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FURTHER RESULTS ON FRAMES

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In [4], the authors defined the term frames, and gave several constructions for frames. We extend our results to show that if (1) t is odd and u is even

or (2) $(t,u) = (2,4)$

or (3) $u = 2$ or 3

then there does not exist a (t,u)-frame.

Also if (1) $u \geq 7$ is odd

or (2) $u \geq 6$ is even and t is even

or (3) $u = 5$ and $(t,210) \neq 1$

or (4) $u = 4$ and 4 divides t

then there exists a (t,u)-frame.

1. *Introduction*

In [4], the authors defined frames, and gave several constructions for frames. We wish to extend these results.

Let T and U be sets with $|T| = t$, $|U| = u$. A tu by tu array S will be called a t-*frame of order* u if it enjoys the following properties:

(1) Each cell is either empty or contains an unordered pair of elements of $U \times T$,

(2) There exist U empty t by t subsquares of S, no two of them containing any cell in the same row or column. These subsquares will be denoted Su_i. (It will usually be convenient to place the Su_i's on the diagonal of S),

(3) A row or column of S which meets Su_i contains each element of $(U \backslash \{u_i\}) \times T$ exactly once, and contains no element of $\{u_i\} \times T$,

(4) Each unordered pair of elements $\{(u_1,t_1),(u_2,t_2)\}$ with $u_1 \neq u_2$, occurs in a unique cell of S. By counting it follows that no pair of type $\{(u,t_1),(u,t_2)\}$ occurs in the array.

We will refer to a t-frame of order u as a (t,u)-frame.

We are interesting in the following question: For what ordered pairs (t,u) does a (t,u) - frame exist?

In [4], a partial answer was given to the above question for u odd. The following was shown.

THEOREM 1.1. *If* $u > 5$ *is odd and there does not exist a t-frame of order* u, *then either*

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(1) $u = 5$ *and* $(t,6) = 1$.

(2) $t = 2$ *or* 6 *and* $u \equiv 3 \bmod 4$.

We will eliminate most of the above possible exceptions, and also obtain some results on frames of even order.

First, we must recall several definitions and results from [4].

Let G be an additive abelian group of order g, and let H be a subgroup of G of order t, with g-t even. A *t-frame starter of order g/t* in G\H (or a *(t,g/t)-frame starter)* is a set of pairs $A = \{\{s_i,t_i\}, 1 \le i \le \frac{g-t}{2}\}$ satisfying the properties:

(1) $\{s_i\} \cup \{t_i\} = G\backslash H$

(2) $\{\pm(s_i-t_i)\} = G\backslash H$.

Let $A = \{\{s_i,t_i\}\}$ and $B = \{\{u_i,v_i\}\}$ be two frame starters. We may assume that $t_i-s_i = v_i-u_i$, for $1 \le i \le \frac{g-t}{2}$. We say that A and B are *orthogonal* frame starters if $u_i-s_i = u_j-s_j$ implies $i = j$, and $u_i-s_i \notin H$ for all i.

A frame starter $A = \{\{s_i,t_i\}\}$ is *strong* if $s_i+t_i = s_j+t_j$ implies $i = j$, and $s_i+t_i \notin H$ for all i.

LEMMA 1.2. *If* $A = \{\{s_i,t_i\}\}$ *is a strong frame starter then* A *and* $-A = \{\{-s_i,-t_i\}\}$ *are orthogonal frame starters.*

LEMMA 1.3. *If there exist a pair of orthogonal t-frame starters in* G\H *with* $|G| = g$ *and* $|H| = t$, *then there exists a (t,u)-frame, where* $u = g/t$.

LEMMA 1.4. *Suppose* $u \equiv 2$ *or* $3 \bmod 4$ *and* t *is odd. Then there does not exist a 2t-frame starter of order* u.

LEMMA 1.5. *Suppose* t *is odd. Then there does not exist a strong t-frame starter of order 5.*

Let v be a positive integer and let K be a set of positive integers. A pair $(X,\mathcal{B})$, where $\mathcal{B}$ is a set of subsets of X, is said to be a (v,K)-PBD (or pairwise balanced design) provided $|X| = v$, $B \in \mathcal{B}$ implies $|B| \in K$, and for any distinct x_1,x_2 in X, there is a unique $B \in \mathcal{B}$ with $\{x_1,x_2\} \subseteq B$. A set A of positive integers is said to be *PBD-closed* if $v \in A$ whenever there exists a (v,A)-PBD.

For t a positive integer, define $F_t = \{u |$ a (t,u)-frame exists$\}$.

LEMMA 1.6. F_t *is PBD-closed.*

LEMMA 1.7. *Suppose a (t,u)-frame exists, and* $s \ne 2$ *or* 6. *Then a (ts,u)-frame exists.*

We close this section by showing that certain frames do not exist.

LEMMA 1.8. *If any of the following holds, then a (t,u)-frame does not* *exist:*

(1) t *is odd and* u *is even*

(2) $u = 2$ *or* 3

(3) $(t,u) = (1,5)$ *or* $(2,4)$.

Proof. In any row of such a frame there must be $t(u-1)$ symbols, an odd number. Since a cell contains 0 or 2 symbols, we have a contradiction. This proves (1).

(2) is trivial.

The case (1,5) is a Room square of order 5, which is known not to exist [9]. Finally, it can be shown by exhaustive search [11] that no (2,4)-frame exists. □

The authors conjecture that, other than the exceptions listed above, all frames exist. In the remainder of the paper we investigate the existence of frames. In Section 2 we determine F_2, and in Section 3 we determine F_4. This is sufficient to determine the existence of all frames of all orders exceeding 5. Frames of orders 4 and 5 are briefly discussed in Section 4.

For applications of frames, the reader is referred to [2], [4], and [10].

2. *2-frames*

In this section we show that $F_2 = \{u \geq 5\}$. We will accomplish this by PBD-closure. Let $K_2 = \{5,6,\ldots,20,22,23,24,27,28,29,32,33,34,39\}$.

LEMMA 2.1. *Suppose* $K_2 \subseteq F_2$. *Then* $F_2 = \{u \geq 5\}$.

Proof. Hanani has shown [5] that there exists a (v,K_2)-PBD if $v \geq 5$. Since F_2 is PBD-closed, we have $F_2 \supseteq \{u \geq 5\}$ (provided $K_2 \subseteq F_2$). But we have already noted that $2,3,4 \notin F_2$. □

Thus we wish to show that $K_2 \subseteq F_2$. It would be nice if we could construct strong (2,u)-frame starters of these orders, but Lemma 2.4 indicates that this is impossible for $u \equiv 2$ or $3 \bmod 4$. We construct 2-frames of orders $u \equiv 2$ or $3 \bmod 4$ by means of intransitive starters and those of orders $u \equiv 3 \bmod 4$ by a direct construction using skew Room squares. 2-frames of orders $u \equiv 1 \bmod 4$ are already known to exist (Theorem 1.1) and 2-frames of orders $u \equiv 0 \bmod 4$ can be constructed by means of strong 2-frame starters.

We first describe the method of constructing frames from intransitive starters. This technique has been used in the construction of Howell designs by Schellenberg and Vanstone [8], and Zhu [12].

Let G be an abelian group of order g, having a subgroup H of order t, with t even (hence g and $g-t$ are also even). We define a pair of *orthogonal intransitive* $(t,g/t)$-*frame starters* (or a pair of $(t,g/t)$-OIFS), in $G\backslash H$, to be a quadruple (A_1,A_2,B_1,B_2) satisfying the following properties:

(1) $A_1 = \{\{s_i,t_i\},1\le i\le\frac{g-3t}{2}\}\cup\{\{r_i\},1\le i\le t\}$,

$A_2 = \{\{s_i,t_i\},\frac{g-3t}{2}+1\le i\le\frac{g-2t}{2}\}$,

$B_1 = \{\{v_i,w_i\},1\le i\le\frac{g-3t}{2}\}\cup\{\{u_i\},1\le i\le t\}$,

$B_2 = \{\{v_i,w_i\},\frac{g-3t}{2}+1\le i\le\frac{g-2t}{2}\}$;

(2) $\{r_i\} \cup \{s_i\} \cup \{t_i\} = G\backslash H$,
$\{u_i\} \cup \{v_i\} \cup \{w_i\} = G\backslash H$;

(3) $\{\pm(s_i-t_i)\} \cup \{\pm(v_i-w_i),\frac{g-3t}{2}+1\le i\le\frac{g-2t}{2}\} = G\backslash H$,

$\{\pm(s_i-t_i),\frac{g-3t}{2}+1\le i\le\frac{g-2t}{2}\} \cup \{\pm(v_i-w_i)\} = G\backslash H$;

(4) $s_i - t_i = v_i - w_i$, for $1 \le i \le \frac{g-3t}{2}$;

(5) if $v_i - s_i = v_j - s_j$ for $1 \le i \le j \le \frac{g-3t}{2}$, then $i = j$,

if $u_i - r_i = u_j - r_j$ for $1 \le i \le j \le t$, then $i = j$,

$v_i - s_i \ne u_j - r_j$ for $1 \le i \le \frac{g-3t}{2}$, $1 \le j \le t$;

(6) $v_i - s_i \notin H$, $1 \le i \le \frac{g-3t}{2}$, and $u_j - r_j \notin H$, $1 \le j \le t$;

(7) any element $s_i - t_i$, or $v_i - w_i$, where $\frac{g-3t}{2} + 1 \le i \le \frac{g-2t}{2}$, has even order.

OIFS are related to frames as follows.

LEMMA 2.2. *If there exists a pair of $(t,g/t)$-OIFS in $G\backslash H$, then there exists a $(t,g/t+1)$-frame.*

Proof. Suppose (A_1,A_2,B_1,B_2) is a pair of OIFS. We describe the construction informally.

First, construct a square array of side g, with first row A_1 and first column B_1, by cycling diagonally, as usual. There are t transversals made up of singletons. Adjoin a new infinite element to each of these.

Now, add t new rows and t new columns to the array. These side t. The first row of this larger array will be A_1 (with the t infinite elements) and A_2; the first column will be B_1 and B_2. We will cycle the pairs of A_2 down the last t columns. The pairs of B_2 will be cycled across the last t rows in a similar way.

This is done as follows. The translates of a given pair of A_2 will use each group element twice. By property 7, we may divide this set of translates into two sets, so that in each set every group element is used exactly once. Since we have t/2 pairs in B_2, the translates yield the contents of the last t columns of the array as desired.

It may be checked that the array described above is a (t,g/t+1)-frame. The properties (1) - (7) are precisely those that ensure that the array is indeed the desired frame. □

EXAMPLE 2.3. *A (2,6)-frame constructed by intransitive starters. Here* $G = \mathbb{Z}_5 \times \mathbb{Z}_2$, $H = \{0\} \times \mathbb{Z}_2$, *and*

$A_1 = \{\{(1,0),(2,0)\},\{(2,1),(4,1)\},\{(3,0)\},\{(1,1)\}\}$

$A_2 = \{\{(4,0),(3,1)\}\}$,

$B_1 = \{\{(3,0),(4,0)\},\{(1,1),(3,1)\},\{(1,0)\},\{(2,1)\}\}$,

$B_2 = \{\{(2,0),(4,1)\}\}$.

0	21 41 20 40	30 ∞_2 31 ∞_2	10 20 11 21	11 ∞_1 10 ∞_1	40 31 41 30
21 ∞_1 20 ∞_1	1	31 01 30 00	40 ∞_2 41 ∞_2	20 30 21 31	00 41 01 40
30 40 31 41	31 ∞_1 30 ∞_1	2	41 11 40 10	00 ∞_2 01 ∞_2	10 01 11 00
10 ∞_2 11 ∞_2	40 00 41 01	41 ∞_1 40 ∞_1	3	01 21 00 20	20 11 21 10
11 31 10 30	20 ∞_2 21 ∞_2	00 10 01 11	01 ∞_1 00 ∞_1	4	30 21 31 20
20 41 21 40	30 01 31 11	40 11 41 00	00 21 01 20	10 31 11 30	∞

We now describe a method of producing intransitive 2-frame starters from ordinary (strong) 2-frame starters.

LEMMA 2.4. *Suppose* $A = \{\{s_i, t_i\}\}$ *is a strong* $(2,u)$*-frame starter in* $G \backslash H$. *Suppose there exist two pairs* $\{a,b\}$ *and* $\{c,d\}$ *in* A *such that*

(1) $a-b$ *and* $c-d$ *both have even order,*

(2) $a+c \neq s_i+t_i \neq b+d$, *for any* i,

(3) $a+c,\ b+d \notin H$.

Then there exists a pair of OIFS in $G \backslash H$, *so* $u+1 \in F_2$.

Proof. Let $A_1 = A\backslash\{\{a,b\},\{c,d\}\} \cup \{\{c\},\{d\}\}$,

$A_2 = \{\{a,b\}\}$,

$B_1 = -A\backslash\{\{-a,-b\},\{-c,-d\}\} \cup \{\{-a\},\{-b\}\}$,

and $B_2 = \{\{-c,-d\}\}$.

It may be easily verified that (A_1,A_2,B_1,B_2) satisfies all the required properties. □

We now show that we can satisfy the hypotheses of the above lemma for $q \equiv 1 \bmod 4$ a prime power.

LEMMA 2.5. *Suppose* $q \equiv 1 \bmod 4$ *is a prime power. Then* $q+1 \in F_2$.

Proof. Let ω be primitive in $GF(q)$, and let $q = 4t+1$. Define $Q = \{1,\omega^2,\ldots,\omega^{2t-2}\}$. We define A, a strong 2-frame starter in $GF(q) \times \mathbb{Z}_2 \backslash \{0\} \times \mathbb{Z}_2$. Let $A = \{\{(y,1),(y\omega,1)\},\{(y\omega,0),(y\omega^2,0)\}$ $\{(-y,1),(-y\omega,0)\},\{(-y\omega,1),(-y\omega^2,0)\} y\in Q\}$. A is a strong frame starter by [4, Theorem 3.4].

Consider $(a,b) = ((-\omega,1),(-\omega^2,0))$,

and $(c,d) = ((-1,1),(-\omega,0))$.

Then $a-b = (\omega^2-\omega,1)$ and $c-d = (\omega-1,1)$ both have second coordinate equal to 1, and so both have even order.

We must check the condition on the sums. We have $a+c = (-1-\omega,0)$ and $b+d = (-\omega-\omega^2,0)$, so neither is in $\{0\} \times \mathbb{Z}_2$.

The sums of pairs in A are

$\{(y(1+\omega),0),(y\omega(1+\omega),0),(-y(1+\omega),1),(-y\omega(1+\omega),1), y\in Q\}$.

Now $-1-\omega = \omega^{2t}(1+\omega)$, and $-\omega-\omega^2 = \omega^{2t}\cdot\omega(1+\omega)$, so neither $a+c$ nor $b+d$ has appeared as a sum. Lemma 2.4 establishes the result. □

We now describe a construction for 2-frames of orders $u \equiv 3 \bmod 4$. In fact, this construction will work for orders $u \equiv 1 \bmod 4$ as well. In [7], Schellenberg and Vanstone use essentially the same construction to show the existence of some Howell designs. This construction uses skew Room squares (defined in [6]) and pairwise orthogonal Latin squares (POLS) containing a common transversal (defined in [3]).

LEMMA 2.6. *Suppose there exists a skew Room square of order* u, *and suppose there exist two POLS of order* u *containing a common transversal. Then* $u \in F_2$.

Proof. Let R be a skew Room square of order u, on symbol set $\infty \cup \{1,2,\ldots,u\}$. Let S be its skew mate, on the same symbol set. We may suppose that R and S are both standardized with respect to ∞, and the diagonal cells contain $\{\infty,1\},\{\infty,2\},\ldots,\{\infty,u\}$ in each case. Let R_i be the array obtained from R by deleting the contents of the diagonal cells, and replacing all symbols x by ordered pairs (x,i). Let S_i be defined similarly.

Now let L and M be two POLS of order u containing a common transversal. Let L and M be described on symbol set $\{1,2,\ldots,u\}$. We may suppose that the common transversal T is formed by the cells containing $(1,1),(2,2),\ldots,(u,u)$, when L and M are superimposed (the first coordinate is the entry in L; the second is the entry in M). Let L_i be formed by deleting T from L, and replacing any symbols x by (x,i), and let M_j be defined similarly. Consider the following array F.

$R_1 \circ S_2$	
	$L_1 \circ M_2$

$R_1 \circ S_2$ is the array formed by superimposing R_1 and S_2; $L_1 \circ M_2$ is the array formed by superimposing L_1 and M_2. It may be verified that F is a $(2,u)$ frame on $\{1,2,\ldots,u\} \times \{1,2\}$. Notice that the symbols $(i,1)$ and $(i,2)$ are missing from rows and columns i and $u+i$, for $1 \le i \le u$. The intersections of these rows and columns form the empty two-by-two subsquares. □

EXAMPLE 2.7. *A (2,7)-frame. An empty two by two subsquare is indicated.*

	61 21	51 41	62 42	31 11	32 22	52 12							
62 22		01 31	61 51	02 52	41 21	42 32							
52 42	02 32		11 41	01 61	12 62	51 31							
61 41	62 52	12 42		21 51	11 01	22 02							
32 12	01 51	02 62	22 52		31 61	21 11							
31 21	42 22	11 61	12 02	32 62		41 01							
51 11	41 31	52 32	21 01	22 12	42 02								
								61 22	51 42	41 62	31 12	21 32	11 52
							21 62		01 32	61 52	51 02	41 22	31 42
							41 52	31 02		11 42	01 62	61 12	51 32
							61 42	51 62	41 12		21 52	11 02	01 22
							11 32	01 52	61 02	51 22		31 62	21 12
							31 22	21 42	11 62	01 12	61 32		41 02
							51 12	41 32	31 52	21 02	11 22	01 42	

LEMMA 2.8. *There exists a skew Room square if* $7 \le n \le 39$ *and* u *is odd.*

POLS containing a common transversal also exist whenever we need them. The following was established in [3].

LEMMA 2.9. *There exist two POLS of order* u *containing a common transversal if* $u \ne 2,3$, *or* 6.

Summarizing the above we have

COROLLARY 2.10. *If* $7 \le u \le 39$ *and* u *is odd, then* $u \in F_2$.

LEMMA 2.11. *For* $u = 8,12,16,20,24,28$, *and* 32, *there exists a strong 2-frame starter in* $\mathbb{Z}_{2u} \setminus \{0,u\}$, *and hence* $u \in F_2$.

Proof. These starters are listed in the appendix. □

We have constructed all the 2-frames we shall need, except for those of orders 22 and 34.

LEMMA 2.12. $\{22,34\} \subseteq F_2$.

Proof. By computer, we have constructed strong 2-frame starters of order u in $\mathbb{Z}_{2u} \setminus \{0,u\}$ for $u = 21$ and 33. These are listed in the appendix. In both cases, we can find two pairs $\{a,b\}$ and $\{c,d\}$ which satisfy the hypotheses of lemma 2.4. These pairs are:

for $u = 21$, $(a,b) = (6,7)$ and $(c,d) = (37,40)$;

for $u = 33$, $(a,b) = (50,51)$ and $(c,d) = (38,39)$. □

Thus we have determined F_2.

THEOREM 2.13. $F_2 = \{u \ge 5\}$.

Proof. Lemma 2.1, Theorem 1.1 and Lemmata 2.5, 2.10, 2.11 and 2.12 prove the result. □

3. *4-frames*

We first show that $F_4 = \{u \ge 4\}$. Again, we use a PBD-closure result. Let $K_4 = \{4,5,\ldots,12,14,15,18,19,23,27\}$.

LEMMA 3.1. *Suppose* $K_4 \subseteq F_4$. *Then* $F_4 = \{u \ge 4\}$.

Proof. Hanani has shown [5] that there exists a (v,K_4)-PBD if $v \ge 4$. □

Now, by Theorem 1.1, we have $u \in F_4$ if $u \ge 5$ is odd, so we need only consider the even orders.

LEMMA 3.2. $4 \in F_4$.

Proof. In [10], a strong 4-frame starter in $\mathbb{Z}_4 \times \mathbb{Z}_4 \setminus \{(0,0),(0,2),(2,0),(2,2)\}$ is given. □

LEMMA 3.3. $\{8,10,12,14,18\} \subseteq F_4$.

Proof. For $u = 8,10,12,14$, and 18, strong 4-frame starters in $\mathbb{Z}_{4u} \setminus \{0,u,2u,3u\}$ are given in the appendix. □

Thus we have only to construct a 4-frame of order 6.

LEMMA 3.4. *There does not exist a strong 4-frame starter of order 6 in* $\mathbb{Z}_{24}\backslash\{0,6,12,18\}$.

Proof. See [1]. □

Thus we construct a (4,6)-frame by means of intransitive starters.

LEMMA 3.5. *There exists a pair of (4,5)-OIFS in* $\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_2\backslash\{0\}\times\mathbb{Z}_2\times\mathbb{Z}_2$ *; hence* $6 \in F_4$.

Proof. Let $A_1 = \{\{(2,0,0),(4,0,0)\},\{(2,0,1),(4,1,0)\},\{(4,0,1),(3,1,0)\},$
$\{(4,1,1),(3,1,1)\},\{(3,0,0)\},\{(1,0,1)\},\{(1,1,0)\},\{(2,1,1)\}\},$

$A_2 = \{\{(3,0,1),(1,1,1)\},\{(1,0,0),(2,1,0)\}\}$

$B_1 = \{\{(1,0,0),(3,0,0)\},\{(1,1,0),(3,0,1)\},$
$\{(2,1,0),(1,0,1)\},\{(2,1,1),(1,1,1)\},\{(4,1,1)\},$
$\{(2,0,1)\},\{(3,1,0)\},\{(4,0,0)\}\}$

$B_2 = \{\{(2,0,0),(4,0,1)\},\{(4,1,0),(3,1,1)\}\}$

It may be verified in a finite amount of time that (A_1,A_2,B_1,B_2) is a pair of (4,5)-OIFS. □

Thus we may describe F_4.

THEOREM 3.6. $F_4 = \{u\geq 4\}$.

Proof. Lemmata 3.1, 3.2, 3.3, and 3.5. □

We are now able to say exactly when (t,u)-frames exist for $u \geq 6$.

THEOREM 3.7. *Let* $u \geq 6$. *Then a (t,u)-frame does not exist if and only if* t *is odd and* u *is even.*

Proof. Suppose first that u is odd, $u \geq 7$.

If $t \neq 2$ or 6, then a (t,u)-frame exists by Theorem 1.1. By Theorem 2.13, $u \in F_2$. Then applying Lemma 1.7 with $t = 2$ and $s = 3$, we have $u \in F_6$.

Thus, suppose u is even, $u \geq 6$. By Theorem 2.13, $u \in F_2$. Applying Lemma 1.7 with $t = 2$ yields a (t,u)-frame for even t unless $t = 4$ or 12. But Theorem 3.6 gives a (4,u)-frame, and applying Lemma 1.7 with $t = 4$, $s = 3$, we obtain a (12,u)-frame. Finally, no (t,u)-frame exists if t is odd and u is even, by Lemma 1.8. □

4. *Frames of order 4 and 5.*

The problem of determining the existence of (t,u)-frames with u = 4 or 5 is made more difficult by the non-existence of (1,5)- and (2,4)-frames. Lemma 1.7 is the only recursive construction we have which preserves the u-value. However, Lemma 1.7 cannot be used to construct any (p,5)- or (2p,4)-frame, for p an odd prime.

Although we cannot completely solve the problem, we obtain some

Theorem 1.1 states that a (t,5)-frame exists if t is divisible by 2 or 3. We will construct (5,5)- and (7,5)-frames; in conjunction with Lemma 1.7, many frames of order 5 will result. Lemma 1.5 indicates that the above frames cannot be constructed by strong frame starters. However, pairs of orthogonal starters do exist.

LEMMA 4.1. *There exist (5,5)- and (7,5)-frames.*

Proof. A pair of orthogonal 5-frame starters (in $\mathbb{Z}_{25}\setminus\{0,5,10,15,20\}$) is given in the appendix. A (7,5)-frame was constructed in [2]. □

LEMMA 4.2. *If $(t,210) \neq 1$, then there exists a (t,5)-frame.*

Proof. If $(t,6) \neq 1$, then Theorem 1.1 implies the result. Thus we may assume that $t = 5s$ or $7s$, with s odd. Then Lemma 1.7 yields the (t,5)-frame. □

Finally we consider frames of order 4.

LEMMA 4.3. *There exists an (8,4)-frame.*

Proof. Two orthogonal 8-frame starters (in $\mathbb{Z}_{32}\setminus\{0,4,8,12,16,20,24,28\}$) are given in the appendix.

LEMMA 4.4. *If $4|t$ then there exists a (t,4)-frame.*

Proof. Lemmata 3.2 and 1.7 yield all the desired frames except for (8,4)- and (24,4)-frames. The (8,4)-frame exists by Lemma 4.3, and the (24,4)-frame exists by setting $t = 8$, $s = 3$, and $u = 4$, in Lemma 1.7. □

5. *Summary*

Thus we have determined the existence (or non-existence) of all (t,u)-frames with $u \neq 4,5$. Partial results for $u = 4$ and 5 are given. To complete these partial results appears to be difficult.

In closing we would like to mention two related problems. First, when can skew frames be constructed? (A frame is *skew* precisely one, of any two cells, symmetric with respect to the diagonal and not in any Su_i, is empty.) Skew frames are of use in constructing skew Room squares. ([1] and [10]).

Another problem is to find examples of "frame-like" objects, which would have all the properties of frames, except the empty sub-squares Su_i need not all be the same size. Such objects could prove useful in the construction of Howell Designs (see [4; Theorem 7.1]) and (if skew), skew Room squares.

11,12	4,6	2,15	1,13	5,10	3,9	7,14	

(2,12)-strong frame starter

22,21	18,16	17,14	6,10	4,23	7,1	8,15	11,3
5,20	9,19	2,13					

(2,16)-strong frame starter

30,31	3,5	17,20	19,23	7,12	21,15	11,4	10,2
1,24	28,18	6,27	13,25	9,22	26,8	14,29	

(2,20)-strong frame starter

10,9	14,16	29,32	7,3	23,18	34,28	6,39	12,4
24,33	21,11	27,38	8,36	26,13	31,17	30,5	19,35
25,2	15,37	22,1					

(2,24)-strong frame starter

8,7	20,18	36,39	6,2	42,47	27,33	23,16	9,17
28,19	44,34	15,4	38,26	45,10	25,11	31,46	30,14
1,32	5,35	40,21	41,13	43,22	29,3	12,37	

(2,28)-strong frame starter

53,52	14,16	9,12	25,21	49,54	29,35	32,39	7,15
47,38	41,51	6,17	43,55	40,27	37,23	34,19	45,5
50,11	8,26	3,22	24,44	31,10	20,42	46,13	4,36
2,33	18,48	30,1					

(2,32)-strong frame starter

57,56	51,53	49,52	37,41	2,7	15,9	22,29	25,33
3,58	28,18	35,46	59,47	23,10	20,6	45,30	17,1
44,61	8,26	31,12	36,16	54,11	21,63	27,50	38,62
39,14	34,60	5,42	4,40	19,48	43,13	24,55	

(2,21)-strong frame starter

6,7	24,26	37,40	35,31	39,34	18,12	4,11	28,36
23,14	29,19	20,9	13,25	15,2	41,27	3,30	17,1
5,22	32,8	10,33	38,16				

(2,33)-strong frame starter

50,51	55,57	41,44	63,59	10,5	8,2	25,18	31,39
29,38	4,14	26,15	16,28	19,32	23,9	21,6	11,27
47,64	36,54	42,61	40,60	58,37	12,56	30,53	3,45
24,65	43,17	7,46	34,62	20,49	22,52	48,13	1,35

26,27	29,31	21,18	23,19	7,12	9,3	15,22	2,25
4,14	6,17	1,13	30,11	10,28	20,5		

(4,12)-strong frame starter

33,34	20,22	44,41	45,1	6,11	47,5	28,35	13,21
4,43	40,30	8,19	10,23	39,25	32,17	18,2	31,14
7,37	27,46	9,29	15,42	16,38	26,3		

(4,10)-strong frame starter

33,32	3,5	26,23	21,25	39,34	17,11	12,19	37,29
26,7	9,38	1,13	22,35	18,4	27,2	24,8	14,31
28,6	15,36						

(4,14)-strong frame starter

7,8	51,49	23,26	20,16	32,27	18,12	36,43	17,9
41,50	39,29	4,15	46,34	45,2	25,40	53,13	30,47
10,48	3,22	1,37	52,31	21,55	5,38	35,11	19,44
24,54	6,33						

(4,18)-strong frame starter

30,31	8,6	9,12	32,28	51,46	41,47	16,23	34,42
67,4	69,59	38,49	44,56	48,61	10,24	3,60	63,7
22,5	33,14	45,65	50,71	35,57	17,66	43,19	62,15
11,37	29,2	26,70	68,39	25,55	21,52	53,13	40,1
20,58	64,27						

(4,22)-strong frame starter

56,57	79,81	25,28	47,51	23,18	59,53	42,49	61,69
7,86	83,73	16,27	46,34	76,1	31,45	17,32	78,62
19,36	29,11	60,41	68,48	87,20	52,75	67,3	84,21
30,4	39,12	5,33	15,74	14,72	71,40	26,58	63,8
10,64	55,2	13,65	43,80	35,85	77,38	54,6	50,9
24,70	37,82						

Two orthogonal (5,5)-frame starters

8,9	17,19	24,2	12,16	1,7	21,3
6,7	1,3	16,19	18,22	2,8	14,21

6,14	13,22	18,4	11,23
9,17	4,13	12,23	24,11

21,22	25,27	14,17	1,6	5,11	23,30
14,15	19,21	27,30	2,7	3,9	26,1
10,19	31,9	2,13	26,7	15,29	3,18
29,6	13,23	11,22	5,18	17,31	10,25

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