applications are given in section 5.

Section 2 describes three recursive constructions for frames. These constructions are quite general, and supersede other constructions which appear in the literature, some of which are indicated as corollaries.

In Section 3 we discuss some starter-adder methods for constructing frames. We describe a method where by intransitive starter-adders can be produced by algebraic techniques: we make use of projecting sets in starter-adders in conjunction with strong orthomorphisms in Galois fields of even order.

Section 4 describes a method for producing Room squares from frames by filling in the holes. This construction is of sufficient generality that several of the most important product theorems for Room squares (e.g. singular direct and indirect products) can be obtained as straightforward corollaries.

In Section 5 we prove several new results, based on the methods described in Sections 2-4. We construct skew frames of types $4^4 2^1$ and $4^4 6^1$, and a skew Room square of side 69. Also, we show that for all $s \equiv 3$ modulo 8, $s > 3$, there exists a Room square of side $3s + 2$ with a subsquare of side s. Such a subsquare is as large as possible.

2. *Three recursive constructions.*

In this section, we describe three recursive constructions for frames. The first construction inflates frames by means of Latin squares; the second utilizes GDDs; the third is a doubling construction.

Suppose F is an $\{S_1,\ldots,S_n\}$-frame, and let L and M be a pair of OLS on symbol set X, both indexed by S.

Define the array F^{LM} by first choosing an ordering, say (a,b), of the contents $\{a,b\}$ of every cell of F, then defining

$$F^{LM}((s,x),(s',x')) = \begin{cases} \text{empty, if } F(s,s') \text{ is empty} \\ \{(a,L(x,x')),(b,M(x,x'))\} \text{ if } F(s,s') = \{a,b\}. \end{cases}$$

CONSTRUCTION 2.1 *If F is an $\{S_1,\ldots,S_n\}$-frame, and L,M are a pair of OLS on symbol set X, then F^{LM} is an $\{S_1\times X,\ldots,S_n\times X\}$-frame. Further, F^{LM} is skew if and only if F is.*

Proof. First, the subsquares $(S_i\times X)^2$ of F^{LM} are empty. Also, it is clear that this construction preserves skewness.

Next, choose a row (s,x) and a symbol (s',x'), where $\{s,s'\} \not\subseteq S_i$, for any $i = 1,2,\ldots,n$. There is a unique t such that $s' \in F(s,t)$, and let $F(s,t) = \{s',t'\}$ for some t'. Suppose first that $\{s',t'\}$ was ordered (s',t'). Now $L(x,y) = x'$ has a unique solution for y, whence $F^{LM}((s,x),(t,y)) = \{(s',x'),(t,y')\}$ for some y'.

If $\{s',t'\}$ was ordered (t',s'), the argument proceeds similarly.

Now let us check the pairs in F_{LM}. Pick two symbols (s,x) and (s',x'), with $\{s,s'\} \not\subseteq S_i$, for any i. There is a unique cell (t,t') such that $F(t,t') = \{s,s'\}$. If the ordering was (s,s'), solve $L(y,y') = x$, $M(y,y') = x'$ for y and y'; then $F^{LM}((t,y),$

$(t',y')) = \{(s,x),(s',x')\}$. The second case proceeds similarly.

Finally, it is clear that no pairs $\{(s,x),(s',x')\}$ occur with $\{s,s'\} \subseteq S_i$, for some i. Thus F^{LM} is the desired frame. □

Let (X,G,A) be a GDD. A *weighting* is a map $w\colon X \to \mathbb{Z}^+ \cup \{0\}$. For any subset $Y \subseteq X$, and w a weighting, let $w(Y)$ denote the multiset $\{w(y)\colon y \in Y\}$. The following construction closely resembles Wilson's fundamental construction for GDDs [16].

CONSTRUCTION 2.2 *Suppose (X,G,A) is a GDD and w is a weighting. Suppose that, for every block $A \in A$, there exists a (skew) frame of type $w(A)$. Then there is a (skew) frame of type $\{\sum_{x \in G} w(x)\colon G \in G\}$.*

Proof. For each $x \in X$, let S_x be a set of size $w(x)$. For $G \in G$, let $S_G = \bigcup_{x \in G} S_x$. By hypothesis, for every $A \in A$, we have an $\{S_x\colon x \in A\}$-frame F_A.

We construct F, an $\{S_G\colon G \in G\}$-frame, by defining

$$F(s,t) = \begin{cases} \text{empty, if } \{s,t\} \subseteq S_G, \text{ for some } G \in G. \\ F_{A(x,y)}(s,t), \text{ otherwise, where } A(x,y) \text{ is the block containing } \{x,y\}, \\ \quad \text{and } s \in S_x,\ t \in S_y. \end{cases}$$

Let us check the necessary properties. First the subsquares S_G^2 are empty. Next pick a row $r \in S_x$ and a symbol $s \in S_y$, where $\{x,y\} \not\subseteq G$ for any $G \in G$. Then s occurs in a unique cell of row r in $F_{A(x,y)}$, and in no other cell of row r.

Now, pick two symbols $s \in S_x$, $t \in S_y$, again with $\{x,y\} \not\subseteq G$ for any $G \in G$. Then $\{s,t\}$ occurs in a unique cell of $F_{A(x,y)}$, and in no other cell of F.

Lastly, let us check that F is skew provided that all the F_A's are. Pick two cells (r,s) and (s,r) with $r \in S_x$, $s \in S_y$, and $(x,y) \not\subseteq G$ for any $G \in G$. Since $F_{A(x,y)}$ is skew, precisely one of cells (r,s) and (s,r) is filled. Thus skewness is preserved. □

Define $F_t = \{u\colon$ there exists a frame of type $t^u\}$ and

$SF_t = \{u\colon$ there exists a skew frame of type $t^u\}$.

COROLLARY 2.3 ([4]) *For any positive integer* t, *the sets* F_t *and* SF_t *are PBD-closed.*

Proof. Let (X,A) be a (v,F_t)-PBD. Then $(X,\{\{x\}: x \in X\},A)$ is a GDD. Define $w: X \to \mathbb{Z}^+ \cup \{0\}$ by setting $w(x) = t$, for all $x \in X$. Apply construction 2.2.2., to obtain a frame of type t^v, where $v = |X|$.

Similarly, SF_t is PBD-closed. □

One drawback to construction 2.1 is that one cannot "double" frames using it: there does not exist a pair of OLS of order 2. This is partially remedied by the following construction.

CONSTRUCTION 2.4 *Suppose the following exist:*

(1) A skew $\{S_1,\ldots,S_n\}$*-frame* F,

(2) A partitioned pair of incomplete Latin squares L,M, *having partition* $\{S_1,\ldots,S_n\}$.

Then an $\{S_1 \times I_2,\ldots,S_n \times I_n\}$ *frame exists, where* $I_2 = \{1,2\}$.

Proof. Define G, on symbol set $(\bigcup_{k=1}^{n} S_k) \times I_2$

$$\text{by } G((s,i),(t,j)) = \begin{cases} \text{empty, if } i \neq j \text{ or } \{s,t\} \subseteq S_k \text{ for some } k, \\ 1 \le k \le n \\ \{(x,1),(y,1)\} \text{ if } F(s,t) = \{x,y\} \text{ and } i=j=1 \\ \{(x,2),(y,2)\} \text{ if } F(t,s) = \{x,y\} \text{ and } i=j=1 \\ \{(L(s,t),1),(M(s,t),2)\} \text{ if } i=j=2. \end{cases}$$

Note that G is well-defined since F is skew.

The other verifications are almost immediate. First pick a symbol (s_k,i), $s_k \in S_k$, $1 \le k \le n$, $1 \le i \le 2$, and a row (s_ℓ,j), $s_\ell \in S_\ell$, $1 \le \ell \le n$, $1 \le j \le 2$. If $k = \ell$, then symbol (s_k,i) does not occur in row (s_ℓ,j), so assume $k \neq \ell$. If $i = j = 1$, then $(s_k,1) \in G((s_\ell,1),(s_m,1))$, where $s_k \in F(s_\ell,s_m)$. If $i = 2$ and $j = 1$, then $(s_k,2) \in G((s_\ell,1),(s_m,1))$, where $s_k \in F(s_m,s_\ell)$. If $i = 1$ and $j = 2$, then $(s_k,1) \in G((s_\ell,2),(s_m,2))$, where $s_k \in M(s_\ell,s_m)$. A similar argument shows that the correct symbols occur in the columns of G.

Let us now check pairs of symbols, say (s_k, i), and (s_ℓ, j), with $k \neq \ell$.

There is a unique cell (t_1, t_2) such that $F(t_1, t_2) = \{s_k, s_\ell\}$. If $i = j = 1$, then $G((t_1,1),(t_2,1)) = \{(s_k,1),(s_\ell,1)\}$. If $i = j = 2$, then $G((t_2,1),(t_1,1)) = \{(s_k,2),(s_\ell,2)\}$. There is a unique cell (t_3, t_4) such that $L(t_3, t_4) = s_k$ and $M(t_3, t_4) = s_\ell$. Thus, if $i = 1$ and $j = 2$, $G((t_3,2),(t_4,2)) = \{(s_k,1),(s_\ell,2)\}$. A similar argument applies if $i = 2$ and $j = 1$.

Thus G is the desired frame. □

3. *Starters and Adders.*

Let G be an additive abelian group, and H a subgroup. Denote $|G| = g$, $|H| = h$, and suppose $g - h$ is even. An $(h, \frac{g}{h})$-*frame starter* in $G \backslash H$ is a set of unordered pairs

$$S = \{\{s_i, t_i\},\ 1 \le i \le \tfrac{g-h}{2}\} \text{ satisfying}$$

(1) $\{s_i\} \cup \{t_i\} = G \backslash H$

(2) $\{\pm(s_i - t_i)\} = G \backslash H$.

Let S be a frame starter in $G \backslash H$, with $S = \{\{s_i, t_i\}\}$. An *adder* for S is an injective mapping $A\colon S \to G \backslash H$ such that $\{s_i + a_i\} \cup \{t_i + a_i\} = G \backslash H$, where $A(s_i, t_i) = a_i$. An adder A is *skew* provided $a_i \neq -a_j$ for any i,j.

Suppose S is a frame starter in $G \backslash H$, and A is an adder. We construct the array F_{SA}, a square array indexed by G, by defining $F_{SA}(x, x - a_i) = \{x + s_i, x + t_i\}$ for $1 \le i \le \frac{g-h}{2}$, and for $x \in G$. Note that at most one unordered pair occurs in each cell of F_{SA}, since A is injective. Also, the subsquares $(H+x)^2$ of F_{SA} are empty since the range of A is $G \backslash H$.

LEMMA 3.1 *Suppose S is a frame starter in $G \backslash H$, and A is an adder. Then F_{SA} is an $\{H+x\colon x \in G\}$-frame of type $h^{g/h}$. Further, if A is skew, then F_{SA} is a skew frame.*

Proof. The cells of row 0 of F_{SA} contain precisely the pairs in S.

Thus the elements occurring in row 0 are those in $G \backslash H$. In row x, for any $x \in G$, the elements of $\{x\} + G\backslash H = G\backslash(H+\{x\})$ occur.

A similar argument applies to the columns. The pairs in column 0 are precisely those $\{s_i+a_i, t_i+a_i\}$, so the elements occurring are those in $G\backslash H$. All other columns are translates of column 0, as was the case for rows.

Next we show the pairs in F_{SA} are precisely those $\{g_1, g_2\}$ with $(g_1, g_2) \in G^2 \backslash \bigcup_{x \in G} (H+x)^2$. For any x, $(x+s_i) - (x+t_i) = s_i - t_i$, so no pair $\{g_1, g_2\}$ occurs if g_1 and g_2 are in the same coset of H. Thus, suppose g_1 and g_2 are in different cosets of H, so $g_1 - g_2 = s_i - t_i$, for some i. Then $\{g_1, g_2\} = F_{SA}(x, x-a_i)$ for $x = g_1 - s_i$. Thus F_{SA} is a frame, as claimed.

Let us consider the skewness of F_{SA}. For any $g_1, g_2 \in G$, cell (g_1, g_2) of F_{SA} is filled if and only if $g_1 - g_2 = a_i$ for some a_i. If A is skew, we cannot have both of cells (g_1, g_2) and (g_2, g_1) filled, for then $g_1 - g_2 = a_i$ and $g_2 - g_1 = a_j$, whence $a_i = -a_j$, contradicting the skew condition. Since F_{SA} contains g^2 cells, exactly $g(\frac{g-h}{2})$ of which are filled, F_{SA} must be skew if and only if A is a skew adder. □

A frame starter $S = \{\{s_i, t_i\}\}$ in $G\backslash H$ is said to be *strong* provided $s_i + t_i \notin H$ for all i, and $s_i + t_i = s_j + t_j$ implies $i = j$. A is called *skew-strong* if (further) $s_i + t_i \neq -(s_j+t_j)$ for any i,j.

LEMMA 3.2 *A strong frame starter* $S = \{\{s_i, t_i\}\}$ *has* $A = \{a_i\}$ *for an adder, where* $a_i = -(s_i+t_i)$. *If* S *is skew-strong, then* A *is skew.*

Proof. $\{s_i+a_i, t_i+a_i\} = \{-s_i, -t_i\}$. □

The frame F_{SA}, arising from a frame starter in $G\backslash H$ and an adder, is constructed by determining a first row $\{\{s_i, t_i\}\}$, a first column $\{\{s_i+a_i, t_i+a_i\}\}$, and then constructing all other rows and columns by taking translates. Thus F_{SA} has G acting on it as an automorphism group, and G is transitive on the rows, and also the columns of F_{SA}.

In the next construction, we produce frames which will have an automorphism group G, say, but the action of G on the rows (and columns) of the frame will not be transitive: thus the name intransitive frame starters.

Let G be an abelian group of order g, having a subgroup H of order h, with $g-h$ even. A $2k$-*intransitive frame starter-adder* in $G\backslash H$ (abbreviated IFSA) is a quadruple (S,C,R,A) where

$$S = \{\{s_i,t_i\}: 1 \le i \le \tfrac{g-h}{2} - 2k\} \cup \{u_i: 1 \le i \le 2k\} \quad \text{(the starter)},$$

$$C = \{\{p_i,q_i\}: 1 \le i \le k\},\ R = \{\{p_i',q_i'\}: 1 \le i \le k\},$$

and A: $S \to G\backslash H$ (the adder) is an injection, satisfying

(1) $\{s_i\} \cup \{t_i\} \cup \{u_i\} \cup \{p_i\} \cup \{q_i\} = G\backslash H,$

$\{s_i+a_i\} \cup \{t_i+a_i\} \cup \{u_i+b_i\} \cup \{p_i'\} \cup \{q_i'\} = G\backslash H$

(where $a_i = A(s_i,t_i)$, $b_i = A(u_i)$),

(2) $\{\pm(s_i-t_i)\} \cup \{\pm(p_i-q_i)\} \cup \{\pm(p_i'-q_i')\} = G\backslash H.$

(3) any element p_i-q_i, or $p_i'-q_i'$, with $1 \le i \le k$, has even order.

Given a $2k$-IFSA (S,C,R,A) in $G\backslash H$, we construct an array $F = F_{SCRA}$ as follows. Let $\infty \notin G$, and define $\Omega = \{\infty\} \times \{1,2,\ldots,2k\}$. For any $x \in G$ and $y \in \Omega$ define $x + y = y$. F will be a square array of side $g + k$, where $|G| = g$, indexed by $G \cup \Omega$.

Now define $F(x,x-a_i) = \{x+s_i, x+t_i\}$ for all $1 \le i \le 2k$, and $x \in G$. Leave all other cells $F(x,y)$, with $(x,y) \in G^2$, empty.

Now suppose d is an element of even order e in G. Define a graph G_d, having vertex set G, joining two vertices x and y by an edge if and only if $(x-y) = \pm d$. The graph G_d thus defined is a disjoint union of cycles of even length e, and thus we may partition $E = E^1 \cup E^2$, where E is the edge set of G_d, and E^1, E^2 are perfect matchings (i.e. G_d has a 1-factorization). Thus, for $1 \le i \le k$, we obtain matchings E_i^1 and E_i^2 with $E_i = E_i^1 \cup E_i^2$, where E_i is the edge set of $G_{(p_i-q_i)}$. Now define, for $1 \le i \le k$ and $x \in G$,

$$F(x,(\infty,2i-1)) = \begin{cases} \{x+p_i, x+q_i\} & \text{if } \{x+p_i,x+q_i\} \in E_i^1 \\ \text{empty, otherwise} \end{cases}$$

$$F(x,(\infty,2i)) = \begin{cases} \{x+p_i, x+q_i\} & \text{if } \{x+p_i,x+q_i\} \in E_i^2 \\ \text{empty, otherwise} \end{cases}$$

Similarly, for $1 \le i \le k$, obtain matchings D_i^1 and D_i^2 from $G_{(p_i'-q_i')}$, and define

$$F((\infty,2i),x) = \begin{cases} \{x+p_i', x+q_i'\} & \text{if } \{x+p_i',x+q_i'\} \in D_i^1 \\ \text{empty, otherwise} \end{cases}$$

$$F((\infty,2i-1),x) = \begin{cases} \{x+p_i', x+q_i'\} & \text{if } \{x+p_i',x+q_i'\} \in D_i^2 \\ \text{empty, otherwise} \end{cases}$$

Lastly leave cells (x,y) of F empty, where $(x,y) \in \Omega^2$.

Note that, since A is injective, at most one ordered pair occurs in each cell of F.

LEMMA 3.3 *Suppose (S,C,R,A) is a 2k-IFSA in $G\backslash H$. Then F_{SCRA} is an $\{H+x:\ x \in G\} \cup \{\Omega\}$-frame of type $h^{g/h}2k^1$.*

Proof. Denote $F = F_{SCRA}$. For any $x \in G$, row x of F contains precisely the symbols $(G\backslash(H+x)) \cup \Omega$ by property (1), and the way F was constructed. In row (∞,i) the symbols which occur are those in G, since the D_i's are all perfect matchings. Similarly the correct symbols occur in the columns of F.

Which pairs occur in F? First, no pair $\{\infty_i,\infty_j\}$ occurs, and no pair $\{x,y\}$ occurs if x and y are in the same coset of H. Secondly, an ∞_i occurs with each $x \in G$, since the equation $u_i + y = x$ has a unique solution y, and then $\{\infty_i,x\} = F(y,y-b_i)$.

Now consider a pair $\{x,y\}$ with x,y in distinct cosets of H. Exactly one of the following holds, by property (2): $x-y = \pm(s_i-t_i)$, $x-y = \pm(p_i-q_i)$, or $x-y = \pm(p_i'-q_i')$, for some i. in the first case, suppose $x-y = s_i-t_i$ (without loss of generality). Then

$$F(z,z-a_i) = \{x,y\}, \quad \text{where } z = x - s_i.$$

If $x-y = p_i-q_i$, then $\{x,y\} = F(z,(\infty,2i-\delta))$ where $z = x-p_i$, and

$$\delta = \begin{cases} 1 & \text{if } \{x,y\} \in E_i^1 \\ 0 & \text{if } \{x,y\} \in E_i^2 . \end{cases}$$

A similar argument applies to the third case.

Finally the subsquares $(H+x)^2$, for $x \in G$, and Ω^2, are empty. This is seen easily by the definition of F, since the range of A is $G\backslash H$. This completes the proof. □

It is natural to ask when the array F_{SCRA} will be skew. We have the following.

LEMMA 3.4 *Suppose* (S,C,R,A) *is a* $2k$ - *IFSA in* $G\backslash H$. *Then the frame* $F = F_{SCRA}$ *can be made skew provided the following extra conditions are satisfied:*

(4) $\{\pm a_i\} \cup \{\pm b_i\} = G\backslash H$.

(5) If $p_i - q_i$ *has order* $2^n m$ *with* m *odd, then* $p_i' - q_i'$ *has order* $2^n m'$ *with* m' *odd, for* $1 \le i \le k$ *(we refer to such an IFSA as a* skew *IFSA).*

Proof. The condition (4) is the same as the one which ensures skewness of a frame F_{SA} constructed from a starter and skew adder. Thus we check only whether the last $2k$ rows of F are skew with respect to the last $2k$ columns of F. This is where we use condition (8). We want to know if for $1 \le i \le k$, E_i^1, E_i^2, D_i^1 and D_i^2 can be constructed so that $\{x+p_i, x+q_i\} \in E_i^1$ if and only if $\{x+p_i', x+q_i'\} \in D_i^1$. Denote $d_i = p_i - q_i$, $e_i = p_i' - q_i'$.

For a given i, construct a graph S, on vertex set $G \times \{1,2\}$, having the edges:

$$\begin{cases} x_1y_1 & \text{iff } x-y = \pm d_i \\ x_2y_2 & \text{iff } x-y = \pm e_i \\ x_1x_2 & \text{for all } x \in G. \end{cases}$$

Thus the edges x_1y_1 yield a subgraph isomorphic to the edge graph of G_{d_i} (vertex x_1 corresponds to edge $p_i + x$, $q_i + x$), and the edges x_2y_2 yield the edge graph G_{e_i}.

We show that S is bipartite. Suppose S has a cycle C of length m, m odd. This yields an equation $kd_i + \ell e_i = 0$, with $k + \ell$ odd. Suppose without loss of generality that k is odd and ℓ is even. Multiply by $f/2$ where f is the (even) order of d_i and e_i, to obtain $\frac{kfd_i}{2} = 0$, a contradition.

Thus we may properly 2-colour the vertices V of S, obtaining a bipartition $V = V_1 \cup V_2$.

Let $V_\ell^k = V_\ell \cap (G\times\{k\})$, for $\ell, k = 1,2$. Then V_1^1 yields E_i^1, V_2^2 yields D_i^1, V_2^1 yields E_i^2, and V_1^2 yields D_i^2, as desired. This completes the proof. □

Next, we describe a construction for IFSAs. First, some definitions are required.

Let $S_1 = \{\{s_i,t_i\}\}$ be a frame starter in $G\backslash H$, and let A_1 be an adder. A *projecting set* of size m is a set $P = \{\{p_i,q_i\}: 1 \le i \le m\}$ of unordered pairs of elements of $G\backslash H$, which satisfies:

(1) $p_i \neq p_{i'} \neq q_{j'} \neq q_j$ for all i, i', j, j'

(2) $|\{p_i,q_i\} \cap \{s_j,t_j\}| \le 1$ for all i,j,

(3) $\{\pm(p_i-q_i)\} \cup \{\pm(p_i+A_1(p_i)-q_i-A_1(q_i))\} =$

$\{\pm(s_j-t_j): \ |\{s_j,t_j\} \cap \{p_i,q_i\}| = 1$ for some $i\}$

(4) the differences $p_i - q_i$ and $p_i + A_1(p_i) - q_i - A_1(q_i)$ all have even order.

If the adder A_1 is skew, a projecting set P is said to be *skew* provided

(5) there exists a bijection α: $P \to \{\{p_i+A_1(p_i),\ q_i+A_1(q_i)\}\}$ such that if $p_i = q_i$ has order $2^n m$ with m odd, then $p_i' + A_1(p_i') - q_i' - A_1(q_i')$ has order $2^n m'$, with m' odd, where $\alpha(p_i,q_i) = \{p_i'+A_1(p_i'), q_i'+A_1(q_i')\}$.

Given a projecting set P of size n we will define a $2n$-IFSA. First, let $J_1 = \{j: \ s_j \in \{p_i,q_i\}$ for some i, $1 \le i \le n\}$, $J_2 = \{j: \ t_j \in \{p_i,q_i\}$ for some i, $1 \le i \le n\}$. Define (S,C,R,A) by $S = \{\{s_j,t_j\}: \ j \notin J_1 \cup J_2\} \cup \{\{s_j\}: \ j \in J_2\} \cup \{\{t_j\}: \ j \in J_1\}$, $C = P$, $R = \{\alpha(p_i,q_i): \ \{p_i,q_i\} \in P\}$, and define $A = A_1$.

LEMMA 3.5 *If* S_1 *is a frame starter in* $G\backslash H$, A_1 *is an adder, and* P *is a projecting set of size* n, *then* (S,C,R,A), *defined above, is a* $2n$*-IFSA. If* P *is skew, then by labelling* $R = \{\{p_i', q_i'\}\}$ *where* $\{p_i', q_i'\} = \alpha(p_i, q_i)$, (S,C,R,A) *is skew.*

Proof. The verifications are routine. □

Suppose P and Q are projecting sets for a frame starter S and adder A. We say that P and Q are *disjoint* provided $P \cap Q = \phi$ and $P \cup Q$ is a projecting set.

The above construction for IFSAs is very flexible when used in conjunction with a multiplication construction for frame starters and adders, which we now describe. This construction has been used by Anderson and Gross [1].

Let G be an additive abelian group. A *strong orthomorphism* is a permutation σ of G such that $\sigma + I$ and $\sigma - I$ are both permutations of G, where I is the identity permutation. Thus $\{\sigma(x) + x: \ x \in G\} = \{\sigma(x) - x: \ x \in G\} = G$. Strong orthomorphisms are known to exist in many groups (see [1], for example). We have the following construction. Suppose S is a frame starter in $G\backslash H$, and A is an adder with $S = \{\{s_i, t_i\}\}$. Let σ be a strong orthomorphism in an abelian group K. Define $S^\sigma = \{\{(s_i,x),(t_i,\sigma(x))\}: \ \{s_i,t_i\} \in A, \ x \in K\}$. $A^\sigma((s_i,x),(t_i,\sigma(x))) = (A(s_i,t_i), -(x+\sigma(x)))$, for all $\{s_i,t_i\} \in S$ and $x \in K$.

LEMMA 3.6 S^σ *and* A^σ, *as described above, are a frame starter and adder in* $(G \times K)\backslash(H \times K)$. *Further, if* A *is skew, then so is* A^σ.

Proof. $\{(s_i,x),(t_i,\sigma(x))\} = (G\backslash H) \times K$ since S is a frame starter in $G\backslash H$ and σ is a permutation. Since A is an adder $\{(s_i+a_i-\sigma(x)), (t_i+a_i,-x)\} = (G\backslash H) \times K$. A^σ is an adder since $\sigma + I$ is a permutation and A is an adder. $\{\pm(s_i-t_i, \ x-\sigma(x))\} = (G\backslash H) \times K$ since S is a frame starter and $\sigma - I$ is a permutation.

Now suppose A is skew. Thus $\{\pm a_i\} = G\backslash H$. Then we have $\{\pm(a_i,-(\sigma(x)+x))\} = (G\backslash H) \times K$, so A^σ is also skew. This completes the proof. □

By itself, the above construction does not yield any new frames. Construction 2.1 enabled us to "multiply" frames by any integer other than 2 or 6, and there is no strong orthomorphism in a group of order 2 or 6. Our interest lies in constructing IFSAs by altering starters and adders by means of projecting sets. Strong orthomorphisms in the additive groups of $GF(2^n)$, $n \geq 2$, are very useful in this context.

In $GF(2^n)$, $n \geq 2$, let α be primitive. The map $\sigma_\alpha: GF(2^n) \to GF(2^n)$ defined by $\sigma_\alpha(x) = \alpha x$ is easily seen to be a strong orthomorphism, since $\alpha \neq 1$.

LEMMA 3.7 *Suppose* $S = \{\{s_j, t_j\}\}$ *is a frame starter, and* A *an adder in* $G\backslash H$, *having a projecting set* $P = \{\{p_i, q_i\}: 1 \leq i \leq m\}$. *Assume* $\{p_i\} \subseteq \{s_j\}$ *and* $\{q_i\} \subseteq \{t_j\}$. *Let* x *be any element of* $GF(2^n)$, *where* $n \geq 2$, *and let* σ *be a strong orthomorphism. Define* $Q_x = \{\{(p_i, x), (q_i, \sigma(x))\}: 1 \leq i \leq m\}$. *Then* Q_x *is a projecting set for the starter* S^σ *and adder* A^σ. *If* P *is skew, then so is* Q_x.

Proof. The construction works since the additive group of $GF(2^n)$ has characteristic 2. Define J_1 and J_2 as before. Then $\{(p_i - q_i, x - \sigma(x))\} \cup \{(p_i + A(p_i) - q_i - A(q_i), \sigma(x) - x)\} = \{(s_j - t_j, x - \sigma(x)): j \in J_1 \cup J_2\}$.

Also, the order of $(y, x - \sigma(x)) \in G \times GF(2^n)$ equals the order of $y \in G$ provided y has even order. Thus skewness is preserved. □

COROLLARY 3.8 *Suppose there exists a (skew) projecting set of size* m *for a frame starter* S *and a (skew) adder* A *in* $G\backslash H$. *Then, for* $1 \leq \ell \leq 2^n$, $n \geq 2$, *there exists a (skew) projecting set of size* ℓm *for the frame starter* S^σ *and Adder* A^σ *in* $(G \times GF(2^n)) \backslash (H \times GF(2^n))$.

Proof. The projecting sets Q_x constructed above are disjoint, for distinct values of x. □

We may prove a result under weaker hypotheses than Lemma 3.1.3. Define a *pre-projecting* set of size m to be a set $P = \{\{p_i, q_i\}: 1 \leq i \leq m\}$ satisfying all the conditions to be a projecting set except possibly (4). That is, we do not require that all the differences $p_i - q_i$ and $p_i + A(p_i) - q_i - A(q_i)$ have even order. A pre-projecting set P is *skew* provided it satisfies conditions (5), allowing, of course, that $n = 0$.

LEMMA 3.9 *Suppose* S *is frame starter,* A *is an adder, and* P *a pre-projecting set in* $G\backslash H$. *Let* x *be any non-zero element of* $GF(2^n)$, *where* $n \geq 2$. *Then* Q_x, *defined as in Lemma 3.1.3, is a projecting set for the starter* S^σ *and* A^σ *in* $(G\times GF(2^n))\backslash(H\times GF(2^n))$. *If* P *is skew, then so is* Q_x.

Proof. $(p_i-q_i, x-\sigma(x))$, for $\{p_i,q_i\} \in P$ and $x \neq 0$, $x \in GF(2^n)$, has even order, so Q_x is a projecting set. Also, skewness is preserved.□

COROLLARY 3.10 *Suppose there exists a (skew) pre-projecting set of size* m *for a frame starter* S *and adder* A *in* $G\backslash H$. *Then for* $1 \leq \ell \leq 2^n-1$, $n \geq 2$, *there exists a (skew) projecting set of size* ℓm *for the frame starter* S^σ *and adder* A^σ *in* $(G\times GF(2^n))\backslash(H\times GF(2^n))$.

Proof. The proof is that of Corollary 3.8, mutatis mutandis. □

4. *Room squares from frames.*

Suppose G is an $\{S_1,\ldots,S_n\}$-frame, and let $T_i \subseteq S_i$ for $1\leq i\leq n$. Denote $S = \bigcup_{i=1}^{n} S_i$ and $T = \bigcup_{i=1}^{n} T_i$. The subarray H of G determined by the cells in $T \times T$ is said to be a $\{T_1,\ldots,T_n\}$-*subframe* if H is a $\{T_1,\ldots,T_n\}$-frame in its own right. If G is skew, it is said to be a *skew* subframe provided it is itself a skew frame.

The following result describes a general method for constructing Room squares from frames.

CONSTRUCTION 4.1 Suppose G is an $\{S_1,\ldots,S_n\}$-frame, and H is a $\{T_1,\ldots,T_n\}$-subframe, where $S = \cup S_i$ and $T = \cup T_i$. Let $a \geq 0$. Suppose the following Room squares exist:

(1) for $1 \leq i \leq n$, a Room square R_i of side $|S_i| + a$ with a sub-square of side $|T_i| + a$,

(2) A Room square R_∞ of side $\sum_{i=1}^{n} |T_i| + a$.

Then a Room square of side $\sum_{i=1}^{n} |S_i| + a$ exists. Further, if G, and R_i for $1 \leq i \leq n$ and $i = \infty$, are skew, then the resulting Room square F is skew.

Proof. Let $\Omega \cap S = \phi$; $|\Omega| = a$, and let $\infty \notin S \cup \Omega$. We may suppose that, for $1 \leq i \leq n$, R_i has symbol set $S_i \cup \Omega \cup \{\infty\}$, and is standardized with respect to ∞.

Define F as follows:

$$F(x,y) = \begin{cases} G(x,y) & \text{if } (x,y) \in S^2 \setminus \bigcup_{i=1}^{n} S_i^2 \\ R_i(x,y) & \text{if } (x,y) \in (S_i \cup \Omega)^2 \setminus (T_i \cup \Omega)^2 \\ R_\infty(x,y) & \text{if } (x,y) \in (\bigcup_{i=1}^{n} T_i \cup \Omega)^2 \end{cases}$$

The above three cases are mutually exclusive and cover all possibilities. It is immediate that the array F is a Room square, and that skewness is preserved. □

Remarks:

(1) It is clear, from the definition of F, that the subframe H of G need not exist. Also, the subsquares of sides $|T_i| + a$ need not exist. (That is, if $|T_i| + a = 3$ or 5, R_i can be taken to be the relevant incomplete Room square (should it exist)).

(2) The Room square F will have various subsquares, depending on how the construction is executed. We will consider the existence of subsquares in several of the corollaries which follow.

We now describe two methods for producing frames with subframes.

COROLLARY 4.2 *In Construction 2.2, there exists a subframe F_A of F, for every block $A \in \mathcal{A}$.*

COROLLARY 4.3 *In Construction 2.1, if L and M contain a pair of sub-OLS on symbol set Y, then F^{LM} contains an $\{S_1 \times Y, \ldots, S_n \times Y\}$-subframe.*

Remark:

If L and M are "missing" the sub-OLS, then F^{LM} is missing the subframe. This can be useful when $|Y| = 2$ or 6.

We are now able to derive several well-known constructions for Room squares as corollaries to Construction 4.1.

COROLLARY 4.4 (The Singular direct product) ([10]). *Suppose there exist:*

(1) a (skew) Room square of side u

(2) a (skew) Room square of side v, *containing a (skew) subsquare of side* w, *with* $v - w \neq 6$.

Then there exists a (skew) Room square of side $u(v-w)+w$, *containing (skew) subsquares of sides* u, v *and* w.

Proof. Start with G, a frame of type 1^u on symbol set $\infty \cup \{1,\ldots,u\}$. Multiply by a pair of OLS, L and M, of side $v - w$ having symbol set $\{1,\ldots,v-w\}$ (Construction 2.1). Finally, apply Construction 3.2.1, with $T = \phi$, $a = w$, to obtain F, a Room square of side $u(v-w)+w$.

R_∞ is a subsquare of side w, and for any i, R_i is a subsquare of side v. We may ensure the existence of s subsquare of side u by stipulating that $L(1,1) = M(1,1) = 1$. Then the subarray indexed by $\{1,\ldots,u\} \times \{1\}$ is a subsquare of side u. □

COROLLARY 4.5 (The Singular indirect product) ([8]). *Suppose there exist:*

(1) a (skew) Room square of side u

(2) a(skew) Room square of side v, *containing (or missing) a (skew) subsquare of side* w

(3) a pair of OLS of side $v-a$ *containing (or missing) a pair of sub-OLS of side* $w-a$ *(where* $0 \leq a \leq w$*)*

(4) a (skew) Room square of side $u(w-a)+a$.

Then there exists a (skew) Room square of side $u(v-a)+a$, *containing (skew) subsquares of sides* u *and* $u(w-a)+a$.

Proof. Start with G, a frame of type 1^u and then multiply by a pair of OLS of order $v-a$ containing (or missing) a pair of sub-OLS of order $w-a$ (2.1). The resulting frame of type $(v-a)^u$ has a subframe (possibly missing) of type $(w-a)^u$. Now apply Construction 4.1. The resulting Room square of side $u(v-a)+a$ has a subsquare of side $u(w-a)+a$ (R_∞), and a subsquare of side u, as in Corollary 4.4. □

A useful modification of the above two corollaries is to start with a frame of type t^u, with $t > 1$, instead of a Room square of side u. The following is obtained.

COROLLARY 4.6 (The frame singular direct product). *Suppose there exist:*

(1) a(skew) frame of type t^u

(2) a (skew) Room square v *containing a (skew) subsquare of side* w

(3) a pair of OLS of order $\frac{v-w}{t}$

Then a (skew) Room square of side $u(v-w)+w$ *exists, containing (skew) subsquares of sides* v *and* w.

Proof. The proof is that of Corollary 4.4, mutatis mutandis. Notice that here we do not have a subsquare of side u. □

COROLLARY 4.7 (The frame singular indirect product) ([2]). *Suppose there exist:*

(1) a (skew) frame of type t^u

(2) a (skew) Room square of side v *containing (or missing) a (skew) subsquare of side* w

(3) a pair of OLS of order $\frac{v-a}{t}$ *containing or missing a pair of sub-OLS of order* $\frac{w-a}{t}$ *(where* $0 \le a \le w$*)*

(4) a (skew) Room square of side $u(w-a)+a$.

Then a (skew) Room square of side $u(v-a)+a$ *exists, containing a (skew) subsquare of side* $u(w-a)+a$.

Proof. The proof is that of Corollary 4.5, mutatis mutandis. □

We derive two further corollaries to Construction 4.1.

COROLLARY 4.8 *Suppose there exists a(skew) frame of type* $t_1^{u_1} \dots t_k^{u_k}$, *and suppose there exists a (skew) Room square of side* t_i+a, *containing a (skew) subsquare of side* a, *for* $1 \le i \le k$. *Then there exists a (skew) Room square of side* $\sum_{i=1}^{k} t_i u_i + a$, *containing (skew) subsquares of side* $t_i + a$, *for* $1 \le i \le k$, *and side* a.

Proof. Let G be an $\{S_1, \dots, S_n\}$-frame of type $t_1^{u_1} \dots t_k^{u_k}$, (where $n = \sum_{i=1}^{k} u_i$). Define $T_i = \phi$, $1 \le i \le n$ and apply Construction 4.1. R_i, $1 \le i \le n$, and R_∞ are subsquares of the resulting square. □

COROLLARY 4.9 *Let* $a \geq 0$. *Suppose there exists a (skew) frame of type* $t_1^1 t_2^{u_2} \ldots t_k^{u_k}$, *and, for* $2 \leq i \leq k$, *a (skew) Room square of side* $t_i + a$ *containing (or missing) a (skew) subsquare of side* a. *Then there exists a (skew) frame of type* $(t_1+a)^1 1^w$ *where* $w = \sum_{i=2}^{k} t_i u_i$. *Further, if a (skew) Room square of side* t_1+a *exists, then a (skew) Room square of side* $t_1 + \sum_{i=2}^{k} t_i u_i + a$ *exists, containing a (skew) subsquare of side* $t_1 + a$.

Proof. This is a slight extension of Construction 4.1. Let G be an $\{S_1,\ldots,S_n\}$ frame of type $t_1^1 t_2^{u_2} \ldots t_k^{u_k}$, where $|S_1| = t_1$, and $1 + \sum_{i=2}^{k} u_i = n$. Define $T_i = \phi$, $1 \leq i \leq n$.

Then, proceed as in Construction 3.2.1, but define

$$F(x,y) = \begin{cases} G(x,y) & \text{if } (x,y) \in S^2 \backslash \bigcup_{i=1}^{n} S_i^2 \\ R_i(x,y) & \text{if } (x,y) \in (S_i \cup \Omega)^2 \backslash \Omega^2,\ 2 \leq i \leq n. \end{cases}$$

It may be checked that F is the desired frame. Now suppose further that a (skew) Room square of side t_1+a exists. Apply Corollary 4.8 with $a = 0$, noting that a (skew) Room square of side one exists. □

5. *Applications*

In [15], a short proof is given that a skew Room square exists for all odd sides exceeding five. The proof depends heavily on the following frames.

LEMMA 5.1 *There exist skew frames of type* 4^4, $4^4 2^1$, 4^5, and $4^4 6^1$.

Proof: It may be checked that S and A, given below, are a starter and skew adder in $(\mathbb{Z}_4 \times \mathbb{Z}_4) \backslash \{(0,0),(0,2),(2,0),(2,2)\}$.

starter	32,11	30,31	21,33	02,13	10,23	12,01
adder	01	23	30	32	11	31

S and A give rise to a skew frame of type 4^4, drawn in Figure 5.1 below (note: this frame was presented in [13], but the picture given there is incorrect). We have three disjoint projecting sets

P_1, P_2, and P_3: $P_1 = \{31,32\}$, $P_2 = \{13,21\}$, $P_3 = \{01,10\}$. They are each skew, since all differences involved have order 4. By Lemmata 3.5, 3,3, and 3.4, the desired skew frames result.

The skew frame of type $4^4 2^1$ is given in Figure 5.2 below. □

As well, a skew Room square of side 69 is required. We give a more general result.

LEMMA 5.3 *For* $1 \leq \ell \leq 3$, *there is a skew frame of type* $12^5 \, 4\ell^1$.

Proof: The following is a starter and skew adder in $\mathbb{Z}_{15} \backslash \{0,5,10\}$:

starter	1,2	9,11	3,6	8,12	13,4	7,14
adder	1	2	6	11	3	7

Then $P = \{\{2,3\},\{4,7\}\}$ is a skew pre-projecting set. Apply Corollary 2.10 with $m = 2$, $n = 2$. □

COROLLARY 5.4 *There exists a skew Room square of side* 69.

Figure 5.1 A skew frame of type 4^4.

<table>
<tr><th></th><th>00</th><th>02</th><th>20</th><th>22</th><th>01</th><th>03</th><th>21</th><th>23</th><th>10</th><th>12</th><th>30</th><th>32</th><th>11</th><th>13</th><th>31</th><th>33</th></tr>
<tr><td>00</td><td colspan="4" rowspan="4">00</td><td></td><td>11
32</td><td>30
31</td><td></td><td>21
33</td><td>03
13</td><td></td><td></td><td></td><td>01
12</td><td></td><td>10
23</td></tr>
<tr><td>02</td><td>13
30</td><td></td><td></td><td>32
33</td><td>01
11</td><td>23
31</td><td></td><td></td><td>03
10</td><td></td><td>12
21</td><td></td></tr>
<tr><td>20</td><td>10
11</td><td></td><td></td><td>31
12</td><td></td><td></td><td>01
13</td><td>23
33</td><td></td><td>30
03</td><td></td><td>21
32</td></tr>
<tr><td>22</td><td></td><td>12
13</td><td>33
10</td><td></td><td></td><td></td><td>21
31</td><td>03
11</td><td>32
01</td><td></td><td>23
30</td><td></td></tr>
<tr><td>01</td><td>12
33</td><td></td><td></td><td>31
32</td><td colspan="4" rowspan="4">01</td><td>02
13</td><td></td><td>11
20</td><td></td><td>22
30</td><td>00
10</td><td></td><td></td></tr>
<tr><td>03</td><td></td><td>10
31</td><td>33
30</td><td></td><td></td><td>00
11</td><td></td><td>13
22</td><td>02
12</td><td>20
32</td><td></td><td></td></tr>
<tr><td>21</td><td></td><td>11
12</td><td>32
13</td><td></td><td>31
00</td><td></td><td>22
33</td><td></td><td></td><td></td><td>02
10</td><td>20
30</td></tr>
<tr><td>23</td><td>13
10</td><td></td><td></td><td>30
11</td><td></td><td>33
02</td><td></td><td>20
31</td><td></td><td></td><td>22
32</td><td>00
12</td></tr>
<tr><td>10</td><td></td><td></td><td>31
03</td><td>13
23</td><td></td><td>20
33</td><td></td><td>11
22</td><td colspan="4" rowspan="4">10</td><td></td><td>21
02</td><td>00
01</td><td></td></tr>
<tr><td>12</td><td></td><td></td><td>11
21</td><td>33
01</td><td>22
31</td><td></td><td>13
20</td><td></td><td>23
00</td><td></td><td></td><td>02
03</td></tr>
<tr><td>30</td><td>11
23</td><td>33
03</td><td></td><td></td><td></td><td>31
02</td><td></td><td>00
13</td><td>20
21</td><td></td><td></td><td>01
22</td></tr>
<tr><td>32</td><td>31
01</td><td>13
21</td><td></td><td></td><td>33
00</td><td></td><td>02
11</td><td></td><td></td><td>22
23</td><td>03
20</td><td></td></tr>
<tr><td>11</td><td>21
30</td><td></td><td>12
23</td><td></td><td></td><td></td><td>32
00</td><td>10
20</td><td>22
03</td><td></td><td></td><td>01
02</td><td colspan="4" rowspan="4">11</td></tr>
<tr><td>13</td><td></td><td>23
32</td><td></td><td>10
21</td><td></td><td></td><td>12
22</td><td>30
02</td><td></td><td>20
01</td><td>03
00</td><td></td></tr>
<tr><td>31</td><td>32
03</td><td></td><td>01
10</td><td></td><td>12
20</td><td>30
00</td><td></td><td></td><td></td><td>21
22</td><td>02
23</td><td></td></tr>
<tr><td>33</td><td></td><td>30
01</td><td></td><td>03
12</td><td>32
02</td><td>10
22</td><td></td><td></td><td>23
20</td><td></td><td></td><td>00
21</td></tr>
</table>

Figure 5.2 A skew frame of type $4^4 2^1$

	00	02	20	22	01	03	21	23	10	12	30	32	11	13	31	33	∞_1	∞_2
00						11 ∞_2	30 ∞_1		21 33	03 13				01 12		10 23	31 32	
02		00			13 ∞_2			32 ∞_1	01 11	23 31			03 10		12 21		33 30	
20					10 ∞_1			31 ∞_2			01 13	23 33		30 03		21 32	11 12	
22						12 ∞_1	33 ∞_2				21 31	03 11	32 01		23 30		13 10	
01	12 ∞_2			31 ∞_1					02 13		11 20		22 30	00 10				32 33
03		10 ∞_2	33 ∞_1							00 11		13 22	02 12	20 32				30 31
21		11 ∞_1	32 ∞_2			01			31 00		22 33				02 10	20 30		12 13
23	13 ∞_1			30 ∞_2						33 02		20 31			22 32	00 12		10 11
10			31 03	13 23		20 33		11 22						21 ∞_2	00 ∞_1		01 02	
12			11 21	33 01	22 31		13 20						23 ∞_2			02 ∞_1	03 00	
30	11 23	33 03				31 02		00 13		10			20 ∞_1			01 ∞_2	21 22	
32	31 01	13 21			33 00		02 11							22 ∞_1	03 ∞_2		23 20	
11	21 30		12 23				32 00	10 20	22 ∞_2			01 ∞_1						02 03
13		23 32		10 21			12 22	30 02		20 ∞_2	03 ∞_1							00 01
11	32 03		01 10		12 20	30 00				21 ∞_1	02 ∞_2				11			22 23
33		30 01		03 12	32 02	10 22			23 ∞_1			00 ∞_2						20 21
∞_1					30 11	32 13	10 31	12 33					00 21	02 23	20 01	22 03	∞	
∞_2	33 10	31 12	13 30	11 32					03 20	01 22	23 00	21 02						

Proof: Start with the skew frame of type $12^5 8^1$, constructed above. Now apply Corollary 4.8 with $a = 1$. Skew Room squares of sides 9 and 13 exist, so one of side 69 may be constructed. □

The remainder of this section is concerned with subsquares in Room squares. We give a numerical example to illustrate how the methods of this paper can be applied to produce Room squares with subsquares: we construct skew Room squares of side 123 with various skew subsquares. It is worth noting that a skew Room square of side 123 was one of the last to be constructed (see [14]), and until quite recently, there was no known example of any Room square of side 123 containing a subsquare of side exceeding 1.

LEMMA 5.5 $0 \le \ell \le 21$, *there exists a skew frame of type* $8^{13}\ 2\ell^1$.

Proof: Consider the following starter and skew adder over $\mathbb{Z}_{13}$:

starter	2,8	12,3	6,11	10,9	5,7	1,4
adder	7	9	8	1	11	13

It is easy to verify that we have the following three disjoint skew pre-projecting sets: $P_1 = \{8,12\}$, $P_2 = \{11,10\}$, and $P_3 = \{7,1\}$. For each of P_1, P_2, and P_3 (independently), we may apply Corollary 3.10 with $m = 1$, $n = 3$. The result is obtained.

COROLLARY 5.6 *There is a skew Room square of side* 123 *having skew subsquares of sides* 9 *and* 19 .

Proof: With $\ell = 7$ in Lemma 5.5, we obtain a skew frame of type $8^{13}18^1$. Apply Corollary 4.8 with $a = 1$, filling in the skew subsquares of side 9 and 19. □

LEMMA 5.7 *There exists a skew Room square of side* 123 *having subsquares of sides* 11 *and* 29.

Proof: Let (X,G,A) be a TD(5,7). Let $G = \{G_i : 1 \le i \le 5\}$, and let x_1, x_2, x_3 be three points in G_5 . Define $w: X \to \{0,2,4\}$ by

$$w(x) = \begin{cases} 4 & \text{if } x \in G_1 \cup G_2 \cup G_3 \cup G_4 \cup \{x_1, x_2\}, \\ 2 & \text{if } x = x_3 \\ 0 & \text{if } x \in G_5 \setminus \{x_1, x_2, x_3\} \end{cases}$$

Apply Construction 2.2 , making use of skew Frames of type 4^4, $4^4 2^1$, and 4^5 (Lemma 5.1). A skew Frame of type $28^4\, 10^1$ is constructed. Now apply Corollary 4.8 with $a = 1$, filling in the skew subsquares of sides 11 and 29 . □

In the remainder of this section we consider Room squares with "large" subsquares.

LEMMA 5.8 *Suppose* F *is an* $\{S_1, \ldots, S_n\}$-*frame with* $|S_1| \geq |S_2| \geq \ldots \geq |S_n|$. *Let* $S = \bigcup_{i=1}^{n} S_i$. *Then* $3|S_1| + |S_2| \leq |S|$, *and, if* $|S|$ *is odd, then* $3|S_1| + |S_2| + 1 \leq |S|$.

Proof: Let s be any element of S_2. The symbol s occurs $|S_1|$ times in the columns indexed by S_1, and $|S_1|$ times in the rows indexed by S_1. Also, s occurs $|S_1|$ times further, once with each element of S_1. Since s occurs a total of $|S| - |S_2|$ times in F, we obtain $3|S_1| + |S_2| \leq |S|$. Now suppose $|S|$ is odd. Then $|S_i|$ is odd, $1 \leq i \leq n$, since $|S| - |S_i|$ must be even. Thus $3|S_1| + |S_2|$ must be even, and the result follows. □

COROLLARY 5.9 (Mullin and Collens [9])

If a Room square of side s *has a subsquare of side* t, *then* $s \geq 3t + 2$.

Proof: A Room square of side s with a subsquare of side t gives rise to a frame F of type $t^1 1^{s-t}$. Since s is odd, Lemma 5.8 yields $3t + 2 \leq s$. □

We shall construct infinite classes of frames of type $t_1^1 t_2^{u_2}$

where $3t_1 + t_2 = t_1+u_2t_2$. We refer to such frames as t_1-*maximum* frames. Using t_1-maximum frames, we can show that for all positive $t > 3$ congruent to 3 modulo 8, there exists a Room square of side $3t + 2$ having a subsquare of side t.

LEMMA 5.10 *Suppose that there exists a* t_1*-maximum frame of type* $t_1^1 t_2^{u_2}$. *Let* $c \geq 0$, *and suppose that there exists a Room square of side* $3\,(\frac{t_2-c}{2})+c$ *containing (or missing) a subsquare of side* $\frac{t_2-c}{2}$. *Then there exists a Room square of side* $3t + c$ *containing a subsquare of side* t, *for* $t = t_1 + \frac{t_2-c}{2}$.

Proof: Apply Corollary 4.9 with $a = \frac{t_2-c}{2}$ and $k = 2$. □

Thus it is desirable to construct t_1-maximum frames. We have such frames already: a frame of type $6^1 4^4$ was produced in Lemma 3.3.1, and $3\cdot 6 + 4 = 22 = 6 + 4\cdot 4$. Also, a frame of type $4n^4$ exists for all $n \geq 1$ by Theorem 2.4.4, and $4\cdot 4n = 3\cdot 4n + 4n$.

LEMMA 5.11 *If* $n \geq 1$, *a frame of type* $6n^1\ 4n^4$ *exists.*

Proof: For $n = 1$, the frame is that one described above. Thus if $n > 1$, $n \neq 2$ or 6, we may obtain the desired frame by multiplication by Latin squares (Construction 2.1).

For $n = 2$, we use the "doubling" construction, Construction 2.4. The frame of type $6^1 4^4$ is skew (see Lemma 5.1), so we need only construct a partitioned pair of incomplete OLS, having a partition of type $6^1 4^4$. This is done using a singular direct product construction for Latin squares. (Note that $22 = 5(6-2)+2$).

Horton [7] has constructed the following six by six array A (cells contain ordered pairs):

		33	44	55	66
		64	35	46	53
34	65	16	52	23	41
45	36	51	13	62	24
56	43	25	61	14	32
63	54	42	26	31	15

Consider A to be partitioned:

$A = \begin{array}{c|c} \phi & R \\ \hline C & T \end{array}$ where R is two by four, C is four by two, and T is four by four. (A can be thought of a pair of incomplete OLS of order 6 missing a pair of sub-OLS of order 2). Let N be the super-position of a pair of OLS of order 4 on symbol set $\{3,4,5,6\}$. For $1 \le i,\ j \le 5$, define C_{ij} (respectively R_{ij}, N_{ij}) by replacing a cell containing (a,b) by (a_i,b_j). For $1 \le i,\ j \le 5$, define T_{ij} by replacing a cell containing (a,b) by (a_i,b_j) if $a,b \ne 1,2$, by (a,b_j) if $a = 1$ or 2, and by (a_i,b) if $b = 1$ or 2.

Consider the array

$$P = \begin{bmatrix} \phi & \phi & R_{54} & R_{42} & R_{35} & R_{23} \\ \phi & \phi & N_{45} & N_{24} & N_{53} & N_{32} \\ C_{35} & N_{43} & \phi & N_{51} & T_{35} & N_{14} \\ C_{54} & N_{25} & T_{54} & \phi & N_{12} & N_{41} \\ C_{23} & N_{52} & N_{31} & N_{15} & \phi & T_{23} \\ C_{42} & N_{34} & N_{13} & T_{42} & N_{21} & \phi \end{bmatrix}$$

It may be verified that P is a partitioned pair of incomplete OLS of type $6^1 4^4$. Thus construction 2.4 yields the frame of type $12^1 8^4$.

Finally for $n = 6$, start with the frame of type $12^1 8^4$ and apply Construction 2.1, multiplying by a pair of OLS of side 3. □

We can now construct an infinite family of Room squares with large subsquares. We need something to start with. The following was obtained by Dinitz [3].

LEMMA 5.12 *There exists an incomplete Room square of side 11 missing a subsquare of side* 3.

This array is presented in Figure 5.3 below.

			48			37	6X			59
			69			5X	38			47
				39	4X			57	68	
67	8X		∞3			04	15	29		
58	79			∞4	03				2X	16
9X		78		06	∞5		24		13	
			05	7X	89	∞6		14		23
	46	3X		25		19	∞7			08
	35	49	1X		26			∞8	07	
34		56			17	28		0X	∞9	
			27	18			09	36	45	∞X

Figure 5.3

An incomplete Room square of side 11 missing a subsquare of side 3.

COROLLARY 5.13 *For* $n \geq 0$ *there exists:*

(1) a Room square of side $3u_n + 2$ *with a subsquare of side* u_n, *where* $u_n = 12\cdot 3^n - 1$, *and*

(2) a Room square of side $3v_n + 2$ *with a subsquare of side* v_n, *where* $v_n = 16\cdot 3^n - 1$.

Proof: By induction on n. First we prove (1). The incomplete Room square of Lemma 5.13, together with the frame of type 4^8, yields a Room Square of side 35 with a subsquare of side 11 (put $t_1 = t_2 = 8$, $u_2 = 3$, and $c = 2$ in Lemma 5.10). This establishes the truth of the proposition for $n = 0$. Assume the proposition is true for $n = \ell - 1 \geq 0$. A frame of type $(24\cdot 3^{\ell})^4$ exists. Apply Lemma 5.10 with $t_1 = t_2 = 24\cdot 3^{\ell-1}$, $u_2 = 3$, and $c = 2$. Since $\frac{t_2 - c}{2} = u_{\ell-1}$, and we have a Room square of side $3u_{\ell-1} + 2$ with a subsquare of side $u_{\ell-1}$, this yields a Room square of side $3t + 2$ with a subsquare of side t for

$$t = t_1 + \frac{t_2 - c}{2} = 24\cdot 3^{\ell-1} + \frac{24\cdot 3^{\ell-1} - 2}{2}$$

$$= 12\cdot 3^{\ell} - 1 = u_{\ell}.$$

Thus the result is shown by induction.

(2) is proven similarly, using the frames of type $4n^4\ 6n^1$. □

We will now prove that all $u \equiv 3$ modulo 8, $u > 3$, there exists a Room square of side $3u + 2$ with a subsquare of side u. This generalizes (1) of Corollary 5.13. It is first necessary to construct some more t_1-maximum frames.

LEMMA 5.14 *For* $q \equiv 1$ *modulo* 4 *a prime power, there exists a strong frame starter in* $(GF(q) \times \mathbb{Z}_2)\backslash(\{0\} \times \mathbb{Z}_2)$, *having a pre-projecting set of size* $\frac{q-1}{2}$.

Proof: We use the following strong frame starter. If $\omega \in GF(q)$ is primitive and $Q = \{\omega^{2i} : 0 \le i \le t-1\}$ where $q = 4t + 1$, then $S = \{\{(x,0), (\omega x,0)\}, \{(-x,0), (-\omega x,1)\}, \{(-\omega x,0), (-\omega^2 x,1)\}, \{(\omega x,1), (\omega^2 x,1)\} : x \in Q\}$ is a strong frame starter (see [4]). Then define $P = \{\{(\omega x,0), (-x,0)\}, \{(-\omega^2 x,1), (\omega x,1)\} : x \in Q\}$. The differences arising from $\pm(p_i-q_i)$ and $\pm(p_i+A(p_i) - (q_i+A(q_i)))$, where $\{p_i,q_i\} \in P$, are those in $\{(x(\omega+1),0), (xw(w+1),0), (x(\omega+1),1), (x\omega(\omega+1),1): x \in Q\}$. The other verications are trivial, so P is a pre-projecting set of size $\frac{q-1}{2} = 2t$. □

Notice that if $\ell = 2^n$ were allowed in Corollary 3.10, we could obtain a projecting set of size $2^{n-1}(q-1)$ for a frame starter-adder in $(GF(q) \times \mathbb{Z}_2 \times GF(2^n))\setminus(\{0\} \times \mathbb{Z}_2 \times GF(2^n))$ for $q \equiv 1$ modulo 4 a prime power and $n \ge 2$.

This would give rise to a t_1-maximum frame of type $t_1^1 t_2^{u_2}$ where $t_1 = 2^n(q-1)$, $t_2 = 2^{n+1}$, and $u_2 = q$. (Thus $3t_1 + t_2 = 3\cdot 2^n(q-1) + 2^{n+1} = t_1 + u_2 t_2$. Even though we cannot use Corollary 3.10 to construct this frame, we can obtain it by other methods.

First, a definition. For integer $\ell \ge 3$, let C_ℓ denote the graph which is a cycle of length ℓ.

For a positive integer n, let $C_\ell[K_{n,n}]$ be the graph constructed by replacing every vertex x of C_ℓ by n vertices $x_1,\ldots,x_n$, and then constructing all edges $x_i y_j$, $1 \le i, j \le n$, whenever xy is an edge of C_ℓ. We define a $C_\ell[K_{n,n}]$-*Room Rectangle* be an ℓn by $2n$ array A in which each cell either is empty of contains an edge of $C_\ell[K_{n,n}]$, such that:

(1) the filled cells of each row of A form a one-factor of some $K_{n,n}$ in $C_\ell[K_{n,n}]$

(2) the filled cells of each column of A form a one-factor of $C_\ell[K_{n,n}]$

(3) each edge of $C_\ell[K_{n,n}]$ occurs in precisely one cell of A.

It has been determined precisely when $C_\ell[K_{n,n}]$- Room rectangles exist. (see Hartman and Stinson [6]).

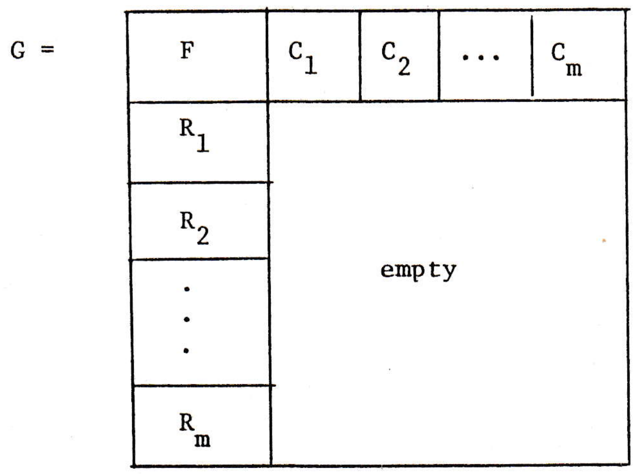
LEMMA 5.15 *Let* $\ell \geq 3$ *and* $n \geq 1$ *be integers. Then a* $C_\ell[K_{n,n}]$*-Room rectangle exists if and only if* ℓn *is even.*

LEMMA 5.16 *Suppose* S *is a frame starter in* $G \backslash H$ *is an adder, and* P *is a pre-projecting set of size* m*. Denote* $|G| = g$ *and* $|H| = h$*. Let* n *be any even positive integer other than two or six. Then a frame of type* $2mn^1 \, ng^{g/h}$ *exists.*

Proof: "Project" P to obtain a quadruple (S,C,R,A), which fails to be a IFSA only in that differences of pairs of elements in R and C may have odd order. Construct an array F_1 from S and A in the usual way. Let L and M be a pair of OLS on symbol set $I_n = \{1,\ldots,n\}$, and denote $F = F_1^{LM}$.

Now consider a pair $\{p_i,q_i\} \in C$, $1 \leq i \leq m$. Corresponding to this pair, we need a gn by 2n array C_i, in which each column is Latin in $G \times I_n$, row (g,j), where $g \in G$ and $1 \leq j \leq n$, is Latin in $\{p_i+g, q_i+g\} \times I_n$, and the unordered pairs occurring in C_i are precisely those $\{(x,j), (y,j')\}$ with $x - y = \pm (p_i-q_i)$, $1 \leq j, j' \leq n$.

Suppose such an array C_i exists for $1 \leq i \leq m$, and a similar array R_i eixsts for 1 i m. Then it is a simple matter to check that the array G, pictured below, is a frame of the desired type.



Thus we must only show that the arrays C_i and R_i exist, $1 \le i \le m$. Let $p_i - q_i = d_i$, and construct the graph D_i on vertex set G, joining x and y if and only if $x - y = \pm d_i$. Then D_i is a disjoint union of cycles of length $e_i \ge 3$. For each cycle B of D_i, we have $C_{e_i}[K_{n,n}]$-Room rectangle A_B. "Stack" these arrays A_B vertically to obtain the desired array C_i. (If necessary, permute the rows of C_i so that the pairs $\{(p_i + g,j), (q_i + g,k)\}$, $1 \le j,k \le n$, $g \in G$, occur in rows $\{g\} \times I_n$.) Thus we can construct the desired frame. □

LEMMA 5.17 *For $u \equiv 1$ modulo 4 a prime power, there exists a frame of type $8^u\ 4(u-1)^1$.*

Proof: Apply Lemma 5.14, and Lemma 5.16 with $g = 2u$, $h = 2$, $m = \frac{u-1}{2}$ and $n = 4$. □

Next, we wish to derive a result similar to that of Lemma 5.14 for $u \equiv 3$ modulo 4. We need another construction.

LEMMA 5.18 *Suppose S is a frame starter in $G \backslash N$, A is a skew adder, and P is a pre-projecting set of size $\frac{g-h}{4}$, where $|G| = g$ and $|H| = h$. Then a frame of type $(g-h)^1 2h^{g/h}$ exists.*

Proof: "Project" P to obtain (S,C,R,A) as in the proof of Lemma 5.17. Note that here S consists entirely of singletons. Construct F_1 from S and A, on symbol set $G \cup \Omega$, where $|\Omega| = \frac{g-h}{2}$. F_1 is skew, and we may define F_2 (the "skew mate") by $F_2(g_1,g_2) = F_1(g_2,g_1)$ for all $g_1,g_2 \in G$. Now define an array F, on symbol set $(G \cup \Omega) \times \{1,2\}$, as follows.

For $x \in G$, $\infty \in \Omega$, and $i = 1,2$, define $D^i(x,\infty)$ to be the two-by-two array

(x,i) (∞,i)	
	(x,i) $(\infty,3-i)$

Superimpose F_1 and F_2, and then replace the contents of every cell (g_1,g_2) by $D^i(x,\infty)$, where $F_i(g_1,g_2) = \{x,\infty\}$. Thus F is a "doubling" of F_1 (this construction enables us to circumvent the requirement $n \neq 2$ of Lemma 5.17). Now F can be completed to a frame exactly as in Lemma 5.17, by making use of the necessary Room rectangles. □

EXAMPLE 5.19 A frame of type $4^5\, 8^1$. We have a skew-strong starter $S = \{\{6,2\}, \{4,3\}, \{8,1\}, \{7,9\}\}$ in $\mathbb{Z}_{10} \setminus \{0,5\}$.

$P = \{\{2,4\}, \{1,7\}\}$ is a skew pre-projecting set.

frame is exhibited in Figure 5.4

LEMMA 5.20 *If there exists a skew-strong starter* $G\setminus\{0\}$, *then there exists a skew-strong frame starter in* $(G \times GF(4))\setminus(\{0\}\times GF(4))$, *having a pre-projecting set of size* $g - 1$, *where* $|G| = g$.

Proof: Let $S_1 = \{\{s_i,t_i\}\}$ be the skew-strong starter in $G\setminus\{0\}$. Let ω be primitive in $GF(4)$. Define $S = \{\{(s_i,x), (t_i,\omega x)\}: x \in GF(4)\}$. Then S is a skew-strong starter in $(G \times GF(4))\setminus(\{0\} \times GF(4))$. Then define $P = \{\{(s_1,0), (t_1,\omega^2)\}$ $\{(s_i,1), (t_i,1)\}\}$. We claim that P is a pre-projecting set.

The adder A associated with S is $A((s_i,x), (t_i,\omega x)) = (-(s_i+t_i), x+\omega x)$. Thus the differences arising from P and $p_i + A(p_i) - q_i - A(q_i)$, $\{p_i,q_i\} \in P$, are those in $\{\pm(s_i-t_i,\omega^2),$ $\pm(s_i-t_i,0), \pm(s_i-t_i,\omega), \pm(s_i-t_i,1)\} = (G\times GF(4))\setminus(\{0\}\times GF(4))$. □

LEMMA 5.21 *For* $u \equiv 3 \bmod 4$ *a prime power exceeding* 3, *a frame of type* $8^u\, 4(u-1)^1$ *exists.*

Proof: Starting with a skew-strong starter in $GF(u)$ (see Mullin and Nemeth [11]), apply Lemma 5.20, and Lemma 5.18 with $g = 4u$, $h = 4$. □

So, to this point, we have constructed a large number of t_1-maximum frames: we have frames of type $8^u\, 4(u-1)^1$ for all prime powers $u > 3$. We now derive a corollary to the GDD construction for

0,5		9′α′		8′β′		6′γ′		3′δ′				9δ		3γ		6β		8α		24′	2′4			17′	1′7		
			9′α		8′β		6′γ		3′δ				9δ′		3γ′		6β′		8α′			24	2′4′			17	1′7′
9α		1,6		0′α′		9′β′		7′γ′		4′δ′				0δ		4γ		7β		35′	3′5			28′	2′8		
	9α′				0′α		9′β		7′γ		4′δ				0δ′		4γ′		7β′			35	3′5′			28	2′8′
8β		0α		2,7		1′α′		0′β′		8′γ′		5′δ′				1δ		5γ		46		4′6′		3′9′		39	
	8β′		0α′				1′α		0′β		8′γ		5′δ				1δ′		5γ′		4′6		46′		3′9		39′
6γ		9β		1α		3,8		2′α′		1′β′		9′γ′		6′δ′				2δ		57		5′7′		4′0′		40	
	6γ′		9β′		1α′				2′α		1′β		9′γ		6′δ				2δ′		5′7		57′		4′0		40′
3δ		7γ		0β		2α		4,9		3′α′		2′β′		0′γ′		7′δ′				6′8′		68		5′1′			51
	3δ′		7γ′		0β′		2α′				3′α		2′β		0′γ		7′δ				6′8		68′		5′1	51′	
		4δ		8γ		1β		3α		5,0		4′α′		3′β′		1′γ′		8′δ′		7′9′		79		6′2′			62
			4δ′		8γ′		1β′		3α′				4′α		3′β		1′γ		8′δ		7′9		79′		6′2	62′	
9′δ′				5δ		9γ		2β		4α		6,1		5′α′		4′β′		2′γ′		80		8′0′		73		7′3′	
	9′δ				5δ′		9γ′		2β′		4α′				5′α		4′β		2′γ		8′0		80′		7′3		73′
3′γ′		0′δ′				6δ		0γ		3β		5α		7,2		6′α′		5′β′		91		9′1′		84		8′4′	
	3′γ		0′δ				6δ′		0γ′		3β′		5α′				6′α		5′β		9′1		91′		8′4		84′
6′β′		4′γ′		1′δ′				7δ		1γ		4β		6α		8,3		7′α′		0′2′			02	95		9′5′	
	6′β		4′γ		1′δ				7δ′		1γ′		4β′		6α′				7′α		0′2	02′			9′5		95′
8′α′		7′β′		5′γ′		2′δ′				8δ		2γ		5β		7α		9,4		1′3′			13	06		0′6′	
	8′α		7′β		5′γ		2′δ				8δ′		2γ′		5β′		7α′				1′3	13′		.	0′6		06′
47		5′8′		69		7′0′		81		9′2′		03		1′4′		25		3′6′		Ω							
4′7′		58		6′9′		70		8′1′		92		0′3′		14		2′5′		36									
	47′		58′		69′		70′		81′		92′		03′		14′		25′		36′								
	4′7		5′8		6′9		7′0		8′1		9′2		0′3		1′4		2′5		3′6								
12		2′3′		34		4′5′		56		6′7′		78		8′9′		90		0′1′									
1′2′		23		3′4′		45		5′6′		67		7′8′		89		9′0′		01									
	12′		23′		34′		45′		56′		67′		78′		89′		90′		01′								
	1′2		2′3		3′4		4′5		5′6		6′7		7′8		8′9		9′0		0′1								

Figure 5.4 A frame of type $4^5 8^1$

frames which enables us to construct t_1-maximum frames recursively. We can then prove a "multiplication" theorem.

LEMMA 5.22 *Suppose* $(X,\mathcal{B})$ *is a resolvable PBD, with parallel classes* $P_1,\ldots,P_r$ such that $|B| = k_i$ *for all* $B \in P_i$, *where* k_i *are integers,* $1 \le i \le r$. *Let* t *be an integer, and suppose that for* $1 \le i \le n$, *there exists a* t_1*-maximum frame of type* $t_i^1\, t^{k_i}$ *(hence* $t_i = \frac{t}{2}(k_i-1)$*). Then a* t_1*-maximum frame of type* $\frac{t}{2}(v-1)^1\, t^v$ *exists.*

Proof: Define a GDD $(Y, \mathcal{G}, \mathcal{A})$ as follows. Let $\Omega = \{\infty_1,\ldots,\infty_r\}$, $Y = X \cup \Omega$, $\mathcal{G} = \{\{x\} : x \in X\} \cup \{\Omega\}$, and $\mathcal{A} = \{B \cup \{\infty_i\} : B \in P_i \subseteq \mathcal{B}\}$. Define a weighting w by $w(x) = t$ if $x \in X$, $w(\infty_i) = \frac{t}{2}(k_i-1)$, $1 \le i \le r$.

Now apply Construction 2.2. For a block $B \in P_i$, we require a frame of type $\frac{t}{2}(k_i-1)^1\, t^{k_i}$, which exists by assumption. The frame constructed has type $t_0^1\, t^v$ where $t_0 = \frac{t}{2}\sum_{i=1}^{r}(k_i-1) = \frac{t}{2}(v-1)$. This frame is t_1-maximum, since

$$\frac{t}{2}(v-1) + vt = \frac{3}{2}tv - \frac{t}{2} = \frac{3t}{2}(v-1) + t\ .$$

Thus the result is proved. □

COROLLARY 5.23 *Let* t *be an integer. Suppose there exist* $m-1$ *MOLS of order* n, *and suppose* t_1*-maximum frames of type* $t_1^1\, t^k$ *exist for* $k = n$ *and* m, *where* $t_1 = \frac{t}{2}(k-1)$. *Then there exists a* t_1*-maximum frame of type* $t_1^1\, t^{nm}$, *where* $t_1 = \frac{t}{2}(nm-1)$.

Proof: By hypothesis there exists a resolvable transversal design RTD(m,n). Hence we can construct a resolvable PBD$(X,\mathcal{B})$ where $|X| = nm$, and $\mathcal{B}$ consists of one parallel class of blocks of size n, and n parallel classes of blocks of size m. Apply Lemma 3.4.17. □

LEMMA 5.24 *For all odd* $u \ge 3$, *a* t_1*-frame of type* $8^u\, 4(u-1)^1$ *exists.*

Proof: For $u = 3$, there exists a frame of type 8^4 (Theorem 1.4), which is t_1-maximum. For $u > 3$, u a prime power, the result follows by Lemmata 5.17 and 5.21.

Let u have prime power factorization $u = \prod_{i=1}^{k} p_i^{\alpha_i}$, where, without loss of generality,
$p_1^{\alpha_1} > p_2^{\alpha_2} > \ldots > p_k^{\alpha_k}$. If $k = 1$, the result is shown above. We proceed by induction on k. The number of OLS of order $u/p_k^{\alpha_k}$ is at least $p_{k-1}^{\alpha_{k-1}} \geq p_k^{\alpha_k} - 1$ (Lemma 1.5).

Apply Corollary 5.23 with $n = u/p_k^{\alpha_k}, m = p_k^{\alpha_k}$. The input frames exist by induction, so a frame of type $8^u\ 4(u-1)^1$ can be constructed. □

The following is our main result.

THEOREM 5.25 *For all $s \equiv 3$ modulo 8, $s > 3$ there exists a Room square of side $3s + 2$ containing a subsquare of side s.*

Proof: Let $u = \frac{s+1}{4}$. Then u is odd and at least 3, so a t_1-maximum frame of type $8^u\ 4(u-1)^1$ exists. Apply Lemma 3.4.2 with $t_1 = 4(u-1)$, $t_2 = 8$, $u_2 = u$, and $c = 2$. We have an incomplete Room square of side 11 missing a subsquare of side 3, so we obtain a Room square of side $3t + 2$ with a subsquare of side t, for

$$t = t_1 + \frac{t_2 - c}{2} = 4(u-1) + \frac{8-2}{2} = 4u - 1 = s, \text{ as desired.}$$ □

6. *Summary*

In this author's opinion, one of the main unresolved problems concerning Room squares is the subsquare problem: for what ordered pairs (s,t) does there exist a Room square of side s containing (or missing, if $t = 3$ or 5) a subsquare of side t? Certainly s and t must be odd positive integers, and $s \geq 3t + 2$. We have demonstrated that for $t \equiv 3$ modulo 8, there is a Room square of side $3t + 2$ containing (or missing, if $t = 3$) a subsquare of side t, so equality can be attained. Also, if $s \geq 6t + 9$ and t is large enough, (s and t odd), there is a Room square of side s with a subsquare of side t.

Thus the following seems reasonable.

CONJECTURE: Let s and t be positive odd integers with $s \geq 3t + 2$. Then there is a Room square of side s containing (or missing, if $t = 3$ or 5) a subsquare of side t if and only if $(s,t) \neq (5,1)$.

References

[1] B.A. Anderson and K.B. Gross, *Starter-adder methods in the construction of Howell designs*, Journal of Australian Math. Soc. 24(A) (1977), 375-384.

[2] B.A. Anderson, R.C. Mullin, and D.R. Stinson, *More skew Room squares*, Utilitas Math 18 (1980), 201-205.

[3] J.H. Dinitz, private communication.

[4] J.H. Dinitz and D.R. Stinson, *The construction and uses of frames*, Ars Combinatoria 10 (1980), 31-53.

[5] J.H. Dinitz and D.R. Stinson, *Further results on frames*, Ars Combinatoria 11, to appear.

[6] A. Hartman and D.R. Stinson, *A note on one-factorizations*, Utilitas Math., to appear.

[7] J.D. Horton, *Sub-latin squares and incomplete orthogonal arrays*, Journal of Combinatorial Theory (A) 16 (1974), 23-33.

[8] R.C. Mullin, *A generalization of the singular direct procuct with applications to skew Room squares*, Journal of Combinatorial Theory (A), to appear.

[9] R.C. Mullin and R.J. Collens, *Some properties of Room squares - a computer search*, Proc. First Louisiana Conf. on Combinatorics, Graph Theory and Computing (1970), 87-111.

[10] R.C. Mullin, J.D. Horton and R.G. Stanton, *A recursive construction for Room designs*, Aequationes Math. 6 (1971), 39-45.

[11] R.C. Mullin and F. Nemeth, *An existence theorem for Room squares*, Canadian Math - Bulleten 12 (1969), 493-497.

[12] R.C. Mullin and W.D. Wallis, *The existence of Room Squares*, Aequationes Math. 13 (1975, 1-7.

[13] D.R. Stinson, *A skew Room square of order 129*, Discrete Math. 31 (1980), 333-335.

[14] D.R. Stinson, *Some results concerning frames, Room squares, and subsquares*, Journal of Australian Math. Soc., to appear.

[15] D.R. Stinson, *The spectrum of skew Room squares*, Journal of Australian Math. Soc. , to appear.

[16] R.M. Wilson, *Constructions and uses of pairwise balanced designs*, Math. Centre Tracts 55 (1974), 18-41.

D.R.Stinson
Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario