Determination of a Covering Number

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Abstract

The exact covering number $g(1,3;18) = 76$.

Let t be a positive integer. An exact t-covering of a finite set X is a family $\mathcal{B}$ of proper subsets of X (called blocks) which satisfies:

(1) every t-subset of X occurs in exactly one block in $\mathcal{B}$

(2) every block has size at least t.

The covering number $g(1,t;v)$ is defined to be the minimum cardinality of $\mathcal{B}$, an exact t-covering of a v-set X. (Such a covering is called a minimum covering). The numbers $g(1,3;v)$ have been investigated in several papers. It is known [2] that $g(1,3;q^2+1) = q^3+q$ for all prime powers $q \geq 3$. Also, $g(1,3;v)$ is known for $4 \leq v \leq 26$, $v \neq 11,13,15,18$, or 23 (see [2][3]). In this paper we show that $g(1,3;18) = 76$.

Lemma 1 $g(1,3;18) \leq 76$.

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Proof: Let $\mathcal{B}$ denote the blocks of a Steiner system $S(3,6,22)$. Pick any block B, and delete all occurrences of four points $x_1,x_2,x_3,x_4 \in B$ in $\mathcal{B}$, and also delete B. The $S(3,6,22)$ has 77 blocks; we obtain an exact 3-covering, of the eighteen points, having 76 blocks. □

The following is established in [4].

Lemma 2 If an exact t-covering $\mathcal{B}$ of a v-set has a block of size k, then $|\mathcal{B}| \geq 1 + \frac{k-1}{v-2}\binom{k}{2}(v-k)$.

Corollary 3 If $g(1,3;v) \leq 75$, then no block in a minimum covering has size exceeding 6.

Proof: The function $f(k) = \frac{k-1}{16}\binom{k}{2}(18-k)$, for $7 \leq k \leq 17$, has a minimum value of 87.625, attained at $k=7$. □

Now, let us suppose there exists a perfect 3-cover $\mathcal{B}$ of 18 points with $b \leq 75$ blocks. By Corollary 3, all blocks have length 3,4,5, or 6. Let the blocks of $\mathcal{B}$ be denoted $L_1,\ldots,L_b$, and let the points be denoted $x_1,\ldots,x_{18}$. For $1 \leq i \leq b$, let k_i denote the length of L_i; for $1 \leq i \leq 18$, let r_i denote the degree of x_i (the number of blocks in which x_i occurs).

We have $\sum_{i=1}^{b} k_i = \sum_{i=1}^{18} r_i = N$, say. For $3 \leq i \leq 6$, suppose there are b_i blocks of length i. Then we have

(0) $3b_3 + 4b_4 + 5b_5 + 6b_6 = N$.

Counting triples, we have

$$6b_3 + 24b_4 + 60b_5 + 120b_6 = 4896,$$

or

(1) $b_3 + 4b_4 + 10b_5 + 20b_6 = 816$.

Of course,

(2) $b_3 + b_4 + b_5 + b_6 = b$.

We may solve (1) and (2) for b_6 in terms of b_3 and b_4, obtaining

$$b_6 = \frac{816 - 10b + 9b_3 + 6b_4}{10} .$$

Then, from (0),

$$N = 3b_3 + 4b_4 + 5(b - b_3 - b_4) + \frac{816 - 10b + 9b_3 + 6b_4}{10}$$

$$= 4b + \frac{816 - 11b_3 - 4b_4}{10} .$$

Since $b \le 75$, we have

<u>Lemma 4</u> If there is an exact 3-cover of 18 points having $b \le 75$ blocks, then $N = \sum_{i=1}^{b} k_i \le \frac{3816 - 11b_3 - 4b_4}{10}$.

For any point x_i of X let $D_i = \{B \backslash \{x_i\} : x_i \in B \in \mathcal{B}\}$. Each D_i, $1 \le i \le 18$ is an exact 2-cover of 17 points (i.e. a pairwise balanced design or finite linear space (FLS)). Further, each D_i has only blocks of length 2, 3, 4 or 5. The following results on finite linear spaces are useful.

Lemma 5 If a finite linear space has 17 points and no block of length exceeding 15, then (1) it has at least 20 blocks

(2) it can be embedded in a projective plane of order 4 if and only if it has 20 or 21 blocks

(3) it does not have exactly 22 blocks.

Proof: (1) is shown in [3]; (2) in [6]; and (3) in [5]. □

Thus, for any point x_i in an exact 3-cover of 18 points, the derived linear space D_i has at least 20 blocks. For $1 \le i \le 18$, let $r_i = 20 + \beta_i$, where $\beta_i \ge 0$ by the above remark. If $\delta = \sum_{i=1}^{18} \beta_i$, then $N = 360 + \delta$, so we have

Lemma 6 Under the above hypotheses, $10\delta + 11b_3 + 4b_4 \le 216$.

Now, for any finite linear space D with 17 points, define C(D), the content of D, to be $C(D) = 10\ell - 200 + \frac{11\ell_2}{3} + \ell_3$, where D has ℓ blocks and ℓ_i blocks of length i, $i \ge 2$. We have the following

Lemma 7 $\sum_{i=1}^{18} C(D_i) = 10\delta + 11b_3 + 4b_4$.

Proof:
$$\sum_{i=1}^{18} C(D_i) = \sum_{i=1}^{18} [10\ell(D_i) + \frac{11\ell_2(D_i)}{3} + \ell_3(D_i)] - 3600$$

$$= 10 \sum_{i=1}^{18} r_i + \frac{11}{3}(3b_3) + 4b_4 - 3600$$

$$= 10N + 11b_3 + 4b_4 - 3600$$

$$= 10\delta + 11b_3 + 4b_4. \qquad \square$$

Corollary 8 $\quad \sum_{i=1}^{18} C(D_i) \le 216$

Let F denote the family of all finite linear spaces with 17 points which have only blocks of length 2,3,4 or 5. For $D \in F$, let $v_{2,3} = v_{2,3}(D)$ denote the number of points which occur in a block of length 2 or 3.

Lemma 9 There are precisely three $D \in F$ with $\ell \le 21$, having the following parameters:

FLS	ℓ	ℓ_2	ℓ_3	ℓ_4	ℓ_5	C	$v_{2,3}$
F_1	20	0	0	16	4	0	0
F_2	21	0	6	8	7	16	15
F_3	21	1	3	11	6	$16\frac{2}{3}$	11

Proof: By Lemma 5, we know that any such FLS can be embedded in a projective plane of order 4. There are precisely three ways to delete four points from a plane of order 4: all four collinear; three collinear; and no three collinear. These yield, respectively, F_1, F_3, and F_2. $\square$

We need to explicitly mention one other FLS of F which we denote by F_4. Its parameters are $\ell = 23$, $(\ell_2,\ell_3,\ell_4,\ell_5) = (3,1,15,4)$,

$C = 42$, and $v_{2,3} = 4$. It can be constructed by replacing any line of length 4 in F_1, say $x_1x_2x_3x_4$ by the four lines $x_1x_2x_3$, x_1x_4, x_2x_4, and x_3x_4. Notice that F_4 does not satisfy condition (2) of the following Lemma.

Lemma 10 Suppose that every $D \in F$ with $\ell > 21$ satisfies

(a) $C(D) \geq 33$.

Further, suppose that every $D \in F$ with $\ell > 21$, other than F_4, satisfies

(b) $(v_{2,3} + 1)C > 216$.

Then $g(1,3;18) \geq 76$.

Proof: First suppose that $D_i \ncong F_1$ for $1 \leq i \leq 18$. Then $b_5 = \frac{16.18}{5}$ is not an integer, a contradiction. Thus some derived FLS has non-zero content. We may suppose that D_1 has the minimum non-zero content, of all the D_i, $1 \leq i \leq 18$. We have three cases.

Case (1): $D_1 \cong F_2$. We have $v_{2,3}(D_1) = 15$, so there are at least 16 points x_i such that D_i has non-zero content. By the minimality of $C(D_1)$ we obtain

$$\sum_{i=1}^{18} C(D_i) \geq 16.C(D_1) = 256 > 216, \text{ contradicting}$$

Corollary 8.

Case (2): $D_1 \cong F_3$. Suppose first that $D_i \cong F_1, F_2$, or F_3, for all i, $1 \leq i \leq 18$. For $1 \leq j \leq 3$, let a_j denote the number of points x_i such that $D_i \cong F_j$. Then $a_2 = 0$ since case (1) does not hold; so

$a_1 + a_3 = 18$. Then $b_5 = ((18-a_3)16+a_3.11)/5 = 288/5 - a_3$ is not an integer, a contradiction.

Thus there is a point, say x_{18}, with $\ell(D_{18}) \geq 22$. We use property (a) of the hypothesis.

Since $C(D_{18}) \geq 33$, then

$$\sum_{i=1}^{18} C(D_i) \geq v_{2,3}.C(D_1) + C(D_{18})$$

$$\geq 11.16\tfrac{2}{3} + 33 = 216\tfrac{1}{3} > 216,$$

a contradiction.

Case (3): $D_1 \not\cong F_1, F_2$ or F_3. If $D_1 \not\cong F_4$, we may apply property (b) of the hypothesis, containing

$$\sum_{i=1}^{18} C(D_i) \geq (v_{2,3}(D_1) + 1)C(D_1) > 216,$$

a contradiction.

Now suppose $D_1 \cong F_4$. We have $\sum_{i=1}^{18} C(D_i) \geq (v_{2,3}(F_4) + 1).C(F_4) = 210$. If $\sum_{i=1}^{18} C(D_i) < 216$ we must have only five points with non-zero content (since $6.42 > 216$), say x_1, x_2, x_3, x_4, x_5.

We may suppose that $x_1x_2x_3x_4$, $x_1x_2x_5$, $x_1x_3x_5$, and $x_1x_4x_5$ are four blocks of the cover, since $D_1 \cong F_4$. Now $x_1, \ldots, x_5$ are the only points of non-zero content, so the blocks in the cover of length 3 and 4 contains only the points $x_1, \ldots, x_5$, so $x_1x_2x_3x_4$ is the only block of length 4. Thus $D_5 \not\cong F_4$.

In D_5 we have $\ell_2 \geq 3$, $\ell_3 = 0$, $\ell \geq 23$, and $42 \leq C \leq 48$. Since $\sum_{j=2}^{5}\binom{j}{2}\ell_j = 136$, $\ell_2 + \ell_3$ is even; so $\ell_2 \geq 4$.

Then $C \geq 30 + 4 \cdot \frac{11}{3} = 44\frac{2}{3}$. Since $C \leq 48$, we must have $\ell_2 = 4$, $\ell_3 = 0$, and $\ell = 23$. Solving for ℓ_4 and ℓ_5, we find they are non-integral. It follows that the hypothesized cover does not exist. $\square$

The remainder of this section is devoted to showing that conditions (a) and (b) of Lemma 10 hold. Notice that the problem is now reduced to establishing certain facts concerning finite linear spaces.

<u>Lemma 11</u> A finite linear space $D \in F$ with $\ell > 21$ has $C(D) \geq 33$.

<u>Proof:</u> By Lemma 5, $\ell \neq 22$. If $\ell \geq 24$, then $C \geq 40$; thus $\ell = 23$, and $C = 30 + \frac{11}{3}\ell_2 + \ell_3$. In order that $C \leq 33$, $\ell_2 = 0$ and $\ell_3 \leq 3$. We have noted that $\ell_2 + \ell_3$ is even. Thus $(\ell_2, \ell_3) = (0,0)$ or $(0,2)$.

Given ℓ_2 and ℓ_3, the system $\sum_{i=2}^{5}\binom{i}{2}\ell_i = 136$, $\sum_{i=2}^{5}\ell_i = 23$ has a unique solution. If $(\ell_2,\ell_3) = (0,0)$, then $\ell_5 < 0$. If $(\ell_2,\ell_3) = (0,2)$, then $(\ell_2,\ell_3,\ell_4,\ell_5) = (0,2,20,1)$. However, the packing number $D(2,4,17) = 20$ (see [1]), so $\ell_4 + \ell_5 \leq 20$. Thus there is no FLS $D \in F$ with $\ell > 21$ and $C(D) < 33$. $\square$

Now, we wish to establish that for all $D \in F$, $D \not\cong F_4$, $(v_{2,3} + 1) \cdot C > 216$.

An FLS is said to be <u>triangle-free</u> if there do not exist three blocks of the form xy, xz, yz.

Let F_Δ denote the collection of FLS $\{F: F \in \mathcal{F},\ F \text{ is triangle-free}\}$. We note that $F_4 \in \mathcal{F}_\Delta$).

Lemma 12 Suppose that for all $D \in \mathcal{F}_\Delta$ with $\ell \geq 22$, other than F_4, $(v_{2,3} + 1)\cdot C > 216$. Then, for all $D \in \mathcal{F}$ with $\ell \geq 22$, $D \not\cong F_4$, We have $(v_{2,3} + 1)\cdot C > 216$.

Proof: Let $D \in \mathcal{F}\backslash F_\Delta$ (thus $D \not\cong F_4$). Then D contains a triangle, say x_1x_2, x_2x_3, x_1x_3. Let D' denote the FLS obtained by deleting x_1x_2, x_2x_3, x_1x_3, and adding the block $x_1x_2x_3$. It can easily be seen that $v_{2,3}(D) = v_{2,3}(D')$ and $C(D) > C(D')$, Keep repeating this process as long as possible, obtaining $D_1 \in \mathcal{F}_\Delta$. Then $(v_{2,3}(D) + 1).C(D) > (v_{2,3}(D_1) + 1).C(D_1)$. If D_1 has $\ell \geq 22$ then $(v_{2,3}(D) + 1).C(D) > 216$ (even if $D_1 = F_4$). If D_1 has $\ell \leq 22$ then $D_1 \cong F_2$ or F_3, since F_1 contains no blocks of length 3. Now $(v_{2,3}(F_2) + 1).C(F_2) > 216$, so we need only consider the case where $D_1 \cong F_3$. Then $v_{2,3}(D) \geq 11$, and $C(D) \geq 30$, (since $\ell(D) \geq 23$), so $(v_{2,3}(D) + 1).C(D) > 216$. This completes the proof. $\square$

Lemma 13 If $D \in \mathcal{F}_\Delta$ has $\ell \geq 22$ and $(v_{2,3} + 1).C \leq 216$ then

(i) $v_{2,3} \leq 5$

(ii) $\ell_3 \leq 2$

(iii) $\ell_2 \leq 4$

Proof: Recall that $\ell \geq 23$ and $C \geq 33$. Then $v_{2,3} \leq \left\lfloor \frac{216}{33} \right\rfloor - 1 = 5$, proving (i).

Next, three blocks of length 3 contain at least six distinct points, since pairs are not repeated.

This proves (ii).

To prove (iii) we use the assumption that D is triangle-free. Assume $\ell_2 \geq 5$; then $v_{2,3} \geq 5$. Since also $v_{2,3} \leq 5$, we have $v_{2,3} = 5$. Then $C \geq 30 + 5 \cdot \frac{11}{3} = 47\frac{2}{3}$, so $(v_{2,3} + 1) \cdot C > 216$, a contradiction. Thus $\ell_2 \leq 4$. □

Lemma 14 If $D \in F_\Delta$ has $\ell \geq 22$, then $(v_{2,3} + 1) \cdot C > 216$, unless $D \cong F_4$.

Proof: Assume $(v_{2,3} + 1).C \leq 216$. We have noted that $\ell \geq 23$, and $\ell_4 + \ell_5 \leq D(2,4,17) = 20$; thus $\ell_2 + \ell_3 \geq 3$. Since $\ell_2 + \ell_3$ is even, $\ell_2 + \ell_3 \geq 4$. Then, by Lemma 13, $\ell_2 + \ell_3 = 4$ or 6, and $(\ell_2,\ell_3) = (4,2)$ $(4,0)$, $(3,1)$, or $(2,2)$.

Assume that $\ell \geq 24$. Then $C \geq 40$, so $v_{2,3} \leq 4$. Hence $\ell_3 \leq 1$, and $\ell_2 \geq 3$ $(\ell_2 + \ell_3 \geq 4)$. Now $C \geq 40 + 3 \cdot \frac{11}{3} = 51$, so $v_{2,3} \leq 3$. But $\ell_2 \geq 3$ also implies that $v_{2,3} \geq 4$, since D is triangle-free. This is a contradiction.

Thus $\ell = 23$. Given ℓ_2 and ℓ_3, we may uniquely determine ℓ_2, ℓ_3, ℓ_4, and ℓ_5. If $(\ell_2,\ell_3) = (3,1)$, then $D \cong F_4$. If $(\ell_2,\ell_3) = (4,0)$ or $(2,2)$, then ℓ_4 and ℓ_5 are non-integral. Finally, if $(\ell_2,\ell_3) = (4,2)$, then $C \geq 30 + 4 \cdot \frac{11}{3} + 2 > 45$, and $v_{2,3} \geq 5$, so $(v_{2,3} + 1) \cdot C > 216$. This completes the proof. □

Theorem 15 $g(1,3;18) = 76$.

Proof: The above Lemmata 10, 11, 12, and 14 prove that $g(1,3;18) \geq 76$, and Lemma 1 establishes that $g(1,3;18) \leq 76$. $\square$

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