A Note on the Covering Numbers $g(1,3;v)$

Sherry Judah, R.C. Mullin and D.R. Stinson

ABSTRACT

An exact (1,3)-cover of order v is a family C of proper subsets of v-set V, each of which has cardinality at least 3, with the property that every unordered triple of distinct elements of V is contained in precisely one member of C. The number $g(1,3;v)$ is defined by $g(1,3;v) = \min\{|C| : C$ is a (1,3)-cover of order $v\}$. The value of $g(1,3;v)$ is known for infinitely values of v, and has been determined for $4 \leq v \leq 26$ with the exception of $v \in \{11,13,19,23,24\}$. Here we show that $g(1,3;19) = 77$, $125 \leq g(1,3;23) \leq 130$ and $g(1,3;24) = 130$.

1. Introduction.

Let V be a v-set of elements (called points), and let C be a collection of proper subsets of V (called blocks). The collection C is said to be a (1,3)-cover of order v if every triple of distinct points occurs in a unique block of C and every block contains at least three points. The number $g(1,3;v)$ is defined to be the least number of blocks which can occur in a (1,3)-cover of order v, that is,

$$g(1,3;v) = \min\{|C| : C \text{ is a minimum cover of order } v\}$$

A (1,3)-cover of order v which contains $g(1,3;v)$ blocks is said to be *minimal.* R.G. Stanton and J.G. Kalbfleisch [8] determined the value of $g(1,3;v)$ for $4 \leq v \leq 10$, and showed that $g(1,3;v) = 0(v^{3/2})$. The values of $g(1,3;v)$ have been determined for an infinite number of values of v by Hartman, Mullin and Stinson [3]. In addition, $g(1,3;v)$ has been determined for all v satisfying $12 \leq v \leq 26$, with the exception of $v \in \{13,19,23,24\}$ (see table 1). It is our purpose here to show that $g(1,3;19) = 77$, $g(1,3;23) \geq 125$, and $g(1,3;24) = 130$.

2. Preliminary results.

A finite linear space G is a pair (P,L) where P is a finite set of objects, called *points,* and L is a family of subsets of P called *lines,* which satisfies the following.

(i) Every pair of distinct points lies in (on) a unique line,

(ii) Every line contains at least two points, and no line contains all points.

A *near-pencil* is a finite linear space in which some line contains all but one of the points. The following is proved in [2].

Lemma 2.1. *Let F be a finite linear space on $v \geq 5$ points which is not a near-pencil. If b denotes the number of lines of F, then $b \geq B(v)$, where*

$$B(v) = \begin{cases} n^2+n+1 & \text{if } n^2+2 \leq v \leq n^2+n+1 \\ n^2+n & \text{if } n^2-n+3 \leq v \leq n^2+1 \\ n^2+n-1 & \text{if } n^2-n+2 = v. \end{cases}$$

An *extended near-pencil* of order v is a (1,3)-cover of a v-set v in which some block contains all but one point of V. Such a cover contains $1+\binom{v-1}{2}$ blocks.

Let C be a (1,3)-cover of order v, and let x be a point of C. Then the set of blocks

$$C_x = \{B\backslash\{x\}: x \in B, B \in C\}$$

is called the derived design of C (with respect to x). Clearly either C_x contains just one block (in which case C is an extended near-pencil), or C_x is a finite linear space on $v-1$ points. It is easily shown that if C_x is a near-pencil on $v-1$ points, then C contains at least $1+\binom{v-1}{2}$ blocks.

The following lemma on binomial coefficients is observed in [3].

Lemma 2.2. *Suppose that* $k_1, k_2, \ldots, k_b$ *are non-negative integers, and*

$$\sum_{i=1}^{b} k_i \geq qb + r,$$

where $0 \leq r < b$ *and* $q \geq 1$. *Then* $\sum_{i=1}^{b}\binom{k_i}{t} \geq r\binom{q+1}{t} + (b-r)\binom{q}{t}$; *with equality if and only if precisely* r *of the* k_i*'s are equal to* $q+1$, *and the remaining* k_i*'s are equal to* q *(hence* $\sum_{i=1}^{b} k_i = qb + r$*).*

Lemma 2.3. *Let* c *be any integer satisfying the inequality* $g(1,3;v) \leq c \leq \binom{v-1}{2}$. *Then the inequality*

$$v(v-1)(v-2) \geq q(q-1)[3v \cdot B(v-1) - 2c(q+1)],$$

where $q = \lfloor v \cdot B(v-1)/c \rfloor$ *and* $B(\cdot)$ *is as in Lemma 2.1, must hold.*

Proof. Suppose that there is a 3-cover of C of a set V, where $|C| \leq c <+ (v-1)(v-2)/2$. For $i = 1,2,\ldots,|C|$, let k_i denote the cardinality of the ith block of C; and if $|C| < c$, let $k_i = 0$ for $|C|+1 \leq i \leq c$. Since C_x cannot be either a single block or a near-pencil for any x in V, we have

$$\sum_{i=1}^{c} k_i \geq vB(v-1) = qc + r,$$

for some r satisfying $0 \leq r < c$. By Lemma 2.2, we have

$$v(v-1)(v-2) = 3!\sum_{i=1}^{c}\binom{k_i}{3}$$

$$\geq r(q+1)q(q-1) + (c-r)q(q-1)(q-2).$$

Substituting $r = vB(v-1) - qc$ and simplifying, we obtain the desired result. □

3. Determination of $g(1,3;v)$, $v = 19,24$.

Lemma 2.3 and modifications thereof can be used in the determination of certain covering numbers. This is demonstrated below.

Lemma 3.1. *The number* $g(1,3;24)$ *is* 130.

Proof. Applying Lemma 2.3 with $v = 24$ and $c = 129$ yields a contradiction; therefore $g(1,3;4) \geq 130$. But $g(1,3;25) = 130$ (see [3]), and the result follows. □

Lemma 3.2. *The number* $g(1,3;19)$ *is* 77.

Proof. It is shown in [9] that $g(1,3;18) = 76$; hence $g(1,3;19) \geq 76$. Now suppose that C is a (1,3)-cover of 19-set V, which contains 76 blocks. We first observe that C can contains most one block of size (cardinality) 3, and that is some point of V occurs in at least 22 blocks, then C cannot contains a block of size 3. Indeed we employ the method of Lemma 2.3, noting that $B(19) = 21$. First assume that C contains at least two blocks of size 3, and let $k_{75} = k_{76} = 3$. Then we find that $\sum_{i=1}^{74} k_i \geq 393 = 5(74) + 24$; thus

$$19 \cdot 18 \cdot 17 - 2(3 \cdot 2 \cdot 1) = 5802 \geq \sum_{i=1}^{74} k_i(k_i-1)(k_i-2) = 5820;$$

a contradiction. Similarly if $k_{76} = 3$ and some point has frequency 19, a contradiction is obtained. Now let us assume that C contains a block B of size 3. Then all other points occur in precisely 21 blocks. Thus the derived designs C_x, $x \in V$ can all be embedded in π, the projective plane of order 4 (see [2]). Thus, any C_x can be obtained from π either by deleting either three collinear points, or a "triangle" of 3 non-collinear points. If x is in B, it follows that B_x contains one line of length 2, twelve lines of length 4 and eight lines of length 5. If x is not in B, then B_x contains three lines of length 3, nine lines of length 4 and nine lines of length 5. Since there are three points in B and sixteen points not in B, we find that the number of blocks in C is

$$|C| = 3 \cdot 1/3 + 16 \cdot 3/4 + (3 \cdot 12 + 16 \cdot 9)/5 + (3 \cdot 8 + 16 \cdot 9)/6$$

which is 77, a contradiction. So C contains no block of size 3. By Lemma 2.4, if C contains a block of size $k \geq 7$, then $|C| > 76$, so C contains no such block. Thus C has only blocks of size 4, 5 and 6. However, $6 \cdot 5 \cdot 4$, $5 \cdot 4 \cdot 3$ and $4 \cdot 3 \cdot 2$ are all divisible by 12, whereas $19 \cdot 18 \cdot 17$ is not so

$$\sum_{i=1}^{76} k_i(k_i-1)(k_i-2) = 19 \cdot 18 \cdot 17$$

cannot be satisfied. Thus $g(1,3;19) \geq 77$. But by [5], $g(1,3;20) = 77$, so $g(1,3;19) = 77$. □

4. A bound for $g(1,3;23)$.

As shown in the next section, $g(1,3;v)$ has now been determined for all $v \leq 26$ with the exception of $v \in \{11,13,23\}$. In this section we show that $g(1,3;23) \geq 125$. (As in the preceding section, we have $g(1,3;23) \leq 130$.) A direct application of Lemma 2.3 yields $g(1,3;23) \geq 123$. This is improved below.

Lemma 4.1. *Any linear space F on 22 points which contains exactly 29 lines contains exactly one line of length 6, twelve lines of length 5 and sixteen lines of length 4 (and no other lines). Conversely any finite linear space on 22 points which has line sizes all of which lie in* {4,5,6} *contains precisely 29 lines.*

Proof. See [2] and [1]. □

Corollary 4.2. *There is no (1,3)-cover C of order 23 all of whose block sizes lie in* {5,6,7}.

Proof. Such a cover would contain exactly 23/7 blocks of size 7, which is absurd. □

Corollary 4.3. *Any* (1,3)*-cover of order* 23 *contains at least three points which lie in more than twenty-nine blocks.*

Proof. Any such cover must contain a block of size 3 or 4, and the derived design of a point in such a block must have at least thirty lines. □

We refer to the number of blocks of a cover C which contain a given point x as the *frequency* of x.

Lemma 4.4. *Any* (1,3)*-cover of order* 23 *contains at least* 124 *blocks.*

Proof. Using arguments similar to those of Lemma 3.2, it is readily shown that if $|C| = 123$, then there can be at most two points of frequency greater than 29 in C, contradicting Lemma 4.3. □

Lemma 4.5. *Any* (1,3)*-cover of order* 23 *which contains a block of size* $k \geq 8$ *contains at least* 130 *blocks.*

Proof. The result follows by applying Lemma 2.4. □

Note that if b_i ($i = 3,4,\ldots,7$) denotes the number of blocks of size i in a (1,3)-cover of order 23, then the b_i satisfy the following equations.

(i) $\sum_{i=3}^{7} b_i = |C|$,

(ii) $\sum_{i=3}^{7} i b_i = 29 \cdot 23 + e$,

(iii) $\sum_{i=3}^{7} \binom{i}{3} b_i = \binom{23}{3}$,

where e is the "excess frequency", that is, $e = \sum_{i=1}^{23} (f_i - 29)$, where f_i is the frequency of the ith point of C.

Lemma 4.6. *There is no* (1,3)*-cover of order* 23 *which contains exactly* 124 *blocks.*

Proof. Assume that such a (1,3)-cover C exists. As noted in Corollary 4.3, C contains at least three points of frequency greater than 29, hence in such a cover we have $e \geq 3$. If we assume that $e \geq 6$, then, bearing in mind that C must contain a block of size 3 or 4, Lemma 2.2 yields a contradiction, namely that $\sum_{i=1}^{124} \binom{k_i}{3} > \binom{23}{3}$, where k_i is the size of the ith block of C. Therefore e must lie in the range $3 \leq e \leq 5$. Thus there are at least 18 points of frequency 29 in C, and since each lies in a block of size 7, so there must be at least three such blocks in C. Moreover if $e = 3$, then we must have $b_3 = 1$, $b_4 = 0$; if $e = 4$ then $b_4 \leq 1$, and if $b_4 = 0$, then $b_3 > 1$; and if $e = 5$ and $b_4 = 0$, then $b_3 > 1$. It is readily verified that there is only one such solution to the above equations, namely $(b_3,b_4,b_5,b_6,b_7,e) = (10,75,44,4,3)$. Should a cover C exist corresponding to this distribution, it must contain exactly 20 points of frequency 29, and each point of B, the block of size 3 must have frequency exactly 30 in C. Thus there is a unique point x of frequency 30 which occurs in exactly one block of size 7 in C, since each point of frequency 29 occurs in precisely one such block. Note that x also occurs in B, the block of size 3 in C. Let c_i denote the number of lines of length i in C_x, the derived design of C with respect to x. Noting that $c_2 = c_6 = 1$, and $c_3 = 0$, counting pairs and lines in C_x yields the equations

$$6c_4 + 10c_5 = 215,$$

$$c_4 + c_5 = 28.$$

Hence $4c_5 = 47$, which is clearly impossible. Therefore no such C exists.

Corollary 4.7. *The number* $g(1,3;23)$ *is at least* 125.

5. Conclusions.

The numbers $g(1,3;v)$ have now been determined for $4 \leq v \leq 26$ with the exceptions of $v \in \{11,13,23\}$. The values of these numbers are exhibited in table 1 with appropriate references. The remaining cases appear to be very difficult. Indeed, it appears that determining $g(1,3;11)$ and $g(1,3;13)$ is beyond the scope of present methods.

Table 1

v	$g(1,3;v)$	reference
4	4	[8]
5	7	[8]
6	11	[8]
7	14	[8]
8	14	[8]
9	29	[8]
10	30	[8]
11	≤ 46	
12	47	[7]
13	?	
14	63	[5]
15	68	[4]
16	68	[5]
17	68	[5]
18	76	[9]
19	77	§3
20	77	[6]
21	77	[6]
22	77	[8]
23	≥ 125	§4
24	130	§3
25	130	[5]
26	130	[5]

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