Classroom Notes

An explicit formulation of the second Johnson bound

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Abstract

In this note, we review the Johnson bounds for packings, and derive an explicit formulation of the second Johnson bound.

A (k, v)*-packing* is defined to be a pair $(X, \mathcal{B})$, where X is a v-set and $\mathcal{B}$ is a set of k-subsets of X, such that no pair of elements of X occurs in more than one k-subset in $\mathcal{B}$. Elements of X are called *points* and elements of $\mathcal{B}$ are called *blocks*. The *packing number* $D(k, v)$ is defined to be the maximum number of blocks in any (k, v)-packing.

Packings and packing numbers have been the object of considerable study; see Mills and Mullin [4] for a recent survey. The "counting" bound on $D(k, v)$ is usually known as the *first Johnson bound* [2, Theorem 2]; it states that

$$D(k,v) \leq \left\lfloor \frac{v}{k} \left\lfloor \frac{v-1}{k-1} \right\rfloor \right\rfloor .$$

This bound tends to be a very good bound when $v \geq k^2 - k + 1$. However, the first Johnson bound is less effective when $v \leq k^2 - k$. For such *subgeometric* packings, the *second Johnson bound* yields much more accurate bounds (see, for example, [7,8,5,6]).

Bulletin of the ICA, Volume 8 (1993) 86-92

We now give a brief description of the second Johnson bound [2, Eq. (6)] (cf. [1, Theorem 8], [3, Ch. 17, Thm. 3], [5, pp. 219-220] and [4, Eq. (10)]). This description follows the approach of Stanton [5].

Suppose $\mathcal{D} = (X, \mathcal{B})$ is a (k, v)—packing, and denote $b = |\mathcal{B}|$. Denote $X = \{x_i : 1 \leq i \leq v\}$, and for $1 \leq i \leq v$, define r_i to be the number of blocks in which x_i occurs. For any block $B \in \mathcal{B}$, define the *weight* of B to be

$$w(B) = b - 1 - \sum_{\{i: x_i \in B\}} (r_i - 1).$$

Evidently, the weight of B is the number of blocks in $\mathcal{B}$ disjoint from B, and hence $w(B) \geq 0$. Define the *weight* of $\mathcal{D}$ to be

$$w(\mathcal{D}) = \sum_{B \in \mathcal{B}} w(B).$$

Substituting, we obtain

$$w(\mathcal{D}) = b(b-1) - \sum_{i=1}^{v} r_i(r_i - 1).$$

Since every block has non-negative weight, so too does $\mathcal{D}$ have non-negative weight. Hence, we obtain the following necessary condition for the existence of a (k, v)—packing with b blocks:

$$b(b-1) \geq \sum_{i=1}^{v} r_i(r_i - 1). \qquad (1)$$

Now, it is clear that

$$\sum_{i=1}^{v} r_i = bk.$$

It is easy to see that $\sum_{i=1}^{v} r_i(r_i - 1)$ is minimized if the r_i's are as equal as possible. More precisely, if we compute $bk = Qv + R$, where R and Q are integers, $0 \leq R \leq v - 1$, then

$$\begin{aligned} \sum_{i=1}^{v} r_i(r_i - 1) &\geq R(Q+1)Q + (v - R)Q(Q-1) \\ &= vQ(Q-1) + 2RQ. \end{aligned}$$

Substituting in Inequality (1), we obtain the following necesssary existence condition for the existence of a (k, v)−packing with b blocks:

$$b(b-1) \geq vQ(Q-1) + 2RQ, \qquad (2)$$

where $bk = Qv + R, 0 \leq R \leq v-1$.

We can use Inequality (2) to obtain an upper bound on $D(k, v)$ by taking $b = 1, 2, \ldots,$ until a contradiction is obtained. If b_0 is the smallest value of b that yields a contradiction in Inequality (2), then $D(k, v) \leq b_0 - 1$. The resulting bound is the second Johnson bound.

The purpose of this note is to obtain an explicit formulation of the second Johnson bound. We will prove two bounds based on Inequality (1). The first form is simpler but slightly weaker than the second. The following bound is shown in [2, Theorem 3] (cf. [1, Eq. (18)], [3, Ch. 17, Thm.2] and [4, Eq. (11)]); the case of equality is discussed in [5, Theorem A].

Theorem 1 $D(k, v) \leq (k-1)v/(k^2 - v)$, *with equality occurring if and only if there exists a* $(\frac{(k-1)v}{k^2-v}, \frac{k^2-k}{k^2-v}, 1)$*−BIBD.*

Proof: Since $\sum_{i=1}^{v} r_i = bk$, Cauchy's inequality tells us that

$$\sum_{i=1}^{v} {r_i}^2 \geq (bk)^2/v.$$

If we substitute this into Inequality (1), then we get

$$b(b-1) \geq (bk)^2/v - bk.$$

This simplifies to give the stated upper bound on $D(k, v)$.

Let's look at when the bound will be met. Equality occurs (in Cauchy's inequality) if and only if $r_i = (bk)/v = (k^2-k)/(k^2-v)$ for $1 \leq i \leq v$. Further, it must be the case that $w(B) = 0$ for every block $B \in \mathcal{B}$. This second condition means that any two blocks intersect in a point.

First, suppose that there exists a (k, v)−packing with $b = (k-1)v/(k^2 - v)$. The dual incidence structure will consist of b points, v blocks of size $(k^2-k)/(k^2-v)$, such that every point occurs in k blocks and every pair of points occurs in one block. Hence it is a BIBD with the stated parameters. Conversely, if a BIBD with these parameters exists, then the dual is a (k, v)−packing with $(k-1)v/(k^2-v)$ blocks. □

Our next objective is to produce an explicit bound using the "$bk = Qv + R$" idea. As far as I know, such a bound has not appeared in the literature. We begin by rewriting Inequality (2). If we replace R by $bk - Qv$, then we obtain the following (equivalent) inequality:

$$b(2kQ - b + 1) \leq vQ(Q + 1). \qquad (3)$$

Consider the function $f(b) = b(2kQ - b + 1)$ for fixed integers k and Q. f is a quadratic function, having its maximum at $b = kQ + 1/2$. However, for integral values of b, f is maximized when $b = kQ$ or $kQ + 1$. Note also (for fixed integers k, v and Q) that we are interested in the integers b such that

$$\frac{Qv}{k} \leq b \leq \frac{Qv + v - 1}{k}.$$

We will call this set of integers the *Q−th interval.*

Lemma 2 *Suppose k and v are positive integers, where $2 \leq k < v < k^2$. Further, suppose $Q \leq (v - k)/(k^2 - v)$ is a positive integer and b is an integer. Then Inequality (3) is satisfied:*

$$b(2kQ - b + 1) \leq vQ(Q + 1).$$

Proof: We have

$$\begin{aligned} b(2kQ - b + 1) &= f(b) \\ &\leq f(kQ) \\ &= kQ(kQ + 1). \end{aligned}$$

However, for $2 \leq k < v < k^2$, we have:

$$\begin{aligned} kQ(kQ + 1) &\leq vQ(Q + 1) \\ \Leftrightarrow Q(k^2 - v) &\leq v - k \\ \Leftrightarrow Q &\leq \frac{v - k}{k^2 - v}. \end{aligned}$$

□

Lemma 3 *Suppose k and v are positive integers, where $2 \leq k < v < k^2$. Further, suppose $Q \geq (k^2 - k)/(k^2 - v)$ is a positive integer. Then $f(Qv/k) \geq vQ(Q + 1)$, and equality occurs if and only if $Q = (k^2 - k)/(k^2 - v)$.*

Proof: We have

$$
\begin{aligned}
f\left(\frac{Qv}{k}\right) &= \frac{Qv}{k}\left(2kQ - \frac{Qv}{k} + 1\right) \\
&= Qv\left(Q\left(2 - \frac{v}{k^2}\right) + \frac{1}{k}\right).
\end{aligned}
$$

However, for $2 \le k < v < k^2$, it is easy to see that

$$
\begin{aligned}
Q\left(2 - \frac{v}{k^2}\right) + \frac{1}{k} &\ge Q + 1 \\
\Leftrightarrow Q\left(1 - \frac{v}{k^2}\right) &\ge 1 - \frac{1}{k} \\
\Leftrightarrow Q &\ge \frac{k^2 - k}{k^2 - v}.
\end{aligned}
$$

□

Lemma 4 *Suppose k and v are positive integers, where $2 \le k < v < k^2$, $Q \ge 1$ is an integer, and b is an integer such that $Qv/k \le b \le 1 + Qv/k$. Then $f(b) > f(Qv/k)$.*

Proof: Since $Qv/k < Qk$, the result follows since f is a quadratic function attaining its maximum at $Qk + 1/2$. □

The behaviour of the second Johnson bound depends on whether $(v - k)/(k^2 - v)$ is an integer. If it is, then we will see that the bound given by Theorem 1 is in fact the same as the second Johnson bound. (We assume throughout that $2 \le k < v < k^2$.)

Case 1: Suppose $Q_0 = (v - k)/(k^2 - v)$ is an integer. From Lemma 2, Inequality (3) is satisfied for any integer b in the Q_0–th interval. Next, observe that $(Q_0 + 1)v/k = (k - 1)v/(k^2 - v) = kQ_0 + 1$ is also an integer. By Lemma 3, for $b = (Q_0 + 1)v/k$, Inequality (3) is met with equality (where $Q = Q_0 + 1$). Finally, by Lemma 4, Inequality (3) is violated for $b = 1 + (Q_0 + 1)v/k$. So, the second Johnson bound in this case is $(k - 1)v/(k^2 - v)$, agreeing with Theorem 1.

Case 2: Suppose $(v - k)/(k^2 - v)$ is not an integer. Let $Q_0 = \left\lceil \frac{v-k}{k^2-v} \right\rceil$. From Lemma 2, Inequality (3) is satisfied for all b in the $(Q_0 - 1)$–st interval; and from Lemmas 3 and 4, Inequality (3) is violated for $b = \lceil (Q_0+1)v/k \rceil$ (with $Q = Q_0 + 1$). So we have to consider the integers b in the Q_0–th interval:

$$\left\lceil \frac{Q_0 v}{k} \right\rceil \leq b \leq \left\lfloor \frac{Q_0 v + v - 1}{k} \right\rfloor .$$

Since Q (= Q_0) is now fixed, we can solve Inequality (3) to obtain an upper bound $b \leq \lfloor b_0 \rfloor$, where

$$b_0 = \frac{2kQ_0 + 1 - \sqrt{(2kQ_0+1)^2 - 4vQ_0(Q_0+1)}}{2}.$$

(From the proof of Lemma 2, Inequality (3) is violated for $b = Q_0 k$, so b_0 is a real number and $b_0 \leq Q_0 k$.)

We need to show that b_0 is in fact the correct value of the second Johnson bound. This will follow if we can show that b_0 is indeed in the Q_0–th interval. By an argument similar to the the proof of Lemma 3, Inequality (3) is satisfied for $b = Q_0 v/k$, so $b_0 \geq Q_0 v/k$.

Hence, let us consider the the value of f when $b = (Q_0+1)v/k$:

$$\begin{aligned} f\left(\frac{(Q_0+1)v}{k}\right) &= \frac{(Q_0+1)v}{k}\left(2kQ_0 - \frac{(Q_0+1)v}{k} + 1\right) \\ &= \frac{(Q_0+1)v}{k^2}\left(2k^2Q_0 - (Q_0+1)v + k\right). \end{aligned}$$

Now, Inequality (3) will be violated if and only if

$$\begin{aligned} \frac{(Q_0+1)v}{k^2}\left(2k^2Q_0 - (Q_0+1)v + k\right) &> vQ_0(Q_0+1) \\ \Leftrightarrow 2k^2Q_0 - (Q_0+1)v + k &> k^2Q_0 \\ \Leftrightarrow Q_0 &< \frac{v-k}{k^2-v}. \end{aligned}$$

Since $Q_0 > (v-k)/(k^2-v)$, it follows that Inequality (3) is violated for $b = (Q_0+1)v/k$. This means that $b_0 < (Q_0+1)v/k$, as desired. So we have proved the following result, which complements Theorem 1:

Theorem 5 *Suppose k and v are integers, $2 \leq k < v < k^2$, and $(v-k)/(k^2-v)$ is not an integer. Let $Q_0 = \left\lceil \frac{v-k}{k^2-v} \right\rceil$. Then*

$$D(k,v) \leq \left\lfloor \frac{2kQ_0 + 1 - \sqrt{(2kQ_0+1)^2 - 4vQ_0(Q_0+1)}}{2} \right\rfloor .$$

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