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THE CONSTRUCTION AND USES OF FRAMES

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ABSTRACT. We define frames, a generalization of Room designs. Several constructions for frames are given. Frames can be constructed directly, by starter methods, and recursively, by means of a Moore construction, and PBD closure. We apply frames to the construction of Howell designs, and Room designs (to improve the lower bounds for the number of pairwise orthogonal symmetric Latin squares).

1. *Introduction*

Let T and U be sets with $|T| = t$, $|U| = u$. A tu by tu array S will be called a t-*frame of order* u if it enjoys the following properties:

(1) Each cell is either empty or contains an unordered pair of elements of $U \times T$,

(2) There exist U empty t by t subsquares of S, no two of them containing any cell in the same row or column. These subsquares will be denoted Su_i. (It will usually be convenient to place the Su_i's on the diagonal of S),

(3) A row or column of S which meets Su_i contains each element of $(U \setminus \{u_i\}) \times T$ exactly once, and contains no element of $\{u_i\} \times T$,

(4) Each unordered pair of elements $\{(u_1,t_1),(u_2,t_2)\}$ with $u_1 \neq u_2$, occurs in a unique cell of S. By counting it follows that no pair of the type $\{(u,t_1),(u,t_2)\}$ occurs in the array.

Informally, a t-frame of order u is a Room square of side tu "missing" u disjoint Room subsquares of order t. (A definition of Room square is given below). The "missing" subsquares need not exist.

It is convenient to index the cells of S by elements of $(U \times T)^2$, so that the cells of the rows and columns meeting any Su_i are $(\{u_i\} \times T) \times (U \times T)$ and $(U \times T) \times (\{u_i\} \times T)$, respectively.

An n-*dimensional* t-*frame of order* u is an n-dimensional cube of side tu, which satisfies property 1 above and such that each two-dimensional projection is a t-frame of order u. Analogous to the two dimensional case label the cells of such a frame by $(U \times T)^n$. For brevity, we may refer to an n-dimensional t-frame of order u as an (n,t,u)-*frame*, and a (2-dimensional) t-frame of order u as a (t,u)-*frame*.

Frames have already been introduced in the literature, but were defined less generally than here. In [16, 21] a "frame" of order u refers to a special type of $(2,2,u)$ frame which possesses a "skew" property describing the distribution of empty cells. Such frames have

EXAMPLE 1.1. *A 3-frame of order 5.*

	00	01	02	10	11	12	20	21	22	30	31	32	40	41	42
00							41 31				40 12	21 42	30 10	32 20	11 22
01		0						42 32		22 40		41 10	12 20	31 11	30 21
02									40 30	42 11	20 41		31 22	10 21	32 12
10	40 20	42 30	21 32							01 41				00 22	31 02
11	22 30	41 21	40 31		1						02 42		32 00		01 20
12	41 32	20 31	42 22									00 40	02 21	30 01	
20		10 32	41 12	00 30	02 40	31 42							11 01		
21	42 10		11 30	32 40	01 31	00 41		2						12 02	
22	12 31	40 11		01 42	30 41	02 32									10 00
30	21 11				20 42	01 22	10 40	12 00	41 02						
31		22 12		02 20		21 40	42 00	11 41	10 01		3				
32			20 10	22 41	00 21		11 02	40 01	12 42						
40				31 21				30 02	11 32	20 00	22 10	01 12			
41					32 22		12 30		31 00	02 10	21 01	20 11		4	
42						30 20	32 01	10 31		21 12	00 11	22 02			

been used in recursive constructions for skew Room squares, for example in [15] and [16].

A *Room square* of odd order u is a square array of side u, each cell of which is either empty or contains an unordered pair of elements chosen from a set of u+1 elements, such that each element occurs exactly once in each row and column, and each pair of elements occurs exactly once in the array. A *Room n-cube* of order u is an n-dimensional cube, each two-dimensional projection of which is a Room square. In [10] and [13] the term Room n-design was used instead of Room n-cube.

Let a Room n-cube be described with symbols $U \cup \{\infty\}$ where $|U| = u$ and $\infty \notin U$. If the contents of all cells containing ∞ are removed, one obtains an $(n,1,u)$ frame. Thus frames are a generalization of Room n-cubes and Room squares.

Room squares have also been generalized in another direction. A *Howell design* $H(S,2n)$ is a square array of side s, each cell of which is either empty or contains an unordered pair of elements chosen from a set of size $2n$, such that each element occurs exactly once in each row and column, and no pair of elements occurs in more than cell. Thus a Room square of order u is an $H(u,u+1)$.

Frames are of use in recursive constructions for Room n-cubes and Howell designs. We consider several such applications of frames in this paper.

Given u, it is natural to ask what the largest $n = \nu(u)$ is so that there exists a Room n-cube of order u. This question can equivalently be described as asking for the largest number n; either of pairwise orthogonal symmetric Latin squares of side u, or of pairwise orthogonal 1-factorizations of K_{u+1}. For definitions and proofs of these equivalences, see [10] and [13].

It is known that $\nu(3) = \nu(5) = 1$, and $\nu(u) \geq 3$, if u is odd, $7 \leq u \leq 1000$, and $u \neq 9$. Much better bounds for ν can be obtained if u is a prime power, and the resulting Room n-cubes are also useful in recursive constructions. We will consider these recursive constructions and derive a list of lower bounds for $\nu(u)$, $u < 1000$, in section 6.

For Howell designs, the general existence question is still open. Frames are of particular use in considering the family of Howell designs $H(s,s+k)$, given any fixed k, as s varies. We consider applications of frames to Howell designs in Section 7.

However, it is important to first construct the frames we shall need. This is done in Sections 2, 3 and 4.

In Section 5 we consider two dimensional frames in more detail.

2. *Frames and Frame Starters*

Just as Room squares and Room n-cubes can be constructed from starters, see [13,18], frames can be constructed from a generalization of starters.

Let G be an additive abelian group of order g, and let H be a subgroup of G of order t, with $g-t$ even. A *t-frame starter of* *order* g/t in $G\backslash H$ (or a *$(t,g/t)$-frame starter*) is a set of pairs $A = \{\{s_i,t_i\}, 1 \le i \le \frac{g-t}{2}\}$ satisfying the properties:

(1) $\{s_i\} \cup \{t_i\} = G\backslash H$

(2) $\{\pm(s_i-t_i)\} = G\backslash H$.

A 1-frame starter is a *starter*.

Let $A = \{\{s_i,t_i\}\}$ and $B = \{\{u_i,v_i\}\}$ be two frame starters. We may assume that $t_i-s_i = v_i-u_i$, for $1 \le i \le \frac{g-t}{2}$. We say that A and B are *orthogonal* frame starters if $u_i-s_i = u_j-s_j$ implies $i = j$, and $u_i-s_i \notin H$ for all i. Several frame starters are *pairwise orthogonal* if each pair of them is orthogonal. Finally, a frame starter $A = \{\{s_i,t_i\}\}$ is *strong* if $s_i+t_i = s_j+t_j$ implies $i = j$, and $s_i+t_i \notin H$ for all i.

The following is immediate.

LEMMA 2.1. *If $A = \{\{s_i,t_i\}\}$ is a strong frame starter then A and $-A = \{\{-s_i,-t_i\}\}$ are orthogonal frame starters.*

The special frame starter $P = \{\{s_i,t_i\}\}$ where $s_i = -t_i$ for all i is called the *patterned frame starter*. It follows that this is only a starter in $G\backslash H$ if $|G|$ is odd. Analogous to the result for strong starters [13] is the following.

LEMMA 2.2. *If there is a strong frame starter in $G\backslash H$, $|G|$ odd, then there are 3 pairwise orthogonal frame starters in $G\backslash H$.*

Proof. Let $A = \{\{s_i,t_i\}\}$ be a strong frame starter in $G\backslash H$. By Lemma 2.1 A and $-A$ are orthogonal frame starters. We show that A and P, the patterned frame starter, are orthogonal. If $\{s_i,t_i\} \in A$ then the pair in P with the same difference is $\{\frac{1}{2}(s_i-t_i), \frac{1}{2}(t_i-s_i)\} = \{u_i,v_i\}$, so $u_i-s_i = -\frac{1}{2}(s_i+t_i)$. Since A is strong and $|G|$ is odd, $-\frac{1}{2}(s_i+t_i) \ne -\frac{1}{2}(s_j+t_j)$ if $i \ne j$. Furthermore $\frac{1}{2}(s_i+t_i) \notin H$ since $s_i+t_i \notin H$. Thus A and P are orthogonal. Similarly $-A$ and P are orthogonal. □

Anderson and Gross [1] have considered a more general notion, "partial starter", of which frame starters are a special case. They limit their investigation to the case of strong partial starters, and applications to Howell designs. We will discuss this further in Section 7.

The connection between orthogonal frame starters and frames is given by the following theorem. This construction is essentially that of Anderson and Gross [1; Theorem 1].

THEOREM 2.3. *If there exist n pairwise orthogonal t-frame starters in $G\backslash H$ with $|G| = g$ and $|H| = t$, then there exists an (n,t,u)-frame, where $u = g/t$.*

Proof. We give the proof for $n = 2$. The general case proceeds in a similar way.

Let $A = \{\{s_i,t_i\}\}$ and $B = \{\{u_i,v_i\}\}$ be two orthogonal frame starters, with $t_i-s_i = v_i-u_i$ for all i. Let $a_i = u_i-s_i = v_i-t_i$.

Now let $K = \{g_1,g_2,...,g_u\}$ be a set of distisnct representatives of the u cosets of H in G. Let $\phi:G\to K\times H$ be the natural bijection defined by $\phi(g) = (k,h)$ if and only if $g = kh$ with $k \in K$, $h \in H$.

Define S, a (t,u) frame on $K\times H$ as follows. For any i, $1 \le i \le \frac{g-t}{2}$, for any $g \in G$ put the ordered pair $\{\phi(u_i+g),\phi(v_i+g)\}$ in cell $\{\phi(g),\phi(g-a_i)\}$. □

We have a comment regarding the proof of the above theorem. The construction is really the same as the standard one for obtaining Room squares from starters. The bijection ϕ was used in order that the frame could be constructed on the direct product of two sets (since that is how we defined frames). We feel that this definition of frame facilitates the description of the recursive constructions of section 4. Note that if $G = G_1\times G_2$, and $H = \{0\}\times G_2$, then ϕ may be taken to be the identity. Many of the frame starters we construct in the next section have this form, so the frame can be constructed very easily from the starter.

In the next section we give several constructions for frame starters of odd order. In the remainder of this section we mention a few limitations to the starter method for frames.

First, notice that if $G\backslash H$ contains an element of order two, then there is no frame starter in $G\backslash H$. For, if $2g = 0$ then $g = -g$, so g cannot appear exactly once as a difference.

Our second observation concerns frame starters of orders $u \equiv 2$ or 3 mod 4. This result is a slight generalization of Anderson [3; Theorems 9 and 10]; the proof is basically a parity argument and goes through without change.

THEOREM 2.4. *Suppose $u \equiv 2$ or 3 mod 4 and t is odd. Then there does not exist a 2t-frame starter of order u.*

Our next observation concerns strong frame starters of order 5.

THEOREM 2.5. *Suppose t is odd. Then there does not exist a strong t-frame starter of order 5.*

Proof. Suppose there exists a strong t-frame starter A in $G\backslash H$, G an abelian group of odd order $5t$ and H a subgroup of order t. Now $G/H \cong \mathbb{Z}_5$, so we may define the canonical homomorphism $\phi: G \to \mathbb{Z}_5$. If $\{s_i, t_i\} \in A$ we say that the pair $\{s_i, t_i\}$ has *type* $(k, \ell) \in \mathbb{Z}_5 \times \mathbb{Z}_5$, where $\ell = k+1$ or $k+2$, and $\{\phi(s_i), \phi(t_i)\} = \{k, \ell\}$. Also, say an element $g \in G$ has *type* $\phi(g) \in \mathbb{Z}_5$.

For $i = 0,1,2,3,4$ and $j = 1,2$, suppose there are a_{ij} pairs of the type $(k, k+j)$. Now $a_{01} = a_{02} = a_{41} = a_{32} = 0$ since no element g of type 0 $(g \in H)$ can occur in A. Also $a_{21} = a_{42} = 0$ since these pairs have sums of type 0. There are t pairs in A with differences of type $\neq 1$, so $t = \sum_{i=0}^{4} a_{i1} = a_{11} + a_{31}$. Also, there are t pairs containing elements of type 1, so $t = a_{01} + a_{11} + a_{12} + a_{42} = a_{11} + a_{12}$. Similarly, there are t pairs containing an element of type 3, so $t = a_{21} + a_{31} + a_{12} + a_{32} = a_{31} + a_{12}$. From the three above equations we obtain $a_{11} = a_{31} = a_{12} = t/2$, a contradiction, since t is odd. □

Even though, for example, a strong 3-frame starter of order 5 does not exist, it is still possible the construct a pair of orthogonal 3-frame starters of order 5.

The following was constructed by hand. These starters generate the (3,5)-frame of Example 1.1.

EXAMPLE 2.6. *Two orthogonal 3-frame starters of order 5 in* $\mathbb{Z}_{15}\backslash\{0,5,10\}$.

difference	s_i, t_i	u_i, v_i	$u_i - s_i = v_i - t_i$
1	1,2	2,3	1
2	9,11	11,13	2
3	3,6	9,12	6
4	8,12	4,8	11
6	13,4	1,7	3
7	7,14	14,6	7

We conclude this section with a strong 2-frame starter of even order. We will make use of this example in a later section.

EXAMPLE 2.7. *A strong 2-frame starter of order 8 in* $\mathbb{Z}_{16}\backslash\{0,8\}$.

$\{11,12\}, \{4,6\}, \{2,15\}, \{1,13\}, \{5,10\}, \{3,9\}, \{7,14\}$.

3. *Some Classes of Frame Starters.*

In this section we construct sets of pairwise orthogonal frame starters.

In order to give a general frame-starter construction, a special scheme in the additive group $(\mathbb{Z}_2)^s$ must first be defined. We make use of the canonical identification between elements in $(\mathbb{Z}_2)^s$ and non-negative integers less than 2^s written in base 2. The reader is cautioned that all arithmetic is still in $(\mathbb{Z}_2)^s$.

A *doubling-scheme* in $(\mathbb{Z}_2)^s$, $\mathcal{D} = (C,D)$, consists of two lists C and D each containing 2^{s+1} elements in $(\mathbb{Z}_2)^s$. One list $C = (c_i | 0 \le i \le 2^{s+1}-1)$ is defined by

$$c_{2i} = i \text{ (written base 2)} \qquad 0 \le i \le 2^{s-1} - 1$$
$$c_{2i+1} = c_{2i} \qquad 0 \le i \le 2^{s-1} - 1$$
$$c_i = i - 2^s \text{ (written base 2)} \qquad 2^s \le i \le 2^{s+1} - 1$$

The other list $D = (d_i | 0 \le i \le 2^{s+1}-1)$ is defined by

$$d_i = i \text{ (written base 2)} \qquad 0 \le i \le 2^s - 1$$
$$d_{2i} = i \text{ (written base 2)} \qquad 2^{s-1} \le i \le 2^s - 1$$
$$d_{2i+1} = d_{2i} \qquad 2^{s-1} \le i \le 2^s - 1$$

The following is a doubling scheme in $(\mathbb{Z}_2)^2$.

EXAMPLE 3.1.

$c_0 = 00$	$d_0 = 00$
$c_1 = 00$	$d_1 = 01$
$c_2 = 01$	$d_2 = 10$
$c_3 = 01$	$d_3 = 11$
$c_4 = 00$	$d_4 = 10$
$c_5 = 01$	$d_5 = 10$
$c_6 = 10$	$d_6 = 11$
$c_7 = 11$	$d_7 = 11$

An important propery of a doubling scheme is given in the following lemma, and can be proven using a simple induction argument.

LEMMA 3.2. *If C and D are as defined above, then* $\{d_i - c_i | 0 \le i \le 2^s - 1\} = \{d_i - c_i | 2^s \le i \le 2^{s+1} - 1\} = (\mathbb{Z}_2)^s$.

This implies $d_i - c_i \ne d_j - c_j$ if $0 \le i,j \le 2^s - 1$ or if $2^s \le i,j \le 2^{s+1}-1$. Also, notice that $d_i - c_i = d_i + c_i$ since the group is $(\mathbb{Z}_2)^s$.

For G a multiplicative group of order $4t$, define a *quarter set* to be a set $Q \subseteq G$ such that $|Q| = t$ and such that there is some $a \in G$ with $Q \cup -Q \cup aQ \cup -aQ = G$. Call a the *multiplier* for Q. As an example, let $G = GF(q)^*$, $G = 4t$, with multiplicative generator ω. Then $Q = \{\omega^{2n} | 0 \le n \le t\}$ is a quarter set, since it is clear that $Q \cup -Q \cup \omega Q \cup -\omega Q = G$. Thus we have

LEMMA 3.3. *If $q = 4t+1$ is a prime power then there is a quarter-set in $GF(q)^*$.*

The following theorem will be a useful tool in constructing Howell designs. It generalizes the result of Anderson [2; Theorem 8] which requires $q = 5$. For $s = 1$ this result was shown in [21].

THEOREM 3.4. *There exists a strong 2^s-frame starter of order $q = 2^k t+1$, q prime power, for all $s \geq 1$, $k \geq 2$, $t \geq 1$, t odd.*

Proof. Let $\mathcal{D} = (C,D)$ be a doubling scheme in $(\mathbb{Z}_2)^s$ and let $K = GF(q) \times \mathbb{Z}_2^s$. Let Q be a quarter set in $GF(q)^*$ with multiplier $a \neq 1$. Define

$$S'_a = \left\{ \left. \begin{matrix} \{(x,c_i),(ax,d_i)\}, & 0 \leq i \leq 2^s-1, & 2|i \\ \{(-x,c_i),(-ax,d_i)\}, & 0 \leq i \leq 2^s-1, & 2\nmid i \\ \{(-ax,c_i),(-a^2x,d_i)\}, & 2^s \leq i \leq 2^{s+1}-1, & 2|i \\ \{(ax,c_i),(a^2x,d_i)\}, & 2^s \leq i \leq 2^{s+1}-1, & 2\nmid i \end{matrix} \right| x \in Q \right\}$$

We will show that S'_a is a strong $(2^s,q)$ frame starter in $K \setminus (\{0\} \times \mathbb{Z}_2^s)$.

First, note that the number of pairs defined is $2^{s+1} \cdot \frac{q-1}{4} = 2^{s-1}(q-1)$, so the number of elements of K in these pairs is $2 \cdot 2^{s-1}(q-1) = 2^s(q-1) = |GF(q) \setminus 0 \times (\mathbb{Z}_2^s)|$. We show that no element of K occurs in more than one pair. From the definition of Q, it is seen that the only possibilities for duplication are if $(x,c_i) = (\pm a^2 z, d_j)$ $x,z \in Q$, $0 \leq i \leq 2^s-1$, $2^s \leq j \leq 2^{s+1}$, or if $(ax,d_i) = (ax,c_j)$ for $0 \leq i \leq 2^s-1$, $2^s \leq j \leq 2^{s+1}-1$. However, by the structure of the doubling-scheme neither of these possibilities can occur. Thus every element in $(GF(q) \setminus 0) \times (\mathbb{Z}_2)^s$ is in exactly one pair in S'_a.

Now, consider the differences between the elements in the pairs of S'_a. Using Lemma 3.2, the set of all differences arising from pairs of type $\{(x,c_i),(ax,d_i)\}$ or $\{(-x,c_i),(-ax,d_i)\}$ is $\pm(a-1)Q \times \mathbb{Z}_2^s$. Thus since $\pm a(a-1)Q \cup \pm(a-1)Q = GF(q) \setminus 0$, every element of $GF(q) \setminus 0 \times \mathbb{Z}_2^s$ occurs as a difference of a pair in S'_a. By counting, it is seen that every element in $GF(q) \setminus 0 \times (\mathbb{Z}_2)^s$ occurs as a difference of exactly one pair in S'_a. Thus we have shown that S'_a is a $(2^s,q)$-frame starter.

In order to show that S'_a is a strong frame starter, consider the sums of the pairs in S'_a. The sums of the first coordinates are $x(a+1)$, $-x(a+1)$, $ax(a+1)$, $-ax(a+1)$, for all $x \in Q$. These are from the sets $(a+1)Q$, $-(a+1)Q$, $(a+1)aQ$ and $-(a+1)aQ$, respectively, which are known to partition the group $GF(q) \setminus 0$. No two sums of pairs in S'_a with different first coordinate can be the same. Now consider two pairs of the form $\{(x,c_i),(ax,d_i)\}$ and $\{(x,c_j),(ax,d_j)\}$ with the same first coordinate sum. From the definition of the starter $i \leq 2^s-1$ if and only if $j \leq 2^s-1$, and therefore by Lemma 3.2, $c_i+d_i \neq c_j+d_j$. Also, since $a \neq -1$, $(0,i)$ is never a sum for any $i \in (\mathbb{Z}_2)^s$. Thus S'_a is a strong $(2^s,q)$-frame starter. □

COROLLARY 3.5. *There is a $(2,2^s,q)$ frame for all $q \equiv 1$ modulo 4 a prime power, and $s \geq 1$.*

EXAMPLE 3.6. *We give a strong 4-frame starter of order 5 in $GF(5)\times\mathbb{Z}_2\times\mathbb{Z}_2$.*
$\{(1,0,0),(2,0,0)\},\{(4,0,0),(3,0,1)\},\{(1,0,1),(2,1,0)\},\{(4,0,1),(3,1,1)\},$
$\{(3,0,0),(1,1,0)\},\{(2,0,1),(4,1,0)\},\{(3,1,0),(1,1,1)\},\{(2,1,1),(4,1,1)\}.$

For the next theorem we will define a different quarter set Q. Let $q = 2^k t+1$ be a prime power with $t \geq 3$, t odd, and $k \geq 2$. Let $G = GF(q)^*$ and $C_0 \subseteq G$ be the subgroup of index 2^k. Let $C_0, C_1, \ldots, C_{2^k-1}$ be the cyclotomic classes of index 2^k (cosets of C_0) and let $\Delta = 2^{k-1}$. Define $Q = C_0 \cup C_2 \cup \ldots \cup C_{\Delta-2}$. For any $a \in C_n$, n odd, $Q \cup -Q \cup aQ \cup -aQ = G$. Let $Q_a = \frac{1}{a-1}Q$, for $a \neq 1$. Thus both Q and Q_a are quarter sets with multiplier a. The following theorem enables one to construct sets of pairwise orthogonal $(2,q)$-frame starters.

THEOREM 3.7. *If $q = 2^k t+1$ is a prime power with $t \geq 3$ odd and $k \geq 2$, then there exist t pairwise orthogonal $(2,q)$-frame starters in $(GF(q)\times\mathbb{Z}_2)\setminus(0\times\mathbb{Z}_2)$.*

Proof. Let Q, Q_a and Δ be as defined above. Also let $\mathcal{D} = (C,D)$ be a doubling scheme in $\mathbb{Z}_2$ (i.e. $c_0 = c_1 = c_2 = 0$, $c_3 = 1$, $d_0 = 0$, and $d_1 = d_2 = d_3 = 1$). For a fixed n odd, $0 < n < 2^k$, and for $a \in C_n$, define S'_a as in Theorem 3.4. That is:

$$S'_a = \left\{ \begin{array}{l|l} \{(x,0),\ (ax,0)\} & \\ \{(-x,0),\ (-ax,1)\} & \\ \{(-ax,0),(-a^2x,1)\} & x \in Q_a \\ \{(ax,1),\ (a^2x,1)\} & \end{array} \right\}$$

As in the proof of Theorem 3.4, for every $a \in C_n$, S'_a is a 2-frame starter of order q. It need only be shown that if $a,b \in C_n$, $a \neq b$, then S'_a is orthogonal to S'_b.

Consider the differences in each pair in S'_a: $(ax,0)-(x,0) = (x(a-1),0) \in Q \times \{0\}$, $(-ax,1)-(-x,0) \in -Q \times \{1\}$, $(-a^2x,1)-(-ax,0) \in -aQ \times \{1\}$, and $(a^2x,1)-(ax,1) \in aQ \times \{0\}$. Q is a quarter-set with multiplier a thus these differences partition $((GF(q) \setminus(\{0\})\times Z_2)$. Also, since $a,b \in C_n$, $aQ = bQ$. So if a pair in S'_a and a pair in S'_b have the same difference then the two pairs must be of the same type (of the four possible types) and the differences must be taken in the same direction. So let

$\{(x,\alpha_1),(ax,\alpha_1')\} \in S_a'$ $\quad$ $\{(z,\alpha_3),(az,\alpha_3')\} \in S_a'$

$\{(y,\alpha_2),(by,\alpha_2')\} \in S_b'$ $\quad$ $\{(w,\alpha_4),(bw,\alpha_4')\} \in S_b'$

such that

$$(ax,\alpha_1')-(x,\alpha_1) = (by,\alpha_2')-(y,\alpha_2) \qquad (1)$$
$$(az,\alpha_3')-(z,\alpha_3) = (bw,\alpha_4')-(w,\alpha_4)$$

We also assume that $(x,\alpha_1) \neq (z,\alpha_3)$. (2)

Then from (1)

$$ax-x = by-y$$
$$az-z = bw-w$$

so $\quad (a-1)(x-z) = (b-1)(y-w)$.

Thus, since $a-1 \neq b-1$, either

(i) $x-z \neq y-w$ and so S_a' is orthogonal to S_b', or

(ii) $x-z = y-w = 0$.

If $x=z$, then $\alpha_1 \neq \alpha_3$ by (2). But since the second coordinate is a function of the first coordinate, $x=z$ implies $\alpha_1 = \alpha_3$ a contradiction. Thus S_a' is orthogonal to S_b' for all $a,b \in C_n$. □

COROLLARY 3.8. *If $q = 2^k t+1$, is a prime power with $t \geq 3$ odd, and $k \geq 2$, then there exists a $(t,2,q)$-frame.*

EXAMPLE 3.9. *Let $q = 29 = 4.7+1$. We have that 2 is a generator of GF(29) and that $C_1 = \{2,3,19,14,21,17,11\}$. We give 2 of the 7 orthogonal (2,29) frame starters constructed by Theorem 3.6.*

$S_2' = \{(1,0),(2,0)\},\{(16,0),(3,0)\},\{(24,0),(19,0)\},\{(7,0),(14,0)\}$
$\{(25,0),(2,1)\},\{(23,0),(17,0)\},\{(20,0),(11,0)\}$
$\{(28,0),(27,1)\},\{(13,0),(26,1)\},\{(5,0),(10,1)\},\{(22,0),(15,1)\}$
$\{(4,0),(8,1)\},\{(6,0),(12,1)\},\{(9,0),(18,1)\}$
$\{(27,0),(25,1)\},\{(26,0),(23,1)\},\{(10,0),(20,1)\},\{(15,0),(1,1)\}$
$\{(8,0),(16,1)\},\{(12,0)(24,1)\},\{(18,0),(7,1)\}$
$\{(2,1),(4,1)\},\{(3,1),(6,1)\},\{(19,1),(9,1)\},\{(14,1),(28,1)\}$
$\{(21,1),(13,1)\},\{(17,1),(5,1)\},\{(11,1),(22,1)\}$.

$S_3' = \{(15,0),(16,0)\},\{(8,0),(24,0)\},\{(12,0),(7,0)\},\{(18,0),(25,0)\}$
$\{(27,0),(23,0)\},\{(26,0),(20,0)\},\{(10,0),(1,0)\}$
$\{(14,0),(13,1)\},\{(21,0),(5,1)\},\{(17,0),(22,1)\},\{(11,0),(4,1)\}$
$\{(2,0),(6,1)\},\{(3,0),(9,1)\},\{(19,0),(28,1)\}$
$\{(13,0),(10,1)\},\{(5,0),(15,1)\},\{(22,0),(8,1)\},\{(4,0),(12,1)\}$
$\{(6,0),(18,1)\},\{(9,0),(27,1)\},\{(28,0),(26,1)\}$
$\{(16,1),(19,1)\},\{(24,1),(14,1)\},\{(7,1),(21,1)\},\{(25,1),(17,1)\}$
$\{(23,1),(11,1)\},\{(20,1),(2,1)\},\{(1,1),(3,1)\}$.

For completeness we list one other class of frame starters.

THEOREM 3.10. *If* $q = 2^k t+1$ *is a prime power with* $t \geq 3$ *odd and* $k \geq 1$ *then there exists* t *pairwise orthogonal 1-frame starters of order* q *(and thus a* $(t,1,q)$*-frame).*

Proof. A 1-frame starter is just a starter. The existence of this class of orthogonal starters was proved in [6]. □

4. *Recursive Constructions for frames.*

In this section we give two recursive constructions for frames. The first construction is a result on the PBD closure of certain classes of frames; the second is a general Moore-type construction.

Let v be a positive integer, and let K be a set of positive integers. A pair $(X,\mathcal{B})$, where $\mathcal{B}$ is a set of subsets of X, is said to be a (v,K)-PBD (or pairwise balanced design) provided $|X| = v$, $B \in \mathcal{B}$ implies $|B| \in K$, and for any distinct x_1, x_2 in X, there is a unique $B \in \mathcal{B}$ with $\{x_1, x_2\} \subseteq B$. A set A of positive integers is said to be *PBD-closed* if $v \in A$ whenever there exists a (v,A)-PBD.

In [16], it was shown that the orders of 2-frames form a PBD-closed set. We give a more general result.

THEOREM 4.1. $F_{d,t} = \{u \mid$ *a* (d,t,u) *frame exists*$\}$ *is PBD-closed.*

Proof. Let $(X,\mathcal{B})$ be a $(v,F_{d,t})$-PBD, and let Y be any set of size t. For any $B \in \mathcal{B}$, let T_B be a $(d,t,|B|)$-frame on $B \times Y$.

We will now construct S, a (d,t,v)-frame on $X \times Y$. Consider a cell $C = ((x_i,y_i), 1 \leq i \leq d)$. If $x_i = x_j$ for some i,j, $1 \leq i \leq j \leq d$, define $S(C)$ to be empty. Otherwise, let $C_0 = \{x_i, 1 \leq i \leq d\}$. If there is no $B \in \mathcal{B}$ such that $C_0 \subseteq B$, then define $S(C)$ to be empty. Otherwise, there is exactly one $B \in \mathcal{B}$ with $C_0 \subseteq B$. Then define $S(C) = T_B(C)$.

It may be checked that S is a (d,t,v)-frame. □

Before describing our second recursive construction, we need to define some terms. We will make use of pairwise orthogonal Latin squares (POLS) and subsquares (sub-POLS). For definitions, see [12]. This construction makes use of frames containing sub-frames. Let B be an (n,k,v)-frame on symbol set $Q \times S$, and let $R \subseteq Q$, $|R| = w$. If the sub-array B_1 induced by $R \times S$ is itself an (n,k,w)-frame (on $R \times S$), we say that B_1 is an (n,k,w)-*sub-frame* of B.

We note that any (n,k,v)-frame contains an $(n,k,1)$-sub-frame (a $k \times k$ empty array), and an $(n,k,0)$-sub-frame. These will be useful in deriving corollaries to the following construction.

THEOREM 4.2. *(A Moore-type construction). Suppose the following exist:*

(1) *An* (n,ℓ,u)*-frame.*

(2) *An (n,k,v)-frame containing an (n,k,w)-sub-frame.*

(3) *n POLS of order $\frac{k(v-w)}{\ell}$.*

Then an $(n,k,u(v-w)+w)$-frame exists.

Proof. Let A be an (n,ℓ,u)-frame on $P \times T$, and let B be an (n,k,v)-frame on $Q \times S$ with an (n,k,w)-sub-frame B_1 on $R \times S$. Here $|T| = \ell$, $|P| = u$, $|S| = k$, $|Q| = v$, $|R| = w$, and $Q \subseteq R$. We will describe D, an $(n,k,u(v-w)+w)$-frame on symbol set $((P\times Q\backslash R)\cup R)\times S$.

Let $Q \setminus R \times S = \bigcup_{t\in T} X_t$ be an arbitrary partition of $Q \setminus R \times S$ into ℓ disjoint sets X_t, each of size $\frac{k(v-w)}{\ell}$.

Let Z be any set of size $\frac{k(v-w)}{\ell}$, and let $\phi_t : X_t \to Z$ be bijections, for $t \in T$. Finally, let $L_1,\ldots,L_n$ be n POLS of order $\frac{k(v-w)}{\ell}$, on symbol set Z, having rows and columns indexed by Z.

We now describe the construction for D.

Pick a cell C. If $C \in (R\times S)^n$, let $D(C) = B_1(C)$. If $C \in ((\{p\}\times Q\backslash R)\cup R)\times S)^n$ for some $p \in P$, but $C \notin (R\times S)^n$, define $D(C) = \{(p,q_1,s_1),(p,q_2,s_2)\}$ if $B(C) = \{(q_1,s_1),(q_2,s_2)\}$, and define $D(C) = \{(p,q,s_1),(r,s_2)\}$ if $B(C) = \{(q,s_1),(r,s_2)\}$.

Suppose C is not one of the cells described above. If $C \notin (P\times Q\backslash R\times S)^n$, leave it empty.

So, let $C = ((p_i,q_i,s_i),1\le i\le n)$, $p_i \in P$, $q_i \in Q\backslash R$, $s_i \in S$, for $1\le i\le n$. Let $C' = ((p_i,t_i),1\le i\le n)$, where $(q_i,s_i) \in X_{t_i}$; $1\le i\le n$. If $A(C')$ is empty, leave cell C of D empty. If not, suppose $A(C') = \{(p,t),(p',t')\}$. If there exist $(q,s) \in X_t$ and $(q',s') \in X_{t'}$ such that $L_i(\phi_t(q,s),\phi_{t'}(q',s')) = \phi_{t_i}(q_i,s_i)$ for $1\le i\le n$, define $D(C) = \{(p,q,s),(p',q',s')\}$; otherwise leave $D(C)$ empty.

This completes the description of D. It may be verified that D is indeed an $(n,k,u(v-w)+w)$ frame. □

For completeness, we state, but do not prove, the following generalization of Theorem 4.2. This is an "indirect" construction, special cases of which have appeared in the literature. See, for example [15]. We will not make use of this more general construction in this paper.

THEOREM 4.3. *Suppose the following exist:*

(1) *An (n,ℓ,u)-frame.*

(2) *An (n,k,v)-frame containing an (n,k,w)-subframe.*

(3) *n POLS of order $\frac{k(v-a)}{\ell}$ containing (or missing) n POLS of order $\frac{k(w-a)}{\ell}$ (where $0\le a\le w$).*

(4) *An $(n,k,u(w-a)+a)$-frame.*

Then an $(n,k,u(v-a)+a)$*-frame exists.*

Note that Theorem 4.2 follows from Theorem 4.3 by putting $a = w$.

In the next section we will investigate the existence of two and three dimensional frames. To close this section we describe a simple application of Theorem 4.2 in constructing higher dimensional frames.

The following corollary of Theorem 4.2, is useful.

COROLLARY 4.4. *Suppose there exist an* (n,ℓ,u)*-frame and* n *POLS of order* $\frac{k}{\ell}$. *Then an* (n,k,u)*-frame exists.*

Proof. In Theorem 4.2, let $v = 1$, $w = 0$. Condition (2) is satisfied trivially, and the result is obtained. □

The following is our result.

THEOREM 4.5. *Suppose* $u = 2^k.t+1$ *is a prime power, and suppose there exist* s *POLS of order* v. *Let* $r = \min\{s,t\}$. *Then:*

(1) There exists an (r,v,u)*-frame.*

(2) If $k > 1$, *there exists an* $(r,2v,u)$*-frame.*

Proof. This follows from Corollary 4.4, Theorem 3.10, and Theorem 3.7. □

5. *Two- dimensional frames.*

In this section we briefly discuss the existence of two-dimensional frames. We will limit our investigation to frames of odd order. Frames of even order will be dealt with in a later paper. We need some results on Room squares, 2-frames and POLS.

LEMMA 5.1. *If* $v \neq 2$ *or* 6, *then there exist two POLS of order* v.

Proof. This was shown by Bose, Shrikhande and Parker in [4]. □

LEMMA 5.2. *If* $u \equiv 1 \bmod 4$, $u \neq 33,57,93,129,$*or* 133, *then there exists a* $(2,u)$*-frame.*

Proof. This result was established in [16]. The frames constructed there were of a special type, having a skew property. The proof depends heavily on a PBD-closure result similar to Theorem 4.1. 2-frames of orders 5,9,13 and 17 are given, and then (v,K)-PBDs with $K = \{5,9,13,17\}$ are constructed to establish the result. □

We first eliminate the exceptions of Lemma 5.2 by constructing strong 2-frame starters of the required orders. In [8], the authors describe a computer algorithm for finding strong starters in cyclic groups. An obvious modiciation of this algorithm is made which enables us to find t-frame starters for $t > 1$. Since $\mathbb{Z}_u \times \mathbb{Z}_2 \cong \mathbb{Z}_{2u}$ if u is odd, we may describe our 2-frame starters in the cyclic groups $\mathbb{Z}_{2u}$, for $u = 33,57,93,129$, and 133. We remark that the frames arising from these strong starters are not frames as defined in [16], since they lack the skew property. (A strong starter $A = \{\{s_i,t_i\}\}$ is *skew* if $s_i+t_i \neq -(s_j+t_j)$ for any i,j).

Thus we may improve Lemma 5.2.

LEMMA 5.3. *If $u \equiv 1 \bmod 4$, then there exists a $(2,u)$-frame.*

Proof. Strong $(2,u)$-frame starters for orders $u = 33,51,93,129$, and 133 are given in the appendix. □

Our main existence results for two-dimensional frames are given in the next theorems.

LEMMA 5.4. *If $u \equiv 1 \bmod 4$, $u > 5$, then there exists a (t,u)-frame for any $t \geq 1$.*

Proof. If $t \neq 2$ or 6, there exist two POLS of order t. Also, a Room square of order u exists, so Corollary 4.4 yields the result. If $t = 2$, Lemma 5.3 gives the result. Finally, if $t = 6$, then apply Corollary 4.4 with $n = 2$, $\ell = 2$, $k = 6$. Lemma 5.3 gives a $(2,u)$-frame, and two POLS of order 3 exist. □

LEMMA 5.5. *If $u \equiv 3 \bmod 4$, $u > 3$, and $t \neq 2$ or 6, then there exists a (t,u) frame.*

Proof. The proof is the first part of the proof of Lemma 5.4. □

LEMMA 5.6. *If $(t,6) \neq 1$, then there exists a $(t,5)$-frame.*

Proof. Suppose first that t is even. Let $t = 2^s.u$ with $s \geq 1$. By Corollary 3.5 there exists a $(2^s,5)$-frame. Since u is odd, there exist two POLS of order u, and thus there is a $(t,5)$ frame.

Thus, assume $t = 3u$ with u odd. Example 1.1 provides a $(3,5)$-frame Since two POLS of order u exist, the result follows. □

Summarizing the above, and recalling Example 1.1, we obtain the following result.

THEOREM 5.7. *If $u \geq 5$ is odd and there does not exist a t-frame of order u, then either*

(1) $u = 5$ and $(t,6) = 1$.

(2) $t = 2$ or 6 and $u \equiv 3 \bmod 4$.

It is trivial to see that there are no frames of order 3, so the two classes above are the only unknown cases for frames of odd side. It is known [18] that no $(1,5)$-frame (i.e. a Room square) exists, but this is the only one of the exceptions of Theorem 5.7 which is known not to exist. Note that Theorems 2.3 and 2.4 give negative results regarding the possibility of constructing these frames by starter methods.

Thus we ask the following two questions: Which (t,u)-frames exist in the following classes?:

(1) $u = 5$ and $(t,6) = 1$

(2) $t = 2$ or 6, $u \equiv 3 \bmod 4$.

Note that a 6-frame could be constructed from a 2-frame if the 2-frame exists, as in the proof of Lemma 5.4.

6. *Frames and Room n-cubes.*

In this section we will consider applications of the recursive construction, Theorem 4.2, to Room n-cubes. Recall that $\nu(u)$ denotes the largest n such that a Room n-cube of order u exists. Let $\nu_t(u)$ denote the largest n such that an (n,t,u)-frame exists. Thus $\nu(u) = \nu_1(u)$. Finally, $N(v)$ denotes the largest number of POLS of order v. We will make use of the following corollaries to Theorem 4.2.

THEOREM 6.1. $\nu(uv) \geq \min\{\nu(u),\nu(v),N(v)\}$.

Proof. Put $\ell = k = 1$ and $w = 0$ in Theorem 4.2. □

THEOREM 6.2. $\nu(u(v-1)+1) \geq \min\{\nu(u),\nu(v),N(v-1)\}$.

Proof. Put $\ell = k = 1$ and $w = 1$ in Theorem 4.2. □

THEOREM 6.3. $\nu(u(v-1)+1) \geq \min\{\nu_2(u),\nu(v),N(\frac{v-1}{2})\}$.

Proof. Put $\ell = 2$, $k = 1$, and $w = 1$ in Theorem 4.2. □

We will use the above three theorems to establish a list of lower bounds for $\nu(u)$, u odd and under 1000. In applying recursive constructions, it is clearly necessary to have something to start with. We will make use of the following result established by the first author in [7].

THEOREM 6.4. $\nu(13) \geq 5$, $\nu(15) \geq 4$, $\nu(17) \geq 4$, $\nu(21) \geq 4$, $\nu(25) \geq 7$, $\nu(29) \geq 13$, $\nu(37) \geq 15$, $\nu(41) \geq 9$, $\nu(53) \geq 17$, $\nu(61) \geq 21$, $\nu(101) \geq 31$.

For prime powers, recall Theorem 3.10, which states that if $q = 2^n.t+1$ is a prime power, then $\nu(q) \geq t$. We also use the following.

THEOREM 6.5. *If $u = 7$, or $11 \leq u \leq 999$ and u is odd, then there exists a strong starter of order u, and hence $\nu(u) \geq 3$.*

Proof. See Stanton and Mullin [20], Dinitz and Stinson [9]. □

We list below in Table 1 lower bounds for $\nu(u)$, for u odd and under 1000. For brevity we omit orders u where we are able to improve the bound of Theorem 6.4, 3.10, or 6.5. We also list the lower bounds needed for POLS in Table 1. The reader is referred to Brouwer [5] for further details regarding these lower bounds for POLS.

Obviously, in many cases, either Theorem 6.2 or 6.3 can be applied. Often, the existence of POLS determines which theorem yields a better bound. For example, we have $815 = 37(23-1)+1$. $\nu(23) \geq 11$, $\nu(37) \geq 15$, $\nu_2(37) \geq 9$, $N(22) \geq 3$, and $N(11) \geq 10$ are the best bounds known. Thus Theorem 6.3 yields $\nu(815) \geq 9$, whereas Theorem 6.2 yields only $\nu(815) \geq 3$. Thus Theorem 6.3 is considerably better in this case.

Table 1

n	Construction	Lower bound for $\nu(u)$	Theorem	Remarks
133	= 11(13-1)+1	5	6.2	$\nu(11)\geq 5, \nu(13)\geq 5, N(12)\geq 5$
143	= 11.13	5	6.1	$\nu(11)\geq 5, \nu(13)\geq 5, N(13)\geq 5$
165	= 15.11	4	6.1	$\nu(11)\geq 5, \nu(15)\geq 4, N(11)\geq 10$
177	= 11(17-1)+1	4	6.2	$\nu(11)\geq 5, \nu(17)\geq 4, N(16)\geq 15$
187	= 11.17	4	6.1	$\nu(11)\geq 5, \nu(17)\geq 4, N(17)\geq 16$
195	= 15.13	4	6.1	$\nu(13)\geq 5, \nu(15)\geq 4, N(13)\geq 12$
205	= 17(13-1)+1	4	6.2	$\nu(17)\geq 4, \nu(13)\geq 5, N(12)\geq 5$
209	= 11.19	5	6.1	$\nu(11)\geq 5, \nu(19)\geq 5, N(19)\geq 18$
221	= 13.17	4	6.1	$\nu(13)\geq 5, \nu(17)\geq 4, N(17)\geq 16$
225	= 15.15	4	6.1	$\nu(15)\geq 4, N(15)\geq 4$
231	= 21.11	4	6.1	$\nu(21)\geq 4, \nu(11)\geq 5, N(11)\geq 10$
247	= 13.19	5	6.1	$\nu(13)\geq 5, \nu(19)\geq 9, N(19)\geq 18$
253	= 11.23	5	6.1	$\nu(11)\geq 5, \nu(23)\geq 11, N(23)\geq 22$
255	= 15.17	4	6.1	$\nu(15)\geq 4, \nu(17)\geq 4, N(17)\geq 16$
273	= 17(17-1)+1	4	6.2	$\nu(17)\geq 4, N(16)\geq 15$
275	= 11.25	5	6.1	$\nu(11)\geq 5, \nu(25)\geq 7, N(25)\geq 24$
285	= 15.19	4	6.1	$\nu(15)\geq 4, \nu(19)\geq 9, N(19\geq 18$
291	= 29(11-1)+1	4	6.3	$\nu_2(29)\geq 7, \nu(11)\geq 5, N(5)\geq 4$
297	= 11.27	5	6.1	$\nu(11)\geq 5, \nu(27)\geq 13, N(27)\geq 13$
299	= 13.23	5	6.1	$\nu(13)\geq 5, \nu(23)\geq 11, N(23)\geq 22$
301	= 25(13-1)+1	5	6.2	$\nu(25)\geq 7, \nu(13)\geq 5, N(12)\geq 5$
305	= 19(17-1)+1	4	6.2	$\nu(19)\geq 9, \nu(17)\geq 4, N(16)\geq 15$
315	= 15.21	4	6.1	$\nu(15)\geq 4, \nu(21)\geq 4, N(21)\geq 4$
319	= 11.29	5	6.1	$\nu(11)\geq 5, \nu(29)\geq 13, N(29)\geq 28$
323	= 17.19	4	6.1	$\nu(17)\geq 4, \nu(19)\geq 9, N(19)\geq 18$
325	= 13.25	5	6.1	$\nu(13)\geq 5, \nu(25)\geq 7, N(25)\geq 24$
337	= 21(17-1)+1	4	6.2	$\nu(21)\geq 4, \nu(17)\geq 4, N(16)\geq 15$
341	= 11.31	5	6.1	$\nu(11)\geq 5, \nu(31)\geq 15, N(31)\geq 30$
345	= 15.23	4	6.1	$\nu(15)\geq 4, \nu(23)\geq 11, N(23)\geq 22$
351	= 13.27	5	6.1	$\nu(13)\geq 5, \nu(27)\geq 13, N(27)\geq 26$
357	= 21.17	4	6.1	$\nu(21)\geq 4, \nu(17)\geq 4, N(17)\geq 16$
369	= 23(17-1)+1	4	6.2	$\nu(23)\geq 11, \nu(17)\geq 4, N(17)\geq 16$
371	= 37(11-1)+1	4	6.3	$\nu_2(37)\geq 9, \nu(11)\geq 5, N(5)\geq 4$
375	= 15.25	4	6.1	$\nu(15)\geq 4, \nu(25)\geq 7, N(25)\geq 4$

377 = 13.29	5	6.1	$\nu(13)\geq 5, \nu(29)\geq 13, N(29)\geq 28$
391 = 17.23	4	6.1	$\nu(17)\geq 4, \nu(23)\geq 11, N(23)\geq 22$
399 = 21.19	4	6.1	$\nu(21)\geq 4, \nu(19)\geq 9, N(19)\geq 18$
405 = 15.27	4	6.1	$\nu(15)\geq 4, \nu(27)\geq 13, N(27)\geq 26$
411 = 41(11-1)+1	4	6.3	$\nu_2(41)\geq 5, \nu(11)\geq 5, N(5)\geq 4$
425 = 17.25	4	6.1	$\nu(17)\geq 4, \nu(25)\geq 7, N(25)\geq 24$
437 = 19.23	9	6.1	$\nu(19)\geq 9, \nu(23)\geq 11, N(23)\geq 22$
445 = 37(31-1)+1	5	6.2	$\nu(37)\geq 15, \nu(13)\geq 5, N(12)\geq 5$
465 = 15.31	4	6.1	$\nu(15)\geq 4, \nu(31)\geq 15, N(31)\geq 30$
469 = 13(37-1)+L	4	6.2	$\nu(13)\geq 5, \nu(37)\geq 15, N(36)\geq 4$
473 = 11.43	5	6.1	$\nu(11)\geq 5, \nu(43)\geq 21, N(43)\geq 42$
475 = 19.25	7	6.1	$\nu(19)\geq 9, \nu(25)\geq 7, N(25)\geq 24$
481 = 13.37	5	6.1	$\nu(13)\geq 5, \nu(37)\geq 15, N(37)\geq 36$
483 = 21.23	4	6.1	$\nu(21)\geq 4, \nu(23)\geq 11, N(23)\geq 22$
493 = 41(13-1)+1	5	6.2	$\nu(41)\geq 9, \nu(13)\geq 5, N(12)\geq 5$
497 = 31(17-1)+1	4	6.2	$\nu(31)\geq 15, \nu(17)\geq 4, N(16)\geq 5$
507 = 11(47-1)+1	4	6.2	$\nu(11)\geq 5, \nu(47)\geq 23, N(46)\geq 4$
513 = 19.27	9	6.1	$\nu(19)\geq 9, \nu(27)\geq 13, N(27)\geq 26$
517 = 11.47	5	6.1	$\nu(11)\geq 5, \nu(47)\geq 23, N(47)\geq 46$
519 = 37(15-1)+1	4	6.3	$\nu_2(37)\geq 9, \nu(15)\geq 4, N(7)\geq 6$
525 = 21.25	4	6.1	$\nu(21)\geq 4, \nu(25)\geq 7, N(25)\geq 26$
527 = 17.31	4	6.1	$\nu(17)\geq 4, \nu(31)\geq 15, N(31)\geq 15$
531 = 53(11-1)+1	4	6.3	$\nu_2(53)\geq 13, \nu(11)\geq 5, N(5)\geq 4$
533 = 13.41	5	6.1	$\nu(13)\geq 5, \nu(41)\geq 9, N(41)\geq 40$
551 = 19.29	9	6.1	$\nu(19)\geq 9, \nu(29)\geq 13, N(29)\geq 28$
555 = 15.37	4	6.1	$\nu(15)\geq 4, \nu(37)\geq 15, N(37)\geq 36$
565 = 47(13-1)+1	5	6.2	$\nu(47)\geq 23, \nu(13)\geq 5, N(12)\geq 5$
567 = 21.27	4	6.1	$\nu(21)\geq 4, \nu(27)\geq 13, N(27)\geq 26$
575 = 23.25	7	6.1	$\nu(23)\geq 11, \nu(25)\geq 7, N(25)\geq 24$
583 = 11.53	5	6.1	$\nu(11)\geq 5, \nu(53)\geq 17, N(53)\geq 52$
589 = 19.31	9	6.1	$\nu(19)\geq 9, \nu(31)\geq 15, N(31)\geq 15$
609 = 21.29	4	6.1	$\nu(21)\geq 4, \nu(29)\geq 13, N(29)\geq 28$
611 = 13.43	5	6.1	$\nu(13)\geq 5, \nu(43)\geq 4, N(43)\geq 42$
615 = 15.41	4	6.1	$\nu(15)\geq 4, \nu(41)\geq 9, N(41)\geq 40$
621 = 23.27	11	6.1	$\nu(23)\geq 11, \nu(27)\geq 13, N(27)\geq 26$
629 = 17.37	4	6.1	$\nu(17)\geq 4, \nu(37)\geq 15, N(37)\geq 36$
637 = 53(13-1)+1	5	6.2	$\nu(53)\geq 17, \nu(13)\geq 5, N(12)\geq 5$
639 = 29(23-1)+1	7	6.3	$\nu_2(29)\geq 7, \nu(23)\geq 11, N(11)\geq 10$
645 = 15.43	4	6.1	$\nu(15)\geq 4, \nu(43)\geq 21, N(43)\geq 47$
649 = 11.59	5	6.1	$\nu(11)\geq 5, \nu(43)\geq 21, N(43)\geq 42$
651 = 21.31	4	6.1	$\nu(21)\geq 4, \nu(31)\geq 15, N(31)\geq 30$
657 = 41(17-1)+1	4	6.3	$\nu(41)\geq 9, \nu(17)\geq 4, N(16)\geq 15$
667 = 23.29	11	6.1	$\nu(23)\geq 11, \nu(29)\geq 13, N(29)\geq 28$
671 = 11.61	5	6.1	$\nu(11)\geq 5, \nu(61)\geq 21, N(61)\geq 60$
675 = 25.27	7	6.1	$\nu(25)\geq 7, \nu(27)\geq 13, N(27)\geq 26$
681 = 17(41-1)+1	4	6.2	$\nu(17)\geq 4, \nu(41)\geq 9, N(40)\geq 4$
685 = 19(37-1)+1	4	6.2	$\nu(19)\geq 9, \nu(37)\geq 15, N(36)\geq 4$
689 = 13.53	5	6.1	$\nu(13)\geq 5, \nu(53)\geq 17, N(53)\geq 52$
697 = 29(25-1)+1	5	6.3	$\nu(29)\geq 7, \nu(25)\geq 7, N(12)\geq 5$
703 = 19.37	9	6.1	$\nu(19)\geq 9, \nu(37)\geq 15, N(37)\geq 15$
705 = 15.47	4	6.1	$\nu(15)\geq 4, \nu(47)\geq 23, N(47)\geq 46$
713 = 23.31	11	6.1	$\nu(23)\geq 11, \nu(31)\geq 15, N(31)\geq 30$
725 = 25.29	7	6.1	$\nu(25)\geq 7, \nu(29)\geq 13, N(29)\geq 28$
737 = 11.67	5	6.1	$\nu(11)\geq 5, \nu(67)\geq 33, N(67)\geq 66$
753 = 47(17-1)+1	4	6.2	$\nu(47)\geq 23, \nu(17)\geq 4, N(17)\geq 16$
755 = 29(27-1)+1	7	6.3	$\nu_2(29)\geq 7, \nu(27)\geq 13, N(13)\geq 12$

767 = 13.59	5	6.1	$\nu(13)\geq5,\nu(59)\geq29,N(59)\geq58$
771 = 11(71-1)+1	5	6.2	$\nu(11)\geq5,\nu(71)\geq35,N(70)\geq6$
775 = 25.31	7	6.1	$\nu(25)\geq7,\nu(31)\geq15,N(31)\geq30$
777 = 21.37	4	6.1	$\nu(21)\geq4,\nu(37)\geq15,N(37)\geq36$
779 = 19.41	9	6.1	$\nu(19)\geq9,\nu(41)\geq9,N(41)\geq40$
781 = 11.71	5	6.1	$\nu(11)\geq5,\nu(71)\geq35,N(71)\geq70$
783 = 27.29	13	6.1	$\nu(27)\geq13,\nu(29)\geq13,N(29)\geq28$
793 = 13.61	5	6.1	$\nu(13)\geq5,\nu(61)\geq21,N(61)\geq60$
795 = 15.53	4	6.1	$\nu(15)\geq4,\nu(53)\geq17,N(53)\geq52$
799 = 17.47	4	6.1	$\nu(17)\geq4,\nu(47)\geq23,N(47)\geq46$
803 = 11.73	5	6.1	$\nu(11)\geq5,\nu(73)\geq9,N(73)\geq72$
805 = 67(13-1)+1	5	6.2	$\nu(67)\geq33,\nu(13)\geq5,N(12)\geq5$
815 = 37(23-1)+1	9	6.3	$\nu_2(37)\geq9,\nu(23)\geq11,N(11)\geq10$
817 = 19.43	9	6.1	$\nu(19)\geq9,\nu(43)\geq21,N(43)\geq42$
837 = 27.31	13	6.1	$\nu(27)\geq13,\nu(31)\geq15,N(31)\geq30$
849 = 53(17-1)+1	4	6.2	$\nu(53)\geq17,\nu(17)\geq4, N(16)\geq15$
851 = 23.27	11	6.1	$\nu(23)\geq11,\nu(37)\geq15,N(37)\geq36$
855 = 61(15-1)+1	4	6.3	$\nu_2(61)\geq15,\nu(15)\geq4,N(7)\geq6$
861 = 21.41	4	6.1	$\nu(21)\geq4,\nu(41)\geq9,N(41)\geq40$
869 = 11.79	5	6.1	$\nu(11)\geq5,\nu(79)\geq39,N(79)\geq78$
871 = 13.67	5	6.1	$\nu(13)\geq5,\nu(67)\geq33,N(67)\geq66$
875 = 19(47-1)+1	4	6.2	$\nu(19)\geq9,\nu(47)\geq23,N(46)\geq4$
885 = 15.59	4	6.1	$\nu(15)\geq4,\nu(59)\geq29,N(59)\geq58$
889 = 37(25-1)+1	5	6.3	$\nu_2(37)\geq9,\nu(25)\geq7,N(12)\geq5$
891 = 11.81	5	6.1	$\nu(11)\geq5,\nu(81)\geq5,N(81)\geq80$
893 = 19.47	9	6.1	$\nu(19)\geq9,\nu(47)\geq23,N(47)\geq23$
899 = 29.31	13	6.1	$\nu(29)\geq13,\nu(31)\geq15,N(31)\geq30$
901 = 17.53	4	6.1	$\nu(17)\geq4,\nu(53)\geq17,N(53)\geq52$
903 = 11(83-1)+1	5	6.2	$\nu(11)\geq5,\nu(83)\geq41,N(82)\geq8$
913 = 11.83	5	6.1	$\nu(11)\geq5,\nu(83)\geq41,N(83)\geq82$
915 = 15.61	4	6.1	$\nu(15)\geq4,\nu(61)\geq21,N(61)\geq60$
921 = 23(41-1)+1	4	6.2	$\nu(23)\geq11,\nu(41)\geq9,N(40)\geq4$
923 = 13.71	5	6.1	$\nu(13)\geq5,\nu(71)\geq35,N(71)\geq70$
925 = 25.37	7	6.1	$\nu(25)\geq7,\nu(37)\geq15,N(37)\geq36$
943 = 23.41	9	6.1	$\nu(23)\geq11,\nu(41)\geq9,N(41)\geq40$
945 = 59(17-1)+1	4	6.2	$\nu(59(\geq29,\nu(17)\geq4,N(16)\geq15$
949 = 13.73	5	6.1	$\nu(13)\geq5,\nu(73)\geq9,N(73)\geq72$
955 = 53(19-1)+1	8	6.3	$\nu_2(53)\geq13,\nu(19)\geq9,N(9)\geq8$
963 = 37(27-1)+1	9	6.3	$\nu_2(37)\geq9,\nu(27)\geq13,N(13)\geq12$
969 = 11(89-1)+1	5	6.2	$\nu(11)\geq5,\nu(89)\geq11,N(88)\geq7$
973 = 81(13-1)+1	5	6.2	$\nu(81)\geq5,\nu(13)\geq5,N(12)\geq5$
979 = 11.89	5	6.1	$\nu(11)\geq5,\nu(89)\geq11,N(89)\geq88$
985 = 41(25-1)+1	5	6.3	$\nu_2(41)\geq5,\nu(25)\geq7,N(12)\geq5$
987 = 21.47	4	6.1	$\nu(21)\geq4,\nu(47)\geq23,N(47)\geq46$
989 = 23.43	11	6.1	$\nu(23)\geq11,\nu(43)\geq21,N(43)\geq42$
999 = 27.37	13	6.1	$\nu(27)\geq13,\nu(37)\geq15, N(37)\geq36$

7. *Frames and Howell designs.*

This section deals with applications of frames to Howell designs.

Suppose X is a set such that $|X| = 2n$. A *Howell design* on X of type $H(s,2n)$ consists of a square array of side s such that (i) each cell is either empty or contains an unordered pair of elements taken from X, (ii) each element of X appears exactly once in each row and each column of the array and (iii) every unordered pair appears in at most one cell of the array. From the definition it is seen that existence requires $n \le s \le 2n-1$. If $Y \subseteq X$ such that $|Y| = 2n - s$ and no pair of elements of Y occur in a cell in the array, then denote this fact notationally by $H^*(s,2n)$.

For information concerning Howell designs, see [14].

As mentioned in the introduction, a Room square of side $2n-1$ is an $H(2n-1,2n)$. Therefore, it is known that $H(2n-1,2n)$ exist for all $n \ne 2,3$. For Howell designs of even side the design most similar to a Room square is a $H(2n,2n+2)$. The existence question for designs of this type has been reduced by Anderson [3] to the following:

(i) Are there designs of type $H(6p,6p+2)$, p prime?

(ii) Are there designs of type $H(24,26)$, $H(48,50)$ and $H(54,56)$?

We will be able to answer (i) in the affirmative for $p \equiv 1(4)$ and (ii) in the affirmative for the $H(48,50)$ and $H(54,56)$.

The following theorem gives the connection between frames and Howell designs. The theorem is similar to a theorem of Anderson and Gross [1; Theorem 1] but stated more directly in terms of the frames. For completeness we give a proof.

THEOREM 7.1. *Suppose there exists a (t,n)-frame and an $H^*(t,t+k)$. Then there exists a $H^*(tn,tn+k)$.*

Proof. Let H be a $H^*(t,t+k)$ on the symbols $\{1,2,\dots,t\} \cup \{\infty_1,\infty_2,\dots,\infty_k\}$. Denote by Hu_i the Howell design H with the symbol n $(1 \le n \le t)$ replaced by the symbol (u_i,n) and the symbol ∞_i $(1 \le i \le k)$ unchanged. Let S be a t-frame of order n. In the empty diagonal square Su_i in S place the Howell design Hu_i. It is easy to check that the resulting square is indeed an $H^*(tu,tu+k)$ on the symbol set $U \times T \cup \{\infty_1,\infty_2,\dots,\infty_k\}$ □

COROLLARY 7.2. *If $n \equiv 1 \bmod 4$ then there is an $H^*(6n,6n+2)$, and an $H^*(6n,6n+4)$.*

Proof. From Lemma 5.4 there is a $(6,n)$ frame for all $n \equiv 1(4)$. Since there are $H^*(6,8)$ and $H^*(6,10)$, see [14], the result follows from

Theorem 7.1. □

Note that Corollary 7.2 implies the existence of an H(54,56).

Using the strong 2-frame starter of order 8 in Example 2.7 to construct a (2,8)-frame and a pair of MOLS of side 3 it is possible by Corollary 4.4 to construct a (6,8)-frame. Again, using the $H^*(6,8)$ and Theorem 7.1 the result that there is an $H^*(48,50)$ follows.

As a final corollary to Theorem 7.1 we have a result which gives many Howell designs.

COROLLARY 7.3. *Suppose* s,t *and* u *are odd,* $s \le t < 1000, u > 7$, $1 \le s \le t$, $s \ne t-2$. *Then there exists an* $H^*(tu, tu+s)$.

Proof. In [9] it is shown that if $s < t < 1000$, t odd and $1 \le s \le t$ except possibly $s = t - 2$ then there is an H(t,t+s). For $t > 5$ odd and $u > 7$ odd by Theorem 5.7 there is a (t,u)-frame. Thus, the result follows by Theorem 7.1.

It has recently come to our attention that Schellenberg and Vanstone have shown the existence of all the designs H(2n,2n+2) mentioned in (i), (ii) above. See [19].

2, 33 STRONG FRAME STARTER

50, 51	55, 57	41, 44	63, 59	10, 5	8, 2	25, 18	31, 39	29, 38	4, 14
26, 15	16, 28	19, 32	23, 9	21, 6	11, 27	47, 64	36, 54	42, 61	40, 60
58, 37	12, 56	30, 53	3, 45	24, 65	43, 17	7, 46	34, 62	20, 49	22, 52
48, 13	1, 35								

2, 57 STRONG FRAME STARTER

5, 4	103,101	112, 1	81, 85	24, 19	42, 36	95,102	26, 34	9, 18	67, 77
58, 47	94,106	113, 12	3, 17	90, 75	91,107	72, 89	15,111	52, 33	68, 88
56, 35	59, 37	105, 82	32, 8	66, 41	71, 45	53, 80	46, 74	40, 69	84, 54
38, 7	60, 28	92, 11	110, 30	64, 29	109, 73	27,104	99, 23	87, 48	39, 79
10, 51	86, 14	22, 65	76, 6	70, 25	96, 50	16, 63	97, 31	62, 13	43, 93
49,100	83, 21	108, 55	44, 98	61, 2	20, 78				

2, 93 STRONG FRAME STARTER

185,184	176,178	36, 33	166,170	71, 66	10, 16	8, 1	38, 46	168,177	135,125
143,132	134,146	94, 81	62, 76	116,101	126,142	5,174	92,110	80, 61	88, 68
53, 32	84,106	24, 47	42, 18	29, 54	171,145	160,133	86, 58	156,127	34, 4
69,100	17, 49	107,140	20,172	52, 87	79,115	99,136	57, 19	150,111	56, 96
7, 48	35, 77	138, 95	163, 21	44, 89	118,164	112, 65	13,151	123, 74	63,113
141, 90	14,148	40,173	182,128	104,159	78, 22	129, 72	25,153	121,180	102,162
169,108	122, 60	154, 91	45,109	23,144	147, 27	73, 6	30, 98	41,158	131, 15
152, 37	67,181	183, 70	124, 12	64,139	55,165	137, 28	161, 83	50,157	39,119
114, 9	179, 97	85, 2	105, 3	167, 82	75,175	43,130	51,149	31,120	155, 59
26,117	103, 11								

2,129 STRONG FRAME STARTER

164,163	234,236	48, 51	249,245	109,104	208,214	148,141	203,211	24, 15	47, 37
33, 22	190,178	250, 5	120,106	65, 50	188,204	158,175	96,114	54, 35	40, 60
195,174	183,161	209,186	216,192	254, 21	125, 99	66, 93	223,251	242,213	230,200
105,136	244,212	152,185	89, 55	231,196	71,107	36, 73	139,177	166,127	179,219
59, 18	210,252	112,155	90, 46	97,142	12, 58	182,135	239, 29	180,131	27,235
39,246	1,207	110, 57	83,137	81, 26	172,228	31,232	202,144	241, 42	247,187
199, 2	147, 85	171,108	28,222	67,132	134, 68	181,248	217,149	146,215	52,122
13, 84	121, 49	243,170	77, 3	162, 87	194,118	88,165	75,255	72,151	193,113
145, 64	94,176	115, 32	153,237	10, 95	100, 14	102,189	168, 80	225, 56	70,160
220, 53	92,184	61,226	44,138	34,197	76,238	159, 62	201, 41	25,124	63,221
16,117	218,116	257,154	111, 7	227, 74	253,101	43,150	79,229	20,169	19,167
128, 17	224, 78	143, 30	91,205	4,119	82,198	9,126	38,156	11,130	86,206
240,103	191, 69	98,233	23,157	123,256	140, 8	133, 6	173, 45		

2,133 STRONG FRAME STARTER

227,228	47, 49	254,251	152,156	204,199	75, 81	35, 42	249,241	46, 55	112,122
239,250	178,166	170,183	67, 53	140,155	117,101	108,125	62, 44	51, 70	100,120
89,110	179,201	31, 54	253, 11	149,174	33, 7	86, 59	190,218	113,142	246, 10
194,163	160,192	259, 26	114, 80	258,223	137,173	115, 78	252,214	181,220	119, 79
172,131	127, 85	196,153	200,244	66,111	128, 82	187,234	213,261	184,135	207,157
16,231	205,257	76,129	219,165	237,182	233,177	134, 77	212,154	150, 91	36, 96
164,103	247, 43	148,211	226,162	102,167	71, 5	197,264	29, 97	126, 57	48,118
235, 40	94, 22	107, 34	198, 6	72,263	215, 25	14,203	262, 74	124, 45	224, 38
63,248	242, 58	60,143	265, 83	61,146	13, 99	202, 23	106, 18	109, 20	195, 19
147,238	260,168	139,232	256, 84	210, 39	93,189	105, 8	236,138	136, 37	216,116
255, 90	21,185	12,175	9,171	230, 69	28,188	73,180	161, 3	225, 68	87,243
98,209	15,169	217, 64	30,144	245,130	41,191	141, 24	222,104	240,121	88,208
123, 2	1,145	50,193	176, 52	206, 65	95,221	32,159	132, 4	92,229	186, 56
158, 27	17,151								

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