On the existence of certain SOLS with Holes

D. R. Stinson
Computer Science and Engineering Department
and Center for Communication and Information Science
University of Nebraska
Lincoln, NE 68588-0115, U.S.A.
stinson@bibd.unl.edu

L. Zhu
Department of Mathematics
Suzhou University
Suzhou 215006
Peoples' Republic of China

Abstract

We consider a pair of MOLS (mutually orthogonal Latin squares) having holes, corresponding to missing sub-MOLS, which are disjoint and spanning. If the two squares are mutual transposes, we say that we have SOLS (self-orthogonal Latin squares) with holes. It is shown that a pair of SOLS with n holes of size $h \geq 2$ exist if and only if $n \geq 4$ and it is also shown that a pair of SOLS with n holes of size 2 and one hole of size 3 exist for all $n \geq 4, n \neq 13, 15$.

As an application, we prove a result concerning intersection numbers of transversal designs with four groups.

1 Introduction

For formal definitions of MOLS (mutually orthogonal Latin squares) with holes, the reader is referred to [8]. Let $HMOLS(h_1{}^{n_1} h_2{}^{n_2} \ldots h_k{}^{n_k})$ denote a pair of MOLS of order $\sum_{i=1}^{k} n_i h_i$ from which n_i sub-MOLS of order h_i are "missing" ($1 \leq i \leq k$), and in which these subsquares are disjoint and spanning. The *type* T of the HMOLS is defined to be $h_1{}^{n_1} h_2{}^{n_2} \ldots h_k{}^{n_k}$. (It is also convenient to think of the type as a multiset.) An $HMOLS(h_1{}^{n_1}$

JCMCC **15** (1994), pp. 33-45

$h_2^{n_2} \ldots h_k^{n_k}$) in which the two squares are mutual transposes is defined to be an $HSOLS(h_1^{n_1} h_2^{n_2} \ldots h_k^{n_k})$.

The following results concerning $HSOLS(h^n)$ have been proved.

Theorem 1.1 *1. There exists an $HSOLS(1^n)$ if and only if $n \geq 4$, $n \neq 6$.*

2. For $h \geq 2$, there exists an $HSOLS(h^n)$ if and only if $n \geq 4$, except possibly for $HSOLS(7^6)$ and $HSOLS(13^6)$.

3. There exists an $HSOLS(1^{v-n}n^1)$ if $v \geq 3n+1$ and $(v,n) \neq (6,1)$ or $(6m+2, 2m)$, where $1 \leq m < 50$.

Proof. 1) is shown in [5]; 2) in [15]; and 3) in [18]. □

For some results on HMOLS, we refer to [2,3,8,9,10,12,13,14,15,16,17].

Here, we construct the previously unknown $HSOLS(7^6)$ and $HSOLS(13^6)$, and we also study the spectrum of $HSOLS(2^n 3^1)$. We show that $HSOLS(2^n 3^1)$ exist for all $n \geq 4$, except possibly for $n = 13, 15$. An application is given in Section 4.

2 Constructions for HSOLS

Our main direct construction is based on difference methods. The following is [15, Lemma 2.3].

Lemma 2.1 *Let G be an Abelian group, let H be a subgroup of G, and let X be any set disjoint from G. Suppose there exists a set of 4-tuples $B \subseteq (G \cup X)^4$ which satisfies the following properties:*

1. *for each i, $1 \leq i \leq 4$, and each $x \in X$, there is a unique $b \in B$ with $b_i = x$ (b_i denotes the ith co-ordinate of b).*
2. *no $b \in B$ has two co-ordinates in X.*
3. *for each i, j ($1 \leq i < j \leq 4$) and each $a \in G \setminus H$, there is a unique $b \in B$ with $b_i, b_j \in G$ and $b_i - b_j = a$.*
4. *$(b_1, b_2, b_3, b_4) \in B$ if and only if $(b_2, b_1, b_4, b_3) \in B$.*

Then there exist $HSOLS(h^{g/h}|X|^1)$, where $g = |G|$ and $h = |H|$.

Figure 1: $HSOLS(2^4 3^1)$

	7	3	c		2	b	a	1	6	5
a		0	4	c		3	b	2	7	6
b	a		1	5	c		4	3	0	7
5	b	a		2	6	c		4	1	0
	6	b	a		3	7	c	5	2	1
c		7	b	a		4	0	6	3	2
1	c		0	b	a		5	7	4	3
6	2	c		1	b	a		0	5	4
2	3	4	5	6	7	0	1			
3	4	5	6	7	0	1	2			
7	0	1	2	3	4	5	6			

Example 1 *Suppose we take $G = \mathbf{Z}_8$, $H = \{0, 4\}$, $X = \{a, b, c\}$, and let B be the following set of twelve 4-tuples:*

0	a	1	2
a	0	2	1
0	b	6	3
b	0	3	6
0	c	5	7
c	0	7	5
0	1	7	a
1	0	a	7
0	2	3	b
2	0	b	3
0	5	2	c
5	0	c	2

These generate the $HSOLS(2^4 3^1)$ depicted in Figure 1.

Lemma 2.2 *Let G, H, and X be as in Lemma 2.1. Suppose there exists a set of 4-tuples $B \subseteq (G \cup X)^4$ which satisfies the following properties:*

1. $\bigcup_{b \in B} \{\{b_1, b_2, b_3, b_4\} : b = (b_1, b_2, b_3, b_4)\} = (G \backslash H) \cup X$

2. *no* $b \in B$ *has two co-ordinates in* X

3. *for each* i, j $(1 \leq i < j \leq 4)$ *and each* $a \in G \backslash H$, *there is a unique* $b \in B$ *with* $b_{i'}, b_{j'} \in G$ *and* $b_{i'} - b_{j'} = a$, *where either* $\{i, j\} = \{i', j'\}$ *or* $\{i, j, i', j'\} = \{1, 2, 3, 4\}$.

Then there exist $HSOLS(h^{g/h}|X|^1)$, *where* $g = |G|$ *and* $h = |H|$.

Proof. Replace each 4-tuple $b = (b_1, b_2, b_3, b_4)$ by four 4-tuples: $b^1 = (b_1, b_2, b_3, b_4)$, $b^2 = (b_2, b_1, b_3, b_4)$, $b^3 = (b_1, b_2, b_4, b_3)$, and $b^4 = (b_2, b_1, b_4, b_3)$. Then, the conditions of Lemma 2.1 are satisfied. □

Remark. The "orthogonal array with holes" obtained from such a collection of 4-tuples is conjugate invariant under the Klein group K_4; see [10].

We now present several further applications of Lemmas 2.1 and 2.2 . In applications of Lemma 2.1, we give only one of each "pair" of 4-tuples.

$HSOLS(2^5 3^1)$ (Lemma 2.1)

$$(0,4,1,3), (0,a,2,1), (0,b,4,8), (0,c,6,9), (0,1,3,a)$$
$$(0,2,8,b), (0,3,7,c)$$

$HSOLS(2^7 3^1)$ (Lemma 2.1)

$$(0,1,2,4), (0,2,1,6), (0,3,11,12), (0,a,5,9), (0,b,8,11)$$
$$(0,c,10,2), (0,8,13,a), (0,9,12,b), (0,10,6,c)$$

$HSOLS(2^9 3^1)$ (Lemma 2.1)

$$(0,1,2,4), (0,2,1,6), (0,3,5,11), (0,4,7,14), (0,5,15,16)$$
$$(0,a,12,15), (0,b,13,5), (0,c,16,12), (0,6,14,a)$$
$$(0,10,17,b), (0,11,6,c)$$

$HSOLS(2^{11} 3^1)$ (Lemma 2.1)

$$(0,1,2,4), (0,2,1,6), (0,3,5,9), (0,4,7,16), (0,5,13,19)$$

$(0,6,21,14),(0,8,15,5),(0,a,16,13),(0,b,18,17),(0,c,20,12)$

$(0,7,17,a),(0,12,10,b),(0,13,9,c)$

$HSOLS(2^{13}11^1)$ (Lemma 2.1) This gives an $HSOLS(2^{17}3^1)$ by filling in the hole of size 11 with $HSOLS(2^4 3^1)$.

$(0,1,3,a),(0,2,5,19),(0,3,9,b),(0,4,14,c),(0,5,25,d)$

$(0,6,20,e),(0,7,23,f),(0,8,7,g),(0,9,10,h),(0,10,22,i)$

$(0,11,18,j),(0,12,16,k),(0,a,21,22),(0,b,17,15),(0,c,8,5)$

$(0,d,4,8),(0,e,24,19),(0,f,15,9),(0,g,11,18),(0,h,6,23)$

$(0,i,1,11),(0,j,2,17),(0,k,12,24)$

$HSOLS(2^{14}13^1)$ (Lemma 2.1). This gives an $HSOLS(2^{19}3^1)$ by filling in the hole of size 13 with $HSOLS(2^5 3^1)$.

$(0,1,3,a),(0,2,8,b),(0,3,15,c),(0,4,7,d),(0,5,6,e)$

$(0,6,13,f),(0,7,17,g),(0,8,16,h),(0,9,27,i),(0,10,19,j)$

$(0,11,22,k),(0,12,25,l),(0,13,12,m),(0,a,24,25)$

$(0,b,21,23),(0,c,18,15),(0,d,23,19),(0,e,10,5)$

$(0,f,11,17),(0,g,1,22),(0,h,4,24),(0,i,2,21)$

$(0,j,26,16),(0,k,9,26),(0,l,20,4),(0,m,5,20)$

$HSOLS(2^{17}15^1)$. This gives an $HSOLS(2^{23}3^1)$ by filling in the hole of size 15 with $HSOLS(2^6 3^1)$.

$(0,1,8,a),(0,2,4,b),(0,3,6,c),(0,4,9,d),(0,5,11,e)$

$(0,6,18,f),(0,7,25,g),(0,8,30,h),(0,9,29,i),(0,10,31,j)$

$(0,11,26,k),(0,12,20,l),(0,13,24,m),(0,14,15,n)$

$(0,15,28,p),(0,16,32,22),(0,a,23,24),(0,b,12,14)$

$(0,c,33,30),(0,d,27,23),(0,e,2,31),(0,f,19,25),(0,g,16,9)$

$(0,h,7,33),(0,i,1,10),(0,j,3,27),(0,k,21,32),(0,l,13,26)$

$(0,m,5,19),(0,n,14,29),(0,p,10,28)$

$HSOLS(2^6 3^1)$ (Lemma 2.2)

$$(0,1,2,4),(0,a,1,5),(0,b,4,9),(0,c,5,2)$$

$HSOLS(2^8 3^1)$ (Lemma 2.2)

$$(0,1,2,4),(0,3,1,7),(0,a,5,10),(0,b,6,13),(0,c,7,11)$$

$HSOLS(2^{10} 3^1)$ (Lemma 2.2)

$$(0,1,2,4),(0,3,1,8),(0,4,7,15),(0,a,4,13)$$

$$(0,b,6,11),(0,c,8,14)$$

$HSOLS(2^{12} 3^1)$ (Lemma 2.2)

$$(0,1,2,4),(0,3,1,7),(0,4,15,10),(0,7,10,18),(0,a,5,15)$$

$$(0,b,7,16),(0,c,8,19)$$

$HSOLS(2^{14} 3^1)$ (Lemma 2.2)

$$(0,1,2,4),(0,3,1,7),(0,4,9,16),(0,5,20,11),(0,11,5,18)$$

$$(0,a,10,20),(0,b,11,19),(0,c,13,25)$$

$HSOLS(2^{14} 11^1)$ (Lemma 2.2) This gives rise to an $HSOLS(2^{18} 3^1)$ by filling in the hole of size 11 with $HSOLS(2^4 3^1)$.

$$(0,1,3,5),(0,a,1,4),(0,b,2,6),(0,c,5,10),(0,d,6,12)$$

$$(0,e,7,15),(0,f,8,17),(0,g,9,19),(0,h,10,21),(0,i,11,27)$$

$$(0,j,12,25),(0,k,13,20)$$

Hence, we have

Lemma 2.3 *There exists an $HSOLS(2^n3^1)$ for*

$$n \in \{4,5,6,7,8,9,10,11,12,14,16,17,18,19,23\}.$$

Lemma 2.4 *There exists an $HSOLS(7^6)$.*

Proof. Apply Lemma 2.1 with the following 4-tuples:

$$(0,1,29,31),(0,2,28,14),(0,3,26,9),(0,4,6,18),(0,6,2,13)$$

$$(0,7,34,1),(0,8,17,33),(0,9,27,g),(0,11,a,3),(0,12,b,21)$$

$$(0,13,c,4),(0,14,d,24),(0,16,e,23),(0,17,f,19),(0,a,16,24)$$

$$(0,b,12,11),(0,c,8,21),(0,d,7,13),(0,e,22,29)$$

$$(0,f,32,34),(0,g,11,7)$$

□

Our recursive construction for HSOLS uses group-divisible designs. A *group-divisible design* (or GDD) is a triple $(X,\mathcal{G},\mathcal{A})$, which satisfies the following properties:

1. $\mathcal{G}$ is a partition of X into subsets called *groups*
2. $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point
3. every pair of points from distinct groups occurs in a unique block.

A *transversal design* $TD(k,n)$ is a GDD with kn points, k groups of size n, and n^2 blocks of size k. (A $TD(k,n)$ is equivalent to $k-2$ MOLS of order n.)

The following construction is essentially [8, Lemma 2.2].

Lemma 2.5 *Suppose $(X,\mathcal{G},\mathcal{A})$ is a GDD and let $w : X \to \mathbf{Z}^+ \cup \{0\}$. Suppose there exist HSOLS of type $w(A)$ for every $A \in \mathcal{A}$. Then there exists HSOLS of type $\{\sum_{x\in G} w(x) : G \in \mathcal{G}\}$.*

We shall use the following specialization.

Corollary 2.6 *Suppose $(X, \mathcal{G}, \mathcal{A})$ is a GDD where every block has size at least four. Suppose also that there is an $HSOLS(2^{|G|}3^1)$ for every $G \in \mathcal{G}$. Then there is an $HSOLS(2^{|X|}3^1)$.*

Proof. Let ∞ be a new point (not in X). Define a new GDD $(X', \mathcal{G}', \mathcal{A}')$, where $X' = X \cup \{\infty\}$, $\mathcal{G}' = \{\{y\} : y \in X'\}$, and $\mathcal{A}' = \mathcal{A} \cup \{G \cup \{\infty\} : G \in \mathcal{G}\}$ (i.e. we add the new point ∞ to each group in $\mathcal{G}$, and form a new GDD with groups of size one). Define the following weighting w of X':

$$\begin{aligned} w(y) &= 2 \quad \text{if } y \in X \\ w(y) &= 3 \quad \text{if } y = \infty. \end{aligned}$$

Now, apply Lemma 2.5. For each $A \in \mathcal{A}$, we require $HSOLS(2^{|A|})$, which exist by Theorem 1.1. For each $A = G \cup \{\infty\}$ ($G \in \mathcal{G}$), we require $HSOLS(2^{|G|}3^1)$, which exists by assumption. We obtain $HSOLS(2^{|X|}3^1)$, as desired. □

Theorem 2.7 *For $h \geq 2$, there exists an $HSOLS(h^n)$ if and only if $n \geq 4$.*

Proof. We need only give an $HSOLS(13^6)$. Take a $TD(6,7)$ having two disjoint blocks A and B. Delete all points in B. Give weight 3 to each point in A and weight 2 to all other points. Since the GDD has block sizes 5 and 6, we need input HSOLS of types $3^6, 2^6, 2^5, 2^4 3^1$ and $2^5 3^1$, which are all known. This gives an $HSOLS(13^6)$. □

3 $HSOLS(2^n 3^1)$

We begin by noting a trivial necessary condition for the existence of HMOLS.

Lemma 3.1 *If there exists $HMOLS(a^n b^1)$, then $n \geq 1 + 2b/a$.*

Proof. Trivial. □

Corollary 3.2 *If there exists an $HSOLS(2^n 3^1)$, then $n \geq 4$.*

In order to close the spectrum of $SOLS(2^n3^1)$, we use the GDD construction (Corollary 2.6). We shall use the following classes of GDDs.

Lemma 3.3 *[1, Lemma 2.5] Suppose there is a* $TD(6,m)$ *and* $4 \le r \le m$. *Then there is a GDD of group-type* $4^{m-r}5^r$ *in which every block has size at least* 4.

Lemma 3.4 *[1, Lemma 2.6] Suppose there is a* $TD(5+r,m)$ *and* $r \ge 1$. *Then there is a GDD of group-type* $4^{m-r}5^r$ *in which every block has size at least* 4.

Lemma 3.5 *For* $n \ge 48$, *there exists an* $HSOLS(2^n3^1)$.

Proof. Write $n = 6m + r$, where m is odd and $4 \le r \le m$ (this can be done in a unique way). There is a $TD(6,m)$ by [4]. Apply Lemma 3.3, obtaining a GDD of group-type $4^{m-r}5^r$ in which every block has size at least 4. Then, apply Corollary 2.6. □

Lemma 3.6 *There is an* $HSOLS(2^n3^1)$ *for* $32 \le n \le 45$.

Proof. Apply Lemma 3.3 with $m = 7, 4 \le r \le 7$; with $m = 8, 4 \le r \le 7$; and with $m = 9, 4 \le r \le 9$. Then, apply Corollary 2.6. □

Lemma 3.7 *There is an* $HSOLS(2^n3^1)$ *for* $n = 46$ *and* 47.

Proof. Apply Lemma 3.4 with $m = 11, r = 2$ and 3. Then, apply Corollary 2.6. □

We present constructions for several other GDDs in Table 1. In each case, we obtain $HSOLS(2^n3^1)$ from Corollary 2.6.

Lemma 3.8 *There is an* $HSOLS(2^n3^1)$ *for* $n = 21, 22$, *and* 31.

Proof. Take a $TD(6,5)$ and let A be a block. Keep the points in A and delete all other points in the last two groups. Give weight 3 to the point which is the intersection of A and the last group. Give weight 2 to other points. We obtain an $HSOLS(2^{21}3^1)$.

In a $TD(8,7)$, keep the points in one block A and delete other points in the last four groups. A similar weighting gives an $HSOLS(2^{31}3^1)$.

In a $(21,5,1)$−BIBD give weight 3 to five points in a block and weight 2 to all other points. We get an $HSOLS(2^{16}15^1)$, and then an $HSOLS(2^{22}3^1)$ by filling in the size 15 hole with an $HSOLS(2^63^1)$. □

Table 1: Constructions of $HSOLS(2^n 3^1)$

n	group-type of GDD	construction
16	4^4	$TD(4,4)$
20	4^5	$TD(5,4)$
24	$5^4 4^1$	$TD(5,5)$ minus a point
25	5^5	$TD(5,5)$
26	$4^4 5^2$	$TD(6,5)$ minus one point each from the first four groups such that no three are in the same block
27	$4^3 5^3$	$TD(6,5)$ minus one point each from the first three groups such that they are not in the same block
28	$5^4 4^2$	$TD(6,5)$ minus one point from each of two groups
29	$5^5 4^1$	$TD(6,5)$ minus a point
30	5^6	$TD(6,5)$

We then obtain the following existence theorem.

Theorem 3.9 *An $HSOLS(2^n 3^1)$ exists if and only if $n \geq 4$, except possibly for $n = 13$ or 15.*

4 An application

$HSOLS(2^n)$ played an essential role in a construction for $2-$perfect $m-$cycle systems [11]. In this section, we give an alternate proof of a known result on intersections of transversal designs using the $HSOLS(2^n 3^1)$ that we have constructed.

Suppose $(X, \mathcal{G}, \mathcal{A}_1)$ and $(X, \mathcal{G}, \mathcal{A}_2)$ are both $TD(4, m)$ (having the same group set). The *intersection* of the two TD's is defined to be the number of common blocks, i.e. $|\mathcal{A}_1 \cap \mathcal{A}_2|$. Define $TI(m)$ to be the set of all possible intersection numbers of two $TD(4, m)$'s. An almost complete determination of the sets $TI(m)$ is proved by Colbourn and Royle in [7]. This is accomplished using incomplete transversal designs. Following the remark made in [7, p. 46], we prove a similar result (for even m) using

$HSOLS(2^n 3^1)$. (Actually, it suffices to use $HMOLS(2^n 3^1)$ in this construction.)

The following result can be proved in a similar manner as the constructions in [7]. We state it without proof.

Theorem 4.1 *Suppose there is an* $HMOLS(2^n 3^1)$. *Let* $0 \le \alpha \le n$, *let* $\beta \in \{0,1,3\}$, *let* $\delta_i \in \{0,2,8\}$ $(1 \le i \le n)$, *and let* $\epsilon \in \{0,1,3,7,15\}$. *Then*

$$\alpha(4n+2) + \beta(2n) + \sum_{i=1}^{n} \delta_i + \epsilon + 1 \in TI(2n+4).$$

Now, simple arithmetic yields the following corollary:

Corollary 4.2 *Suppose* $m \ge 12$ *is even,* $1 \le t \le m^2$, *and* $t \ne m^2 - s$ *where*

$$s \in \{1,2,3,4,5,7,9,10,11,13,17,19\}.$$

Then $t \in TI(m)$.

Remarks.

1. It is easy to see that $0 \in TI(m)$ for any $m \ge 3$, $m \ne 6$.
2. From the proof of [6, Lemma 2.1], it follows that $m^2 - s \notin TI(m)$ if $s \in \{1,2,3,4,5,7\}$.
3. In [7, Lemma 3.6], it is proved that if $m = 9$ or $m \ge 12$, $m \ne 14$, $0 \le t \le m^2$, and $t \ne m^2 - s$, where

$$s \notin \{1,2,3,4,5,7,10,11,13,19\},$$

then $t \in TI(m)$. So the case $m = 14$ is the only "new" case covered by Theorem 4.1.

Acknowledgements

D. R. Stinson's research is supported by NSF Grant CCR-9121051. L. Zhu's research is supported by an NSFC grant.

References

[1] A. Assaf, E. Mendelsohn, and D. R. Stinson. On resolvable coverings of pairs by triples. *Utilitas Math.* **32** (1987), 67–74.

[2] F. Bennett, L. Wu, and L. Zhu. Conjugate orthogonal Latin squares with equal-sized holes. *Ann. Discrete Math.* **34** (1987), 65-80.

[3] F. Bennett, L. Wu, and L. Zhu. On the existence of COLS with equal-sized holes. *Ars Combin.* **26B** (1988), 5-36.

[4] Th. Beth, D. Jungnickel, and H. Lenz. *Design Theory*, Bibliographisches Institut, Zurich, 1985.

[5] R. K. Brayton, D. Coppersmith, and A. J. Hoffman. Self-orthogonal Latin squares, *Teorie Combinatorie, Proc. Rome Conf.* (1976), 509-517.

[6] C. J. Colbourn, D. G. Hoffman, and C. C. Lindner. Intersections of $S(2,4,v)$ designs. *Ars Combin.* **33** (1992), 97-111.

[7] C. J. Colbourn and G. F. Royle. Supports of $(v,4,2)$ designs. *Le Matematiche* **XLV** (1990), 39-60.

[8] J. H. Dinitz and D. R. Stinson. MOLS with holes. *Discrete Math.* **44** (1983), 145-154.

[9] E. R. Lamken. The existence of three orthogonal partitioned incomplete Latin squares of type t^n. *Discrete Math.* **89** (1991), 231-251.

[10] C. C. Lindner, R. C. Mullin, and D. R. Stinson. On the spectrum of resolvable orthogonal arrays invariant under the Klein group K_4. *Aequationes Math.* **26** (1983), 176-183.

[11] C. C. Lindner and C. A. Rodger. 2$-$perfect $m-$cycle systems. *Discrete Math.* **104** (1992), 83-90.

[12] R. C. Mullin and D. R. Stinson. Holey SOLSSOMs. *Utilitas Math.* **25** (1984), 159-169.

[13] R. C. Mullin and L. Zhu. The spectrum of $HSOLSSOM(h^n)$ where h is odd. *Utilitas Math.* **27** (1985), 157-168.

[14] D. R. Stinson and L. Zhu. On sets of three MOLS with holes. *Discrete Math.* **54** (1985), 321-328.

[15] D. R. Stinson and L. Zhu. On the existence of MOLS with equal-sized holes. *Aequationes Math.* **33** (1987), 96-105.

[16] D. R. Stinson and L. Zhu. On the existence of three MOLS with equal-sized holes. *Australasian Journal of Math.* **4** (1991), 33-47.

[17] L. Zhu. Existence of holey SOLSSOMs of type 2^n. *Congr. Numer.* **45** (1984), 295-304.

[18] L. Zhu. Existence of incomplete self-orthogonal latin squares $ISOLS(6m+2, 2m)$. To appear in *Ars Combin.*