

# Packing Costas Arrays

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## Dedicated to Ralph Stanton

### Abstract

A Costas latin square of order  $n$  is a set of  $n$  disjoint Costas arrays of the same order. Costas latin squares are studied here from a construction as well as a classification point of view. A complete classification is carried out up to order 27. In this range, we verify the conjecture that there is no Costas latin square for any odd order  $n \geq 3$ . Various other related combinatorial structures are also considered, including near Costas latin squares (which are certain packings of near Costas arrays) and Vatican Costas squares.

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# 1 Introduction

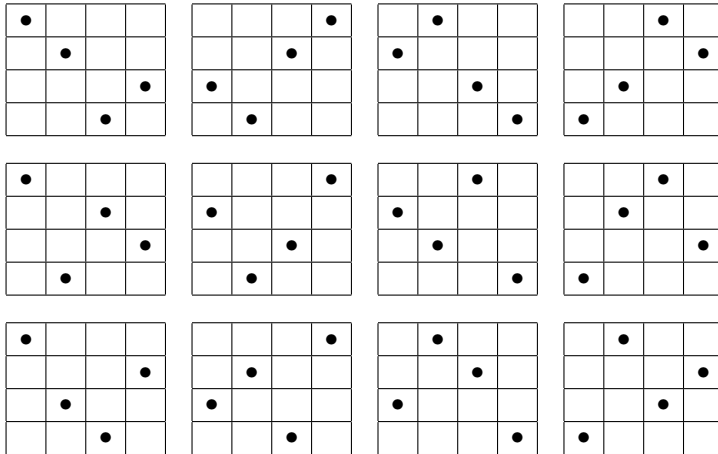
A *Costas array* of order  $n$  (or *side*  $n$ ) is an  $n \times n$  array of dots and empty cells such that

1. there are  $n$  dots and  $n(n - 1)$  empty cells, with exactly one dot in each row and column, and
2. all the segments between pairs of dots differ in length or in slope.

Costas arrays were introduced by J.P. Costas; see [1, 4, 7] for early results and historical remarks. Extensive surveys of Costas arrays can be found in [6, 12].

Two Costas arrays of order  $n$  are *disjoint* if there is no cell in which both arrays have a dot. Let  $D(n)$  denote the maximum number of mutually disjoint Costas arrays of order  $n$ . The main topic of this paper is the study of  $D(n)$ . Obviously  $D(n) \leq n^2/n = n$ . We are particularly interested in cases where this upper bound on  $D(n)$  is attained; in those cases, we say that we have a *full set* of mutually disjoint Costas arrays.

**Example 1.1.** *We present all the Costas arrays of order 4. The four arrays in the first row are mutually disjoint.*



For notational convenience, Costas arrays are often presented using a certain one-line notation. Given a Costas array of order  $n$ , let  $\pi(i) = j$  whenever the array contains a dot in cell  $(i, j)$ . A Costas array of order  $n$  can be presented as the permutation  $\pi = (\pi(1), \pi(2), \dots, \pi(n - 1), \pi(n))$ .

We call this the *permutation representation* of a Costas array. Using this notation, the Costas arrays in the first row of Example 1.1 have permutation representations  $(1, 2, 4, 3)$ ,  $(4, 3, 1, 2)$ ,  $(2, 1, 3, 4)$  and  $(3, 4, 2, 1)$ , respectively.

A *Costas latin square* of order  $n$ , denoted  $\text{CLS}(n)$ , is a latin square of order  $n$  such that for each symbol  $i$ ,  $1 \leq i \leq n$ , a Costas array results if a dot is placed in the cells containing symbol  $i$ . Clearly a  $\text{CLS}(n)$  is equivalent to  $n$  disjoint Costas arrays of order  $n$ . Costas latin squares were first defined and studied by Etzion [3].

**Example 1.2.** *The first four disjoint Costas arrays of order 4 given in Example 1.1 are mutually disjoint and lead to the following  $\text{CLS}(4)$ .*

1	3	4	2
3	1	2	4
2	4	3	1
4	2	1	3

Etzion [3] defines a *near Costas array* of order  $n \geq 2$  to be an  $n \times n$  array of dots and empty cells such that

1. there are  $n - 1$  dots and  $n^2 - n + 1$  empty cells, with at most one dot in each row and column, and
2. all the segments between pairs of dots differ in length or in slope.

As with Costas arrays, we will say that two near Costas arrays of order  $n$  are *disjoint* if there is no cell in which both arrays have a dot.

Let  $D_{\text{near}}(n)$  denote the maximum number of near Costas arrays of order  $n$ . Observe that  $D_{\text{near}}(n) \leq \lfloor \frac{n^2}{n-1} \rfloor = n + 1$  for all  $n \geq 3$ . A set of  $n + 1$  mutually disjoint near Costas arrays of order  $n \geq 3$  is called a *full set* of arrays. In this case, if the filled cells of the  $i$ th near Costas array is given the symbol  $i$ , then superimposing all these near Costas arrays yields an  $n \times n$  array on the symbol set  $S = \{1, 2, \dots, n + 1\}$  with exactly one empty cell. Such a square is termed a *near Costas latin square* of order  $n$ , and abbreviated as an  $\text{NCLS}(n)$ . A somewhat stronger definition could also be given, where one of the  $n + 1$  arrays is a Costas array and there is no empty cell.

**Example 1.3.** A near Costas latin square of order 6 (a NCLS(6)).

6	1	7	4	2	3
5		6	3	1	7
7	2	1	5	3	4
3	5	4	1	6	2
4	6	5	2	7	1
2	4	3	7	5	6

The roles of the rows, columns and symbols of a latin square can be interchanged. By interchanging the roles of rows and symbols, a Costas latin square of order  $n$  can be transformed into an (equivalent) latin square  $L = (l_{i,j})$  with the property that for any  $1 \leq k \leq n - 1$ ,  $l_{i,j+k} - l_{i,j} = l_{i,m+k} - l_{i,m}$  implies that  $j = m$ . We call such a square a *row-Costas latin square*. Analogously, interchanging the roles of columns and symbols, we get the definition of *column-Costas latin squares*.

We define one final related concept. A *Vatican square* of order  $n$  is a latin square  $L = (l_{i,j})$  of order  $n$  with the property that for any fixed  $d$ ,  $1 \leq d \leq n - 1$ , all ordered pairs of the form  $(l_{i,j}, l_{i,j+d})$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq n - d$  are distinct. A *Vatican Costas square* is a latin square that is both a Vatican square and a Costas latin square. Vatican squares were introduced in [8] while Vatican Costas squares were first considered by Etzion in [3]. In a Vatican Costas square one can interchange the roles of the rows, columns, and symbols to get other equivalent structures.

The paper is organized as follows. In Section 2 some basic constructions for Costas arrays are surveyed and applied to the problem of constructing Costas latin squares; near Costas arrays and near Costas latin squares are also considered. A computational study is carried out in Section 3 to enumerate Costas latin squares of small order; also other related structures defined above are considered.

## 2 Constructions

### 2.1 Costas Latin Squares

One way to construct Costas latin squares would seem to be to start with an initial Costas array of order  $n$  and then move each of the dots to the right  $k$  cells (mod  $n$ ) to construct the  $k$ th array (where  $0 \leq k \leq n - 1$ ). Certainly these arrays will be mutually disjoint, but in general they will not be Costas arrays. However, this technique will succeed provided that

we start with a singly periodic Costas array, which we now define.

Let an  $n \times n$  array be *left-right extended* by putting a dot in cell  $(i, j+n)$  if and only if there is a dot in cell  $(i, j)$ . A *singly periodic Costas array* of order  $n$  is an  $n \times n$  array such that every  $n \times n$  window in its left-right extension is a Costas array of order  $n$ . The following result is now obvious.

**Theorem 2.1.** *If there exists a singly periodic Costas array of order  $n$ , then there exists a CLS( $n$ ).*

Next, we describe the well known *Welch construction* for Costas arrays. Let  $p$  be prime and  $\alpha$  be a primitive element in the field  $\mathbb{F}_p$ . Let  $n = p - 1$ . A Costas array of order  $n$  is obtained by placing a dot at  $(i, j)$  if and only if  $i = \alpha^j$ , for  $0 \leq j < n$ , and  $i = 1, \dots, n$ . We get the following result, which is implicit in [5] and explicit in [3].

**Corollary 2.1.** *There exists a CLS( $p - 1$ ) for all primes  $p$ .*

*Proof.* It is known that Costas arrays constructed via the Welch construction are singly periodic. Apply Theorem 2.1.  $\square$

*Remark.* In [5], this method was used to construct a Vatican square.

Using 2 for the primitive element in  $\mathbb{F}_5$ , one obtains the following Costas array of order 4, given on the left below. (Note here that the columns are labeled 0, 1, 2, 3, while the rows are labeled 1, 2, 3, 4 from the bottom.) This Costas array is singly periodic. On the right is the CLS(4) obtained from this array. It is interesting to note that this Costas latin square is different from any dihedral transformation of the CLS(4) presented in Example 1.1 (e.g., consider the front and back diagonals).

		•	
			•
	•		
•			

3	4	1	2
2	3	4	1
4	1	2	3
1	2	3	4

Unfortunately, the only known singly periodic Costas arrays of order  $n$  come from the Welch construction when  $n = p - 1$ . It is also known that a singly periodic Costas array of order  $n$  does not exist if  $n > 1$  is odd, or if  $n \in \{8, 14, 20\}$ .

Corollary 2.1 gives an infinite class of CLS( $n$ ). Etzion [3] gave another infinite class which contains no new orders but which are inequivalent to

the previous class. Etzion's second infinite class is based on the *Golomb construction*, which we now describe.

Let  $\alpha$  and  $\beta$  be primitive elements in a finite field  $\mathbb{F}_q$ . Denote  $n = q - 2$ . For  $1 \leq i \leq n$ ,  $1 \leq j \leq n$ , place a dot in cell  $(i, j)$  of an  $n \times n$  array if and only if  $\alpha^i + \beta^j = 1$ . Denote the resulting array by  $G(\alpha, \beta)$ . Then  $G(\alpha, \beta)$  is a Costas array of order  $n$ .

Etzion's infinite class is constructed as follows. Suppose that  $q = 2^r$  and suppose that  $q - 1$  is a (Mersenne) prime. Let  $\alpha \in \mathbb{F}_q$  be a primitive element. Note that  $\alpha^i$  is primitive for  $1 \leq i \leq n$  because  $q - 1$  is prime. It is not difficult to verify that the  $n$  Costas arrays  $G(\alpha, \alpha^i)$  are disjoint, for  $1 \leq i \leq n$ . This yields a CLS( $2^r - 2$ ) whenever  $2^r - 1$  is a Mersenne prime.

## 2.2 Near Costas Latin Squares

We begin with a simple observation.

**Lemma 2.2.** *Suppose there exists a CLS( $n$ ). Then  $D_{near}(n) \geq n$ .*

*Proof.* Given a CLS( $n$ ), we have  $n$  disjoint Costas arrays of order  $n$ . Remove one dot from each of these  $n$  arrays, yielding  $n$  disjoint near-Costas arrays of order  $n$ .  $\square$

**Corollary 2.3.** *If  $p$  is a prime, then  $D_{near}(p - 1) \geq p - 1$ .*

We now provide another way of obtaining  $n - 1$  near-Costas arrays of order  $n - 1$ . This method works whenever  $n$  is a prime power; it is based on the *Lempel construction* which is the special case of the Golomb construction where  $\beta = \alpha$ . We will use the following slight variation: Let  $q$  be a prime power and label the rows and columns of  $q - 1$  square arrays of side  $q - 1$  by the integers  $0, \dots, q - 2$ . The arrays will be named as  $A_c$ ,  $c \in \mathbb{F}_q^*$ . For each such array  $A_c$ , place a dot in cell  $(i, j)$  whenever  $\alpha^i + \alpha^j = c$ . Clearly these  $q - 1$  arrays are all disjoint and symmetric. Each array contains  $q - 2$  dots which miss exactly one row and column (namely, row and column  $x$ , where  $\alpha^x = c$ ).

We now show that each of these arrays is a near-Costas array. This can be done by modifying the proof of [4, Theorem 2]. Suppose that  $A_c$  contains dots in cells  $(i, j)$ ,  $(i + u, j + v)$ ,  $(i', j')$  and  $(i' + u, j' + v)$ , where  $i \neq i'$ ,  $j \neq j'$  and  $u, v \neq 0$ . Then

$$\alpha^i + \alpha^j = \alpha^{i+u} + \alpha^{j+v} = \alpha^{i'} + \alpha^{j'} = \alpha^{i'+u} + \alpha^{j'+v} = c.$$

We then obtain

$$\alpha^{i+u} + \alpha^v(c - \alpha^i) = \alpha^{i'+u} + \alpha^v(c - \alpha^{i'}) = c,$$

and therefore

$$\alpha^{i+u} - \alpha^{i+v} = \alpha^{i'+u} - \alpha^{i'+v} = c(1 - \alpha^v).$$

We have  $c \neq 0$ , and  $\alpha^v \neq 1$  follows because  $v \neq 0$ . Hence,

$$\alpha^i(\alpha^u - \alpha^v) = \alpha^{i'}(\alpha^u - \alpha^v) \neq 0.$$

Notice that  $u \neq v$  because  $\alpha^i(\alpha^u - \alpha^v) \neq 0$ . However, since  $u \neq v$ , we obtain  $\alpha^i = \alpha^{i'}$ , which is a contradiction because  $i \neq i'$ .

Summarising the above discussion, we have the following theorem.

**Theorem 2.4.** *If  $q$  is a prime power, then  $D_{near}(q-1) \geq q-1$ .*

*Remark.* Theorem 2.4 is more general than Corollary 2.3 in that it applies to prime powers rather than just primes.

**Example 2.1.** *Let  $q = 7$  and take  $\alpha = 3$ . We present the superposition of  $A_1, \dots, A_6$ :*

2	4	3		5	6
4	6	5	2		1
3	5	4	1	6	
	2	1	5	3	4
5		6	3	1	2
6	1		4	2	3

Next, we use the Golomb construction to construct disjoint near Costas arrays of order  $q-2$ , for certain prime powers  $q \equiv 3 \pmod{4}$ . Since  $q \equiv 3 \pmod{4}$  is a prime power,  $-1$  is a quadratic non-residue modulo  $q$ . Suppose also that  $q = 2p + 1$  where  $p$  is an odd prime. Then  $\mathbb{F}_q$  has  $\phi(2p) = p-1$  primitive roots, say  $\alpha_k$ ,  $k = 1, \dots, p-1$ .

Denote  $\gamma = \alpha_k$ ,  $\delta = \alpha_\ell$  and  $\epsilon = \alpha_m$ , where  $\ell \neq m$ . We will show that  $G(\gamma, \delta)$  and  $G(\gamma, \epsilon)$  contain exactly one common dot. Suppose to the contrary that  $\gamma^i + \delta^j = \gamma^i + \epsilon^j = 1$ . Then  $\delta^j = \epsilon^j$ . Denote  $\zeta = \delta/\epsilon$ ; then  $\zeta^j = 1$ . It is clear that  $\text{ord}(\zeta) \mid 2p$ , so  $\text{ord}(\zeta) \in \{1, 2, p, 2p\}$ . We consider each possibility in turn.

**case 1:**  $\text{ord}(\zeta) = 1$ . Here  $\zeta = 1$ , so  $\delta = \epsilon$ , which is a contradiction.

**case 2:**  $\text{ord}(\zeta) = 2$ . Here  $\zeta = -1$  and  $\delta = -\epsilon$ . However,  $-1$  is a quadratic non-residue, so  $\delta$  and  $\epsilon$  cannot both be primitive elements. This is a contradiction.

**case 3:**  $\text{ord}(\zeta) = p$ . Here  $\zeta$  is a quadratic residue and  $\delta^j = \epsilon^j = -1$ . This implies that  $j = p$  and  $i$  is the unique element such that  $\gamma^i = 2$ .

**case 4:**  $\text{ord}(\zeta) = 2p$ . Here  $\zeta$  is a primitive element and  $\delta^j = \epsilon^j = 1$ . This implies that  $j = 2p$ , which is a contradiction because  $j \leq 2p - 1$ .

It follows from this discussion that the  $p - 1$  Costas arrays  $G(\alpha_1, \alpha_i)$  ( $1 \leq i \leq p - 1$ ) all contain a single common dot. On removing this dot, we obtain  $p - 1$  disjoint near-Costas arrays of order  $2p - 1$ . Summarising, we have the following.

**Theorem 2.5.** *Suppose that  $q \equiv 3 \pmod{4}$  is a prime power and suppose that  $(q - 1)/2$  is prime. Then  $D_{\text{near}}(q - 2) \geq (q - 3)/2$ .*

Theorem 2.5 is illustrated by the following example.

**Example 2.2.** *Let  $q = 11$ , so  $n = 9$  and  $p = 5$ . The four primitive elements modulo 11 are 2, 8, 7 and 6. We present the superposition of  $G(2, 2)$ ,  $G(2, 8)$ ,  $G(2, 7)$ ,  $G(2, 6)$ . The unique common dot is in position (1, 5). If this dot is deleted, then we get four disjoint near Costas arrays of order 9.*

				•				
8		2				6		7
	2		8		7		6	
7		6				2		8
2		7				8		6
	6		7		8		2	
	7		2		6		8	
	8		6		2		7	
6		8				7		2

*Remark.* It may be the case that various Costas arrays of the form  $G(\alpha, \alpha)$  are disjoint. When  $q = 11$ , one finds that the four Costas arrays  $G(2, 2)$ ,  $G(8, 8)$ ,  $G(7, 7)$  and  $G(6, 6)$  are disjoint. The resulting array is presented in the next example.



**Example 2.3.** *The four disjoint Costas arrays  $G(2,2)$ ,  $G(8,8)$ ,  $G(7,7)$  and  $G(6,6)$ .*

6	7		8	2				
7		2	6		8			
	2	8		7	6			
8	6					2	7	
2		7				8		6
	8	6					2	7
			2	8		7	6	
			7		2	6		8
				6	7		8	2

It is not obvious how to derive a simple algebraic formula to determine when Costas arrays of the form  $G(\alpha, \alpha)$  are disjoint. So at present we have no theorem that generalises the previous example.

### 3 Classification of Squares

The concept of equivalence (or isomorphism) is central in any classification of combinatorial structures. Two Costas arrays are said to be *equivalent* if there is a mapping in the dihedral group of rotations and reflections of the square (which has order 8) that maps one array onto the other. Two Costas latin squares are *equivalent* if there is a mapping in the same group that maps every Costas array of one square onto a Costas array of the other. (Note that with this definition, the numbers given to the Costas arrays in the Costas latin square are irrelevant.)

The set of all mappings that map a Costas array (or Costas latin square) onto itself forms the *automorphism group* of the respective structure. The definitions of equivalence and automorphism groups for near Costas arrays and near Costas latin squares are analogous.

#### 3.1 Costas Latin Squares

Let us first have a brief look at Costas latin squares with very small parameters. It is trivial to construct CLS(1) and CLS(2); these are unique up to equivalence. There is no CLS(3) since no Costas array of order 3 can contain a dot in the middle cell. However, it is easy to construct two disjoint Costas arrays of order 3, so it follows that  $D(3) = 2$ .

A CLS(4) is presented in Example 1.2. In the next case,  $n = 5$ , it turns out that there is no Costas latin square of order 5. This can still be done by

hand (for example, with a case-by-case argument). The following example shows that there are four disjoint Costas arrays of order 5, so it follows that  $D(5) = 4$ .

**Example 3.1.** *Here are four disjoint Costas arrays of order 5, given in permutation representation:*

$$(1, 4, 5, 3, 2), (2, 5, 3, 4, 1), (3, 1, 2, 5, 4) \text{ and } (4, 3, 1, 2, 5).$$

*It is interesting to note that a 5th Costas array which is disjoint to the other four arrays in three cells also exists. This Costas array has permutation representation*

$$(5, 2, 4, 3, 1).$$

*If we superimpose these five Costas arrays, we get the following array.*

1	2	3	4	5
3	5	4	1	2
4	3	2	5	1
	4	1, 5	2	3
2, 5	1		3	4

We know of no better way than a computer search to handle orders greater than 5. For a given order  $n$ , the largest possible number of mutually disjoint Costas arrays can be found by solving the following instance of the maximum clique problem. Form a graph  $G$  with one vertex for each Costas array and with an edge between two vertices exactly when the corresponding arrays are disjoint. Any clique in  $G$  corresponds to a set of mutually disjoint Costas arrays, so we may now search for the size of a maximum clique. The *Cliquer* software [10] was used to search for (maximum) cliques in the current work. All Costas arrays of order between 1 and 27 can be obtained in electronic form from [11].

Whenever there are  $n$  disjoint Costas arrays of order  $n$ , one can find all cliques of size  $n$  in the above mentioned graph to get all possible Costas latin squares. However, utilizing the fact that the arrays then cover all the cells of the square, such cases are for performance reasons preferably considered as instances of the exact cover problem. In the *exact cover problem*, we are given a set  $S$  and a collection  $U$  of its subsets, and the task is to form a partition of  $S$  by using sets in  $U$ . Here,  $S$  is the set of all cells of the square, and  $U$  has one set for each Costas array (consisting of the cells where there are dots). The *libexact* library [9] was used here to solve instances of the exact cover problem.

The computational results for all orders up to 27 are summarized in Table 1. The columns in the table are the order  $n$ , the total number of Costas arrays  $N_c$ , the number of equivalence classes of Costas arrays  $N_{ce}$ , the largest number of disjoint Costas arrays  $D(n)$ , and, if a full set of disjoint Costas arrays exists, the total number of Costas latin squares  $N_l$  and the number of equivalence classes of Costas latin squares  $N_{le}$ . The entries for  $N_c$  and  $N_{ce}$  have been taken from [2, 6, 12].

Table 1: Computational results for Costas arrays

$n$	$N_c$	$N_{ce}$	$D(n)$	$N_l$	$N_{le}$
1	1	1	1	1	1
2	2	1	2	1	1
3	4	1	2		
4	12	2	4	7	3
5	40	6	4		
6	116	17	6	124	26
7	200	30	6		
8	444	60	8	312	85
9	760	100	8		
10	2160	277	10	128	30
11	4368	555	10		
12	7852	990	12	16346	3761
13	12828	1616	$\leq 12$		
14	17252	2168	$\leq 13$		
15	19612	2467	$\leq 14$		
16	21104	2648	16	32768	8256
17	18276	2294	$\leq 16$		
18	15096	1892	18	5832	756
19	10240	1283	$\leq 18$		
20	6464	810	$\leq 19$		
21	3536	446	11		
22	2052	259	22	200	30
23	872	114	9		
24	200	25	8		
25	88	12	5		
26	56	8	6		
27	204	29	8		

It is easy to determine the automorphism groups of the classified Costas latin squares. The results were then validated by applying the orbit-

stabilizer theorem to get the total number, which coincides with the number in the column Nl of Table 1.

Note that there exists a CLS(8); this was also observed in [3]. This is the only order  $n$  for which there exists a CLS( $n$ ) but  $n + 1$  is not a prime number. An example of a CLS(8) can be found in the Appendix.

Table 1 lends additional numerical evidence to the truth of the conjecture of Etzion [3, Conjecture 2] that there is no CLS( $n$ ) for any odd  $n \geq 3$ .

### 3.2 Disjoint Near Costas Arrays

For near Costas arrays, one can carry out a similar computational study as for Costas arrays in the previous subsection. In this case exact cover cannot be used in a direct way to find full sets of disjoint arrays. One possibility is to use a clique search to find full sets, another is to use the framework of exact cover for all  $n^2$  possibilities for the empty cell in the near Costas latin square.

In any case, the number of near Costas latin squares grows very quickly; already for order 5 this number is 978982. Consequently, we will not be able to present an extensive table like that in Table 1. Instead, in the Appendix, we give just an example of a near Costas latin square for each order up to  $n = 8$ .

### 3.3 Vatican Costas Squares

For some of the structures defined in the introduction, the mappings in the definitions of equivalence must map rows to rows (resp. columns to columns), that is, the mapping is in a dihedral group of order 4 rather than 8. Vatican Costas squares, which we consider here, are such structures (so this specification should be added to the definition on [3, p. 149]).

We consider one representative from each equivalence class of Costas latin squares classified in the current work (cf. column Nle in Table 1), and check whether it is a Vatican Costas square. This check is also carried out for the square obtained by a 90 degree rotation. For the orders up to 27 for which Costas latin squares exist, that is,  $n = 2, 4, 6, 8, 10, 12, 16, 18$  and 22, the number of equivalence classes of Vatican Costas squares is 1, 1, 3, 0, 2, 2, 4, 3 and 5, respectively. One square from each equivalence class is given in the Appendix. Etzion [3] already classified such squares of order 6. It is interesting to note that many of these squares have the property that all of the columns are translates of a single column. Also note that  $n = 8$  is the only order for which there is a Costas latin square but no Vatican Costas square (coincidentally, perhaps, it is also the only order  $n$  where

there is a Costas latin square but  $n + 1$  is not a prime).

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## Appendix

### A.1 A Costas latin square of side 8

1	5	7	4	2	8	3	6
2	4	3	6	7	5	1	8
3	6	1	8	5	4	2	7
6	3	8	2	4	7	5	1
8	1	5	7	6	3	4	2
4	8	2	1	3	6	7	5
5	7	6	3	1	2	8	4
7	2	4	5	8	1	6	3

### A.2 Near Costas latin squares of orders 2 through 8

	2
1	3

	2	1
4	1	3
3	4	2

	2	1	3
4	5	2	1
3	1	5	4
5	4	3	2

	1	3	5	2
4	2	6	1	3
6	5	1	2	4
5	4	2	3	6
3	6	5	4	1

	1	3	4	7	5
6	2	7	1	3	4
5	4	6	2	1	3
7	5	1	3	2	6
4	7	2	5	6	1
3	6	5	7	4	2

	1	2	3	4	6	5
3	2	1	8	5	7	6
4	5	8	7	1	2	3
7	4	6	2	3	5	8
5	6	7	1	8	3	4
8	7	5	6	2	4	1
6	8	3	4	7	1	2

	1	2	4	5	7	9	3
3	2	1	8	9	6	4	5
4	9	6	3	8	1	5	7
9	7	3	1	2	4	8	6
5	8	7	6	4	9	2	1
8	4	5	2	7	3	6	9
7	6	9	5	1	8	3	2
6	5	8	7	3	2	1	4

### A.3 Vatican Costas squares

$n = 2$

1	2
2	1

$n = 4$

1	2	4	3
2	3	1	4
3	4	2	1
4	1	3	2

$n = 6$

1	2	3	4	5	6
3	6	2	5	1	4
5	3	1	6	4	2
2	4	6	1	3	5
6	5	4	3	2	1
4	1	5	2	6	3

1	3	2	5	6	4
2	4	3	6	1	5
3	5	4	1	2	6
4	6	5	2	3	1
5	1	6	3	4	2
6	2	1	4	5	3

1	2	3	4	5	6
4	1	5	2	6	3
3	6	2	5	1	4
2	4	6	1	3	5
6	5	4	3	2	1
5	3	1	6	4	2

$n = 10$

1	2	9	3	5	10	8	4	7	6
2	3	10	4	6	1	9	5	8	7
3	4	1	5	7	2	10	6	9	8
4	5	2	6	8	3	1	7	10	9
5	6	3	7	9	4	2	8	1	10
6	7	4	8	10	5	3	9	2	1
7	8	5	9	1	6	4	10	3	2
8	9	6	10	2	7	5	1	4	3
9	10	7	1	3	8	6	2	5	4
10	1	8	2	4	9	7	3	6	5

1	4	5	7	3	8	2	10	9	6
2	5	6	8	4	9	3	1	10	7
3	6	7	9	5	10	4	2	1	8
4	7	8	10	6	1	5	3	2	9
5	8	9	1	7	2	6	4	3	10
6	9	10	2	8	3	7	5	4	1
7	10	1	3	9	4	8	6	5	2
8	1	2	4	10	5	9	7	6	3
9	2	3	5	1	6	10	8	7	4
10	3	4	6	2	7	1	9	8	5

$n = 12$

1	2	5	3	10	6	12	4	9	11	8	7
2	3	6	4	11	7	1	5	10	12	9	8
3	4	7	5	12	8	2	6	11	1	10	9
4	5	8	6	1	9	3	7	12	2	11	10
5	6	9	7	2	10	4	8	1	3	12	11
6	7	10	8	3	11	5	9	2	4	1	12
7	8	11	9	4	12	6	10	3	5	2	1
8	9	12	10	5	1	7	11	4	6	3	2
9	10	1	11	6	2	8	12	5	7	4	3
10	11	2	12	7	3	9	1	6	8	5	4
11	12	3	1	8	4	10	2	7	9	6	5
12	1	4	2	9	5	11	3	8	10	7	6

1	6	9	11	10	2	8	4	5	3	12	7
2	7	10	12	11	3	9	5	6	4	1	8
3	8	11	1	12	4	10	6	7	5	2	9
4	9	12	2	1	5	11	7	8	6	3	10
5	10	1	3	2	6	12	8	9	7	4	11
6	11	2	4	3	7	1	9	10	8	5	12
7	12	3	5	4	8	2	10	11	9	6	1
8	1	4	6	5	9	3	11	12	10	7	2
9	2	5	7	6	10	4	12	1	11	8	3
10	3	6	8	7	11	5	1	2	12	9	4
11	4	7	9	8	12	6	2	3	1	10	5
12	5	8	10	9	1	7	3	4	2	11	6

$n = 16$

1	3	16	5	12	2	6	7	15	14	10	4	13	8	11	9
2	4	1	6	13	3	7	8	16	15	11	5	14	9	12	10
3	5	2	7	14	4	8	9	1	16	12	6	15	10	13	11
4	6	3	8	15	5	9	10	2	1	13	7	16	11	14	12
5	7	4	9	16	6	10	11	3	2	14	8	1	12	15	13
6	8	5	10	1	7	11	12	4	3	15	9	2	13	16	14
7	9	6	11	2	8	12	13	5	4	16	10	3	14	1	15
8	10	7	12	3	9	13	14	6	5	1	11	4	15	2	16
9	11	8	13	4	10	14	15	7	6	2	12	5	16	3	1
10	12	9	14	5	11	15	16	8	7	3	13	6	1	4	2
11	13	10	15	6	12	16	1	9	8	4	14	7	2	5	3
12	14	11	16	7	13	1	2	10	9	5	15	8	3	6	4
13	15	12	1	8	14	2	3	11	10	6	16	9	4	7	5
14	16	13	2	9	15	3	4	12	11	7	1	10	5	8	6
15	1	14	3	10	16	4	5	13	12	8	2	11	6	9	7
16	2	15	4	11	1	5	6	14	13	9	3	12	7	10	8

1	7	6	13	10	12	8	3	11	16	4	2	5	14	15	9
2	8	7	14	11	13	9	4	12	1	5	3	6	15	16	10
3	9	8	15	12	14	10	5	13	2	6	4	7	16	1	11
4	10	9	16	13	15	11	6	14	3	7	5	8	1	2	12
5	11	10	1	14	16	12	7	15	4	8	6	9	2	3	13
6	12	11	2	15	1	13	8	16	5	9	7	10	3	4	14
7	13	12	3	16	2	14	9	1	6	10	8	11	4	5	15
8	14	13	4	1	3	15	10	2	7	11	9	12	5	6	16
9	15	14	5	2	4	16	11	3	8	12	10	13	6	7	1
10	16	15	6	3	5	1	12	4	9	13	11	14	7	8	2
11	1	16	7	4	6	2	13	5	10	14	12	15	8	9	3
12	2	1	8	5	7	3	14	6	11	15	13	16	9	10	4
13	3	2	9	6	8	4	15	7	12	16	14	1	10	11	5
14	4	3	10	7	9	5	16	8	13	1	15	2	11	12	6
15	5	4	11	8	10	6	1	9	14	2	16	3	12	13	7
16	6	5	12	9	11	7	2	10	15	3	1	4	13	14	8

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
5	10	15	3	8	13	1	6	11	16	4	9	14	2	7	12
8	16	7	15	6	14	5	13	4	12	3	11	2	10	1	9
6	12	1	7	13	2	8	14	3	9	15	4	10	16	5	11
13	9	5	1	14	10	6	2	15	11	7	3	16	12	8	4
14	11	8	5	2	16	13	10	7	4	1	15	12	9	6	3
2	4	6	8	10	12	14	16	1	3	5	7	9	11	13	15
10	3	13	6	16	9	2	12	5	15	8	1	11	4	14	7
16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
12	7	2	14	9	4	16	11	6	1	13	8	3	15	10	5
9	1	10	2	11	3	12	4	13	5	14	6	15	7	16	8
11	5	16	10	4	15	9	3	14	8	2	13	7	1	12	6
4	8	12	16	3	7	11	15	2	6	10	14	1	5	9	13
3	6	9	12	15	1	4	7	10	13	16	2	5	8	11	14
15	13	11	9	7	5	3	1	16	14	12	10	8	6	4	2
7	14	4	11	1	8	15	5	12	2	9	16	6	13	3	10

1	3	8	5	4	10	14	7	15	6	2	12	13	16	11	9
2	4	9	6	5	11	15	8	16	7	3	13	14	1	12	10
3	5	10	7	6	12	16	9	1	8	4	14	15	2	13	11
4	6	11	8	7	13	1	10	2	9	5	15	16	3	14	12
5	7	12	9	8	14	2	11	3	10	6	16	1	4	15	13
6	8	13	10	9	15	3	12	4	11	7	1	2	5	16	14
7	9	14	11	10	16	4	13	5	12	8	2	3	6	1	15
8	10	15	12	11	1	5	14	6	13	9	3	4	7	2	16
9	11	16	13	12	2	6	15	7	14	10	4	5	8	3	1
10	12	1	14	13	3	7	16	8	15	11	5	6	9	4	2
11	13	2	15	14	4	8	1	9	16	12	6	7	10	5	3
12	14	3	16	15	5	9	2	10	1	13	7	8	11	6	4
13	15	4	1	16	6	10	3	11	2	14	8	9	12	7	5
14	16	5	2	1	7	11	4	12	3	15	9	10	13	8	6
15	1	6	3	2	8	12	5	13	4	16	10	11	14	9	7
16	2	7	4	3	9	13	6	14	5	1	11	12	15	10	8

$n = 18$

1	2	14	3	17	15	7	4	9	18	13	16	6	8	12	5	11	10
2	3	15	4	18	16	8	5	10	1	14	17	7	9	13	6	12	11
3	4	16	5	1	17	9	6	11	2	15	18	8	10	14	7	13	12
4	5	17	6	2	18	10	7	12	3	16	1	9	11	15	8	14	13
5	6	18	7	3	1	11	8	13	4	17	2	10	12	16	9	15	14
6	7	1	8	4	2	12	9	14	5	18	3	11	13	17	10	16	15
7	8	2	9	5	3	13	10	15	6	1	4	12	14	18	11	17	16
8	9	3	10	6	4	14	11	16	7	2	5	13	15	1	12	18	17
9	10	4	11	7	5	15	12	17	8	3	6	14	16	2	13	1	18
10	11	5	12	8	6	16	13	18	9	4	7	15	17	3	14	2	1
11	12	6	13	9	7	17	14	1	10	5	8	16	18	4	15	3	2
12	13	7	14	10	8	18	15	2	11	6	9	17	1	5	16	4	3
13	14	8	15	11	9	1	16	3	12	7	10	18	2	6	17	5	4
14	15	9	16	12	10	2	17	4	13	8	11	1	3	7	18	6	5
15	16	10	17	13	11	3	18	5	14	9	12	2	4	8	1	7	6
16	17	11	18	14	12	4	1	6	15	10	13	3	5	9	2	8	7
17	18	12	1	15	13	5	2	7	16	11	14	4	6	10	3	9	8
18	1	13	2	16	14	6	3	8	17	12	15	5	7	11	4	10	9

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
3	6	9	12	15	18	2	5	8	11	14	17	1	4	7	10	13	16
9	18	8	17	7	16	6	15	5	14	4	13	3	12	2	11	1	10
8	16	5	13	2	10	18	7	15	4	12	1	9	17	6	14	3	11
5	10	15	1	6	11	16	2	7	12	17	3	8	13	18	4	9	14
15	11	7	3	18	14	10	6	2	17	13	9	5	1	16	12	8	4
7	14	2	9	16	4	11	18	6	13	1	8	15	3	10	17	5	12
2	4	6	8	10	12	14	16	18	1	3	5	7	9	11	13	15	17
6	12	18	5	11	17	4	10	16	3	9	15	2	8	14	1	7	13
18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
16	13	10	7	4	1	17	14	11	8	5	2	18	15	12	9	6	3
10	1	11	2	12	3	13	4	14	5	15	6	16	7	17	8	18	9
11	3	14	6	17	9	1	12	4	15	7	18	10	2	13	5	16	8
14	9	4	18	13	8	3	17	12	7	2	16	11	6	1	15	10	5
4	8	12	16	1	5	9	13	17	2	6	10	14	18	3	7	11	15
12	5	17	10	3	15	8	1	13	6	18	11	4	16	9	2	14	7
17	15	13	11	9	7	5	3	1	18	16	14	12	10	8	6	4	2
13	7	1	14	8	2	15	9	3	16	10	4	17	11	5	18	12	6



1	6	12	11	9	17	13	16	5	14	7	4	8	18	2	3	15	10
2	7	13	12	10	18	14	17	6	15	8	5	9	1	3	4	16	11
3	8	14	13	11	1	15	18	7	16	9	6	10	2	4	5	17	12
4	9	15	14	12	2	16	1	8	17	10	7	11	3	5	6	18	13
5	10	16	15	13	3	17	2	9	18	11	8	12	4	6	7	1	14
6	11	17	16	14	4	18	3	10	1	12	9	13	5	7	8	2	15
7	12	18	17	15	5	1	4	11	2	13	10	14	6	8	9	3	16
8	13	1	18	16	6	2	5	12	3	14	11	15	7	9	10	4	17
9	14	2	1	17	7	3	6	13	4	15	12	16	8	10	11	5	18
10	15	3	2	18	8	4	7	14	5	16	13	17	9	11	12	6	1
11	16	4	3	1	9	5	8	15	6	17	14	18	10	12	13	7	2
12	17	5	4	2	10	6	9	16	7	18	15	1	11	13	14	8	3
13	18	6	5	3	11	7	10	17	8	1	16	2	12	14	15	9	4
14	1	7	6	4	12	8	11	18	9	2	17	3	13	15	16	10	5
15	2	8	7	5	13	9	12	1	10	3	18	4	14	16	17	11	6
16	3	9	8	6	14	10	13	2	11	4	1	5	15	17	18	12	7
17	4	10	9	7	15	11	14	3	12	5	2	6	16	18	1	13	8
18	5	11	10	8	16	12	15	4	13	6	3	7	17	1	2	14	9

$n = 22$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
7	14	21	5	12	19	3	10	17	1	8	15	22	6	13	20	4	11	18	2	9	16
3	6	9	12	15	18	21	1	4	7	10	13	16	19	22	2	5	8	11	14	17	20
21	19	17	15	13	11	9	7	5	3	1	22	20	18	16	14	12	10	8	6	4	2
9	18	4	13	22	8	17	3	12	21	7	16	2	11	20	6	15	1	10	19	5	14
17	11	5	22	16	10	4	21	15	9	3	20	14	8	2	19	13	7	1	18	12	6
4	8	12	16	20	1	5	9	13	17	21	2	6	10	14	18	22	3	7	11	15	19
5	10	15	20	2	7	12	17	22	4	9	14	19	1	6	11	16	21	3	8	13	18
12	1	13	2	14	3	15	4	16	5	17	6	18	7	19	8	20	9	21	10	22	11
15	7	22	14	6	21	13	5	20	12	4	19	11	3	18	10	2	17	9	1	16	8
13	3	16	6	19	9	22	12	2	15	5	18	8	21	11	1	14	4	17	7	20	10
22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
16	9	2	18	11	4	20	13	6	22	15	8	1	17	10	3	19	12	5	21	14	7
20	17	14	11	8	5	2	22	19	16	13	10	7	4	1	21	18	15	12	9	6	3
2	4	6	8	10	12	14	16	18	20	22	1	3	5	7	9	11	13	15	17	19	21
14	5	19	10	1	15	6	20	11	2	16	7	21	12	3	17	8	22	13	4	18	9
6	12	18	1	7	13	19	2	8	14	20	3	9	15	21	4	10	16	22	5	11	17
19	15	11	7	3	22	18	14	10	6	2	21	17	13	9	5	1	20	16	12	8	4
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11	22	10	21	9	20	8	19	7	18	6	17	5	16	4	15	3	14	2	13	1	12
8	16	1	9	17	2	10	18	3	11	19	4	12	20	5	13	21	6	14	22	7	15
10	20	7	17	4	14	1	11	21	8	18	5	15	2	12	22	9	19	6	16	3	13

1	3	17	5	2	19	20	7	11	4	10	21	15	22	18	9	8	13	16	6	14	12
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3	5	19	7	4	21	22	9	13	6	12	1	17	2	20	11	10	15	18	8	16	14
4	6	20	8	5	22	1	10	14	7	13	2	18	3	21	12	11	16	19	9	17	15
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9	11	3	13	10	5	6	15	19	12	18	7	1	8	4	17	16	21	2	14	22	20
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11	13	5	15	12	7	8	17	21	14	20	9	3	10	6	19	18	1	4	16	2	22
12	14	6	16	13	8	9	18	22	15	21	10	4	11	7	20	19	2	5	17	3	1
13	15	7	17	14	9	10	19	1	16	22	11	5	12	8	21	20	3	6	18	4	2
14	16	8	18	15	10	11	20	2	17	1	12	6	13	9	22	21	4	7	19	5	3
15	17	9	19	16	11	12	21	3	18	2	13	7	14	10	1	22	5	8	20	6	4
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