A NOTE ON ONE-FACTORIZATIONS

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ABSTRACT. The graph $C_n[K_{m,m}]$ is defined to have vertex set $\mathbb{Z}_n \times \mathbb{Z}_m$ and $(a,i)(b,j)$ is an edge whenever $a - b \equiv \pm 1 \pmod n$. Parker has shown that these graphs have a one-factorization if and only if mn is even. We construct special one-factorizations of $C_n[K_{m,m}]$ (when mn is even) which are useful in the construction of Room squares with large subsquares.

1. *Introduction.*

In this note a *graph* is a graph without multiple edges or loops. The number of edges incident with a vertex v is called the *degree* of v; a graph is k-*regular* if every vertex has degree k. The number of vertices is referred to as the *order* of a graph.

Let G be a graph of even order $2n$. A *one-factor* of G is a set of n independent edges, that is, a 1-regular subgraph of order $2n$. (A one-factor is sometimes called a perfect matching.) A *one-factorization* of a k-regular graph G is a set $\{F_1, F_2, \ldots, F_k\}$ of one-factors which partitions the edges of G. A one-factorization may be thought of as a k-colouring of the edges of G so that each vertex is incident with precisely one edge of each colour.

For any graph G and any positive integer m we define $G[K_{m,m}]$ to be the graph formed by replacing each vertex v of G by m vertices $v_1, v_2, \ldots, v_m$. Each edge vw of G is then replaced by all the m^2 edges $v_i w_j$, where $1 \le i, j \le m$.

Let G be a k-regular graph of order n (so kn is even) and let m be a positive integer. We define a $G[K_{m,m}]$-Room rectangle to be a $knm/2$ by km array A, each cell of which is either empty or contains an edge of $G[K_{m,m}]$, which satisfies

(1) each edge of $G[K_{m,m}]$ occurs in precisely one cell of A;

(2) the filled cells of any column of A form a one-factor of $G[K_{m,m}]$;

(3) the filled cells of any row of A form a one-factor of one of the $K_{m,m}$'s which make up $G[K_{m,m}]$.

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Note that condition (2) requires that $G[K_{m,m}]$ have a one-factorization, so necessarily nm (the order of $G[K_{m,m}]$) must be even.

The study of $G[K_{m,m}]$-Room rectangles was motivated by a construction method for Room squares due to the second author [5]. Although the existence question has been solved for Room squares (see Mullin and Wallis [2]), the existence questions for (Room) subsquares of Room squares remain open. It is easily seen that if a Room square of side u contains a subsquare of side v, then $u \geq 3v+2$. By using the Room rectangles we construct here, it can be shown that for infinitely many values of v, there exists a Room square of side $3v+2$ with a subsquare of side v.

The cycle C_n of length $n \geq 3$ is defined to be the 2-regular graph with vertex set $\mathbb{Z}_n$ and edges xy for each pair satisfying $x-y \equiv \pm 1 (\mathrm{mod}\ n)$.

Parker [3] has shown that the graph $C_n[K_{m,m}]$ has a one-factorization if and only if nm is even. Under the same conditions we show that there exists a $C_n[K_{m,m}]$-Room rectangle. We also show the existence of $G[K_{m,m}]$-Room rectangles for some more general graphs G.

2. *The Existence of* $C_n[K_{m,m}]$*-Room Rectangles.*

In this section we show that a $C_n[K_{m,m}]$-Room rectangle exists if and only if mn is even. We begin with some examples and direct constructions.

Example 2.1. A $C_3[K_{2,2}]$-Room rectangle.

$0_1 1_2$			$0_2 1_1$
	$1_1 2_2$		$1_2 2_1$
		$2_1 0_2$	$2_2 0_1$
	$0_2 1_2$	$0_1 1_1$	
$1_1 2_1$		$1_2 2_2$	
$2_2 0_2$	$2_1 0_1$		

LEMMA 2.2. *A* $C_n[K_{2,2}]$*-Room rectangle exists for all odd* $n \geq 3$.

Proof. We give a constructive proof based on the idea of inserting four rows at a time into Example 2.1 and increasing the length of the cycle by 2.

Let $n = 2j+1$ for some $j \geq 1$.

The first two rows of the array are identical to the first two rows of Example 2.1. The next $2(j-1) = n-3$ rows will have the form

	$k_1(k+1)_2$		$k_2(k+1)_1$	$k = 2,3,\ldots,n-2$.

The next row is

		$(2j)_1\ 0_2$	$(2j)_2\ 0_1$

replacing the third row of Example 2.1.

The next two rows of the array are identical to rows 4 and 5 of Example 2.1.

The next $2(j-1)$ rows have the form

$(2k)_2(2k+1)_2$		$(2k)_1(2k+1)_1$		$k = 1,2,\ldots,j-1$.
$(2k+1)_1(2k+2)_1$		$(2k+1)_2(2k+2)_2$		

The final row is

$(2j)_2\ 0_2$	$2j_1\ 0_1$		

replacing the final row of Example 2.1. The array thus constructed is a $C_{2j+1}[K_{2,2}]$-Room rectangle. □

LEMMA 2.3. *A* $C_n[K_{2,2}]$*-Room rectangle exists for all even* $n \geq 4$.

Proof. The proof is by direct construction. Let $n = 2j$ for some $j \geq 2$. The first $n = 2j$ rows are

		$k_1(k+1)_2$	$k_2(k+1)_1$	$k = 0,1,2,\ldots,n-1$.

The final 2j rows are

$(2k)_1 (2k+1)_1$	$(2k)_2 (2k+1)_2$			$k = 0,1,2,\ldots,j-1.$
$(2k+1)_2 (2k+2)_2$	$(2k+1)_1 (2k+2)_1$			

The array thus constructed is a $C_{2j}[K_{2,2}]$-Room rectangle. □

We now give an example of a $C_3[K_{4,4}]$-Room rectangle. It is constructed using Example 2.1 embedded in the top left hand subsquare.

Example 2.4. A $C_3[K_{4,4}]$-Room rectangle.

$0_1 1_2$	$0_4 1_4$	$0_3 1_3$	$0_2 1_1$				
$1_3 2_3$	$1_1 2_2$	$1_4 2_4$	$1_2 2_1$				
$2_4 0_4$	$2_3 0_3$	$2_1 0_2$	$2_2 0_1$				
$0_3 1_4$	$0_2 1_2$	$0_1 1_1$	$0_4 1_3$				
$1_1 2_1$	$1_3 2_4$	$1_2 2_2$	$1_4 2_3$				
$2_2 0_2$	$2_1 0_1$	$2_3 0_4$	$2_4 0_3$				
				$0_1 1_3$	$0_2 1_4$	$0_3 1_2$	$0_4 1_1$
				$0_4 1_2$	$0_3 1_1$	$0_1 1_4$	$0_2 1_3$
				$1_1 2_3$	$1_2 2_4$	$1_3 2_2$	$1_4 2_1$
				$1_4 2_2$	$1_3 2_1$	$1_1 2_4$	$1_2 2_3$
				$2_1 0_3$	$2_2 0_4$	$2_3 0_2$	$2_4 0_1$
				$2_4 0_2$	$2_3 0_1$	$2_1 0_4$	$2_2 0_3$

LEMMA 2.5. *A* $C_n[K_{4,4}]$*-Room rectangle exists for all odd* $n \geq 3$.

Proof. The proof is by a direct construction similar to those of Lemmas 2.2 and 2.3.

Let $n = 2j+1$ for some $j \geq 1$. The first two rows of the array are identical to those of Example 2.4. The next $2(j-1)$ rows have the form

(*)

$(2k)_4(2k+1)_4$	$(2k)_1(2k+1)_2$	$(2k)_3(2k+1)_3$	$(2k)_2(2k+1)_1$				
$(2k+1)_3(2k+2)_3$	$(2k+1)_1(2k+2)_2$	$(2k+1)_4(2k+2)_4$	$(2k+1)_2(2k+2)_1$				

$k = 1,2,\ldots,j-1.$

The next row is

(†)

$(2j)_4\ 0_4$	$(2j)_3\ 0_3$	$(2j)_1\ 0_2$	$(2j_2)\ 0_1$				

.

The next two rows of the array are identical to rows 4 and 5 of Example 2.4. The next $2(j-1)$ rows have a form obtained from (*) by interchanging the subscripts (1 and 3) and (2 and 4). The next row is obtained from (†) by the same interchange. The final $2n$ rows are constructed, as in Example 2.4, from the n subarrays of the form

				$k_1(k+1)_3$	$k_2(k+1)_4$	$k_3(k+1)_2$	$k_4(k+1)_1$
				$k_4(k+1)_2$	$k_3(k+1)_1$	$k_1(k+1)_4$	$k_2(k+1)_3$

$k = 0,1,2,\ldots,n-1.$

The array constructed is a $C_{2j+1}[K_{4,4}]$-Room rectangle. □

We now give a recursive construction for $G[K_{m,m}]$-Room rectangles. We assume that the reader is familiar with the definitions of Latin squares and orthogonal Latin squares (see for example [6]).

LEMMA 2.6. *If a* $G[K_{m,m}]$*-Room rectangle exists and a pair of orthogonal Latin squares of order* ℓ *exist then a* $G[K_{\ell m,\ell m}]$*-Room rectangle exists.*

Proof. Notice that $K_{\ell m,\ell m} = K_{m,m}[K_{\ell,\ell}]$, so we may think of the vertices of $G[K_{\ell m,\ell m}]$ as the vertices of $G[K_{m,m}]$ subscripted by elements of the set $\{1,2,\ldots,\ell\}$.

Let A be a $G[K_{m,m}]$-Room rectangle and let L and M be a pair of orthogonal Latin squares of side ℓ. We form a new array from A by the following process. Each empty cell of A is replaced by an $\ell\times\ell$ empty array. Each cell of A containing an edge xy of the graph $G[K_{m,m}]$ is replaced by an $\ell\times\ell$ array whose $(i,j)^{th}$ entry is $x_{L(i,j)}\ y_{M(i,j)}$, where $L(i,j)$ and $M(i,j)$ are the $(i,j)^{th}$ entries from L and M respectively.

The new array is clearly a $G[K_{\ell m,\ell m}]$-Room rectangle by the orthogonality of L and M and the structure inherited from the array A. □

We are now in a position to prove the major result of this section.

THEOREM 2.7. *A* $C_n[K_{m,m}]$*-Room rectangle with* $n \geq 3$, $m \geq 1$ *exists if and only if* mn *is even.*

Proof. As noted in the introduction the conditon that mn is even is clearly necessary for the existence of a $C_n[K_{m,m}]$-Room rectangle. We shall establish sufficiency in two cases, according to the parity of n.

If n is odd, then m is even. When $m = 2$, the result is Lemma 2.2. When $m = 2\ell$ and $\ell \neq 2$ or 6 we may apply Lemma 2.7 since Bose, Shrikhande, and Parker [1] have established the existence of a pair of orthogonal Latin squares of order ℓ provided $\ell \neq 2$ or 6. When $m = 4$ the result is Lemma 2.4, and when $m = 12$ the result is obtained by applying Lemma 2.7, since $12 = 4 \cdot 3$.

If n is even then a $C_n[K_{1,1}]$-Room rectangle is just a one-factorization of the even cycle C_n, which exists. Lemma 2.7 then yields the result for every $m \geq 1$ except $m = 2$ or 6. When $m = 2$ the result is just Lemma 2.3, and this, together with Lemma 2.7, yields the result for $m = 6 = 2 \cdot 3$. □

3. *The Existence of* $G[K_{m,m}]$*-Room Rectangles.*

In this section we shall give necessary and sufficient conditions for the existence of $G[K_{m,m}]$-Room rectangles when G is any 2-regular graph. We also give methods for construction of these rectangles for many cases where G is k-regular and $k \geq 3$.

We begin with two simple decomposition lemmas.

LEMMA 3.1. *Let* G *be a* k*-regular disconnected graph. A* $G[K_{m,m}]$*-Room rectangle exists if and only if* $G_i[K_{m,m}]$*-Room rectangles exist for each component* G_i *of* G.

Proof. A $G[K_{m,m}]$-Room rectangle may be constructed by "stacking" the component rectangles one above another. The component rectangles may always be recovered since each row of the $G[K_{m,m}]$-Room rectangle comes from some edge of G which belongs to a unique component. □

LEMMA 3.2. *Let* G *be a* k-*regular graph with edge-disjoint subgraphs* $G_1, G_2, \ldots, G_n$ *such that*

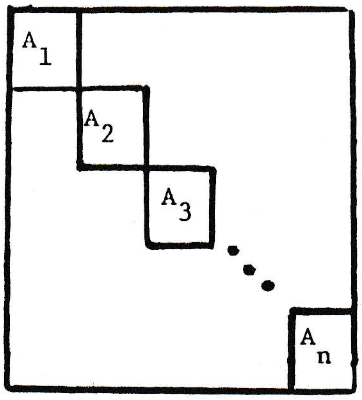
(1) $G = G_1 \cup G_2 \cup \ldots \cup G_n$,

(2) each G_i *is a spanning regular subgraph, and*

(3) a $G_i[K_{m,m}]$-*Room rectangle exists for each* i.

Then a $G[K_{m,m}]$-*Room rectangle exists.*

Proof. Let A_i be a $G_i[K_{m,m}]$-Room rectangle for each i; then the array



is a $G[K_{m,m}]$-Room rectangle. □

Note that this lemma was used implicitly in Example 2.4 and Lemma 2.5.

The following is immediate.

LEMMA 3.3. *If* G *is a 2-regular graph then a* $G[K_{m,m}]$-*Room rectangle exists if and only if* m *is even or all the components of* G *are even cycles.*

By analogy with 1-factorization we say that a graph G has a k-*factorization* for some positive integer k if G can be partitioned into spanning k-regular subgraphs.

THEOREM 3.4. *If* G *is a regular graph of even degree and* m *is even, then a* $G[K_{m,m}]$-*Room rectangle exists.*

Proof. Petersen [4] has shown that a regular graph of even degree has a 2-factorization. Apply Lemmas 3.2 and 3.3. □

REFERENCES

[1] R. C. Bose, E. T. Parker, and S. S. Shrikhande, *Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture*, Canad. J. Math. *12* (1960), 189-203.

[2] R. C. Mullin and W. D. Wallis, *The existence of Room squares*, Aequationes Math. *13* (1975), 1-7.

[3] E. T. Parker, *Edge coloring numbers of some regular graphs*, Proc. Amer. Math. Soc. *37* (1939), 423-424.

[4] J. Petersen, *Die Theorie der regularen Gräphen*, Acta Math. *15* (1891), 193-220.

[5] D. R. Stinson, *Some constructions for frames, Room squares, and subsquares*, Ars Combinatoria, to appear.

[6] A. P. Street and W. D. Wallis, *Combinatorial Theory: An Introduction*, Charles Babbage Research Centre, St. Pierre, Manitoba, 1977.

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