Skew Squares of Low Order

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Abstract. It is shown that for all odd $v \geq 1565$ there exists a skew Room square of side v. Moreover, for $7 \leq v \leq 1565$ there are at most 31 odd v for which no such square exists.

1. Introduction.

Thus we assume the definitions and terminology which occur in [3] and [4]. The following theorem appears there as well.

Theorem 1.1. (i) if $v \equiv 1 \bmod 6$ and $v \geq 46017$

or

(ii) if $v \equiv 3$ or $5 \bmod 6$ and $v \geq 17301$

then there exists a skew Room square of side v.

Our purpose here is twofold. We wish to improve the result above to show that if $v > 1565$ and v odd then $v \in SS$, where $SS = \{v : \exists$ a skew square of side $v\}$, as in [3]. Further we wish to outline a method of obtaining this result without employing a computer. We will assume the table of v not known to be in $oa(10)$ as obtained in [4] to be given. Although this table was constructed by computer, for the small values employed here that paper is essentially constructive. It is written so that for any particular value of t claimed to be in $oa(10)$, a method for producing 8 mutually orthogonal latin squares of side t can be obtained readily by using a sieving technique through the tables presented there. Further a

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"computer-free" proof (relying partially on that table for a few small values) of the following theorem is given there.

Theorem 1.2. If v is positive and $v \equiv 1 \bmod 8$ and $\{65,129\} \in SS$, then $v \in SS$.

Since that time J. Dinitz [2] has shown that $65 \in SS$. Hence if $129 \in SS$, then $v \in SS$ for all positive $v \equiv 1 \bmod 8$, $v \in SS$. The result cited as Theorem 1.1 was proved without employing a computer except to prove that for positive $v \equiv 1 \bmod 8$ (with the possible exception of $v = 129$), $v \in SS$. Thus the existence of a skew Room square of side 129 is sufficient to yield a "computer-free" proof of Theorem 1.1.

In the present paper, apart from obtaining members of oa(10) as mentioned above, the results given do not rely on a computer.

2. Squares of side $v \equiv 1 \bmod 6$.

The following is shown in [3].

Theorem 2.1. Suppose $m \neq 16$, $m \in oa(10)$ and $\{6m+1, 6t+1\} \subset SS$. Then there exists a skew square of side $56m + 6t + 1$. (It is easily verified that this square has subsquares of side $6m+1$ and $6t+1$).

In view of the above, it is clearly important to consider which values $v \equiv 1 \bmod 6$ belong to SS. To this end, we list several results and constructions concerning the existence of skew (Room) squares. The Lemmata 2.2 - 2.7 form an update of the results cited in [3], based on the results obtained there.

Lemma 2.2. For v an odd prime power, $v \neq 3, 5$, $v \in S$.

Lemma 2.3. Suppose there is a skew Room square of side v_2 which contains a skew Room subsquare of side v_3.

i) If $v_2 - v_3 \neq 6$ and if there is a skew Room square of side v_1 then there is a skew Room square of side $v_1(v_2-v_3) + v_3$ which contains skew Room subsquares of sides v_1, v_2 and v_3. This result also holds for $v_3 = 0$.

ii) If $v_3 \neq 0$ and $v_2 - v_3 \neq 12$, then there is a skew Room square of side $5(v_2-v_3) + v_3$ which contains skew Room subsquares of sides v_2 and v_3.

Lemma 2.4. If v is an odd positive integer and $v \notin SS$, then $v = 3n$ or $v = 5n$ or $v = 75n$, where $(n,15) = 1$.

Lemma 2.5. If v is odd, and if $7 \leq v \leq 67$, and if $v \neq 55$, then $v \in SS$.

Lemma 2.6. If $v \equiv 1 \bmod 12$ and v is positive, then $v \in SS$.

Lemma 2.7. If $v \equiv 1 \bmod 8$ is positive and $v \neq 129$, then $v \in SS$.

Lemma 2.8. If $v \equiv 1 \bmod 10$ is positive, then $v \in SS$.

The following is proved in [5].

Lemma 2.9. If $v \equiv 7 \bmod 24$ and $v > 0$, then $v \in SS$. Moreover such squares contain subsquares of order 7.

Other constructions based on pairwise balanced designs are given below. (For definitions, see [4]).

Theorem 2.10. Suppose that there exists integers m and t such that $0 \leq t \leq m$, $m \in oa(18)$, and $\{6m+1, 6t+1\} \subset SS$. Then $112m + 6t + 1 \in S$.

Proof. The proof is based on the fact that $\{16,17\} \subset oa(7)$ as is that of the similar theorem in [3], mutatis mutandis. □

Theorem 2.11. Suppose that there exist integers m and t such that $0 \leq t \leq m$ and that $m \in oa(18)$. Then $176m + 10t + 1 \in SS$.

Proof. The proof is based on the fact that $\{16,17\} \subset oa(11)$ and is that of the similar theorem in [3], mutatis mutandis. □

Theorem 2.12. Suppose that there exist integers m and t such that $0 \leq t \leq m$, and that $m \in oa(14)$. If $\{6m+1,6t+1\} \subset SS$, then $84m + 6t + 1 \in SS$.

Proof. This is as based on the fact that $\{12,13\} \subset oa(7)$. Proceed as above. □

Lemma 2.13. If m and t are integers such that $0 \leq t \leq m$, $m \in oa(43)$ and if $\{m,m+6t\} \in SS$ then $43m + 6t \in SS$.

Proof. Employing $EG(7,2)$ one obtains a group divisible design of type $42\{1\} + \{7\}$ by using a flat of order 7. Moreover, since $43 \in SS$ there is a group divisible design consisting of one block of size 43 and group type $43\{1\}$. This gives rise as above to a group divisible design with blocks of size 43 and 7 and groups of size m and $m + 6t$. □

The following is extremely useful for constructing skew Room squares of smaller orders which are multiples of three.

Lemma 2.14. If m and t are integers such that $0 \leq t \leq m$, $m \in oa(9)$, and $m + 6t \in SS$, then $51m + 6t \in SS$.

Proof. There is a group divisible design of group type $\{1\} + 8\{7\}$ and blocks sizes 7 and 9 obtainable from a resolvable orthogonal array $OA(8,7)$

to which a new point has been added. Moreover since $7 \in oa(10)$, there is a group divisible design with group type $9\{7\}$ and block sizes 7 and 9. As before this yields a design with block sizes 7 and 9 and group sizes $7m$ and $7m + 6t$. However if $7m + 6t \in SS$, then m is odd, and it was shown in [4] that $7m \in SS$ for all odd positive m. □

The following theorem is proved in [5].

<u>Theorem 2.15</u>. Let K be a PBD closed set. Suppose that there exists a PBD $(v,\{K\})$ which contains a flat of order w. Suppose that a is an integer such that $0 \le a \le w$, and that there exists $n - 2$ mutually orthogonal latin squares of side $v - a$ which contain $n - 2$ common subsquares of order $w - a$ for some $n \in K$. If $n(w-a) + a \in K$, then $n(v-a) + a \in K$.

<u>Lemma 2.16</u>. $\{695,2165,2995,3695,4435\} \subset SS$.

<u>Proof</u>. Note that $695 = 7(113-16) + 16$. Since $16 \in oa(7)$ there is a PBD of order $7.16 + 1$ with block sizes 7 and 17. Let F be a flat (block) of size 17. Let $a = 16$ as in the above theorem. Since $97 \in oa(5)$ the required latin squares exist.

For the remaining cases except 2165 we present the required data in numerical form.

v	w	a	n	v-a with sub w-a	Order of design constructed
$209 = 11 \times 19$	11	10	15	$199 \in oa(15)$	2995
$533 = 9.56 + 6.3 + 1$	7	6	7	$527 \in oa(7)$	3695
$405 = 15.27$	15	2	11	$403 = 13.31$	4435

For 2165, we proceed as follows. Start with $EG(7,2)$ taking a flat of order 7. Applying the theorem once, we obtain $307 = 7(49-6) + 6$ (since

$43 \in oa(7))$. This PBD can be shown to have all blocks of size 7 except for one of size 13. Using a well known construction for latin squares, we can obtain 5 mutually orthogonal latin squares of side 307 with common subsquares of side 7.

Now $323 = 21.23$. Thus there is a PBD of order 323 with a flat of order 23. Since $2165 = 7(323-16) + 16$ there is a PBD of order 2165 with block sizes from SS. □

Lemma 2.17. Suppose that $v = 6p + 1 \in SS$ for all primes p satisfying $7 \leq p \leq 1931$ except possibly for $p \in \{19,59,199\}$. If $v \equiv 1 \bmod 6$ and $1 \leq v < 11593$, then $v \in SS$, except possibly for $v \in \{115,355,1195\}$.

Proof. By lemma 2.6, we need only consider integers of the form $6t + 1$ where t is odd. Moreover by lemma 2.8, we may assume that $(t,5) = 1$. Since $3^{\alpha} \in SS$ for $\alpha = 2,3,4,...,$ and since $6.3 + 1 \in SS$, we have $v = 3^{\alpha}(6.3t+1-1) + 1 = v = 6.3^{\alpha+1} + 1 \in SS$ for $\alpha \geq 2$. But $6.9 + 1 \in SS$, therefore $v = 6.3^{\alpha+1} + 1 \in SS$ for all $\alpha \geq 1$. Moreover if $s > 0$ and $(s,15) = 1$, then $v = 6.3^{\alpha}s + 1 \in SS$ for $\alpha \geq 1$ since $s \in SS$. Thus we may assume that $(3,t) = 1$. If $t = 1$, then $6t + 1 \in SS$. Let $P = \{p: p \text{ prime}, 1 \leq p \leq 1931, p \notin \{19,59,199\}\}$. Suppose t is divisible by a prime $p \in P$. Then $v = 6t + 1 = (t/p)(6p+1-1) + 1 \in SS$. Thus if $1 \leq 6t + 1 \leq 8389$ and $v \notin SS$, then t is a product of primes in $\{19,59,199\}$ and $t \leq 1931$. By Lemma 2.9, we may assume that the number of primes (counting multiplicities) is odd. But $19^3 > 1931$. □

Lemma 2.17. Suppose that $v = 6p + 1$ where p is a prime satisfying $7 \leq p \leq 1931$. Then $6p + 1 \in SS$ except possibly for $p \in \{19,59,199\}$.

Proof. In view of the previous lemmata we need only consider those cases where $6p + 1 \equiv 0 \bmod 5$ and both conditions $6p + 1 \equiv 7 \bmod 24$ and $6p + 1 \equiv 0 \bmod 25$ fail to hold. That is we need only consider primes $p \equiv 9 \bmod 10$ where $p \not\equiv 4 \bmod 25$ and $p \not\equiv 1 \bmod 4$. These cases are treated below.

p	6p+1	construction
139	835	9(99-7) + 7, 99 = 7(15-1) + 1
239	1435	35.41
359	2155	84.25 + 6.9 + 1
419	2515	5(547-55) + 55, 547 = 56.9 + 6.7 + 1
439	2635	85.31, 85 = 7(13-1) + 1
499	2995	lemma 2.16
599	3595	39(99-7) + 7, 99 = 7(15-1) + 1
619	3715	112.32 + 6.25 + 1
659	3955	35.113
719	4315	84.49 + 6.33 + 1
739	4435	lemma 2.16
839	5035	19.265, 265 = 33(9-1) + 1
859	5155	5(1051-25) + 25, 1051 = 25(43-1) + 1
919	5515	27(211-7) + 7, 271 = 7(31-1) + 1
1019	6115	7(883-11) + 11, 883 = 9(99-1) + 1, 99 = 9.11
1039	6235	43.145, 145 = 9(17-1) + 1
1259	7555	51(155-7) + 7, 155 = 7(23-1) + 1, 51 = 5(11-1) + 1
1319	7915	67(127-9) + 9, 127 = 9(15-1) + 1
1399	8395	23.365, 365 = 13(29-1) + 1
1439	8695	7(1243-1) + 1
1499	8995	35.257

1619	9715	67.145, 145 = 9(17-1) + 1
1699	10195	171.56 + 103.6 + 1
1759	10555	171.56 + 163.6 + 1

This completes the proof. □

Corollary. If $v \equiv 1 \bmod 6$, $1 \leq v < 1593$, and $v \in \{115,355,1195\}$ then $v \in SS$.

Lemma 2.18. Suppose that there exists an increasing sequence of integers $M^* = (m_1, m_2, \ldots, m_n)$ such that

(i) $m_i \in oa(10)$, $i = 1,2,\ldots,n$;

(ii) $6m_i + 1 \in SS$, $i = 1,2,\ldots,n$;

(iii) $56m_i + 1 \leq 62m_{i-1} + 1$ for $i = 1,2,\ldots,n$;

(iv) $6m_n + 1 \geq 8389$; and

(v) $m_i \equiv 0 \bmod 3$, $i = 1,2,\ldots,n$.

If $v \equiv 1 \bmod 6$ and v satisfies $56m_i + 1 \leq v \leq 62m_n + 1$, then $v \in SS$ with the possible exception of $v \in \{56m + w : m \in M^*, w \in W = \{115,355,1195\}, w \leq 6m + 1$ and $m \equiv 0 \bmod 5\}$,

Proof. This is a direct application of theorem 2.1 and the corollary of lemma 2.17 except for the condition that $m \equiv 0 \bmod 5$ in the definition of exceptional cases. But if $v \equiv 1 \bmod 6$ is positive and $v \notin SS$, then $v \equiv 0 \bmod 5$. Since all members of $W \equiv 0 \bmod 5$, $v = 56m + w \equiv 0 \bmod 5$ for $w \in M^*$ implies that $m \equiv 0 \bmod 5$. □

Lemma 2.19. Suppose that $v \equiv 1 \bmod 6$. If for $11593 \leq v \leq 50593$, then $v \in SS$.

Proof. Consider the following table.

m	56m+1	62m+1
207	11593	12835
225	12601	13951
243	13609	15057
267	14953	16495
288	16129	17857
297	16633	18415
321	17977	19903
351	19657	21763
387	21673	23995
423	23688	26227
465	26041	28831
513	28729	31807
549	30745	34039
606	33937	37573
669	37465	41479
738	41329	45747
816	45697	50593

This covers all cases with the possible exception of $v \in \{12955, 13795, 26395, 27235\}$.

But

$$12955 = 127(103-1) + 1, \quad 26395 = 83(319-1) + 1,$$

$$13795 = 11.19(67-1) + 1, \quad 27235 = 17.89(19-1) + 1.$$

The lemma follows. □

Theorem 2.20. If $v \equiv 1 \bmod 6$, $v > 0$, and $v \notin \{115,355,1195\}$, then $v \in SS$. The proof is immediate from Theorem 1.1 and lemmata 2.17 and 2.18. □

3. Squares of side $v \equiv 5 \bmod 6$.

If v is positive, $v \equiv 5 \bmod 6$, and $v \notin SS$, then $v \equiv 0 \bmod 5$ by lemma 2.4. With this fact in mind we examine the spectrum of skew Room squares of side $\equiv 5 \bmod 6$.

Lemma 3.1. Suppose that there exists an increasing sequence of integers $M^* = (m_1,m_2,\ldots,m_n)$ such that

(i) $m_i \in oa(10)$, $i = 1,2,\ldots,n$;

(ii) $6m_i + 1 \in SS$, $i = 1,2,\ldots,n$;

(iii) $56m_i + 1 \le 62m_{i-1} + 1$ for $i = 2,3,\ldots,n$;

(iv) $6m_n + 1 \le 2094$;

(v) $m_i \equiv 2 \bmod 3$, $i = 1,2,\ldots,n$.

If $v \equiv 5 \bmod 6$ and v satisfies $56m_i + 1 \le v \le 62m_n + 1$, then $v \in SS$ with the possible exception of $v \in \{56m + w : m \in M^*, m \equiv 0 \bmod 5, w \in \{115,355,1195\}, w \le 6m + 1\}$.

Proof. This is the proof of lemma 2.18, mutatis mutandis. □

Lemma 3.2. Let p be a prime such that $7 \le p \le 239$. Then $5p \in SS$ except for $p \in Q$ where $Q = \{19,23,43,67,71,79,103,223,239\}$.

Proof. For $7 \le p \le 13$, this is covered in §2. By lemma 2.7, we need not consider $p \equiv 5 \bmod 8$.

p	5p	p	5p
17	85 = 7(13-1) + 1	137	685 = 19(37-1) + 1
31	155 = 11(15-1) + 1	139	695 = lemma 2.16
47	235 = 13(19-1) + 1	151	755 = 29(27-1) + 1
59	295 = 21(15-1) + 1	163	815 = 37(23-1) + 1
73	365 = 13(29-1) + 1	167	835 = 9(99-7)+7, 99 = 7(15-1) + 1
83	415 = 23(19-1) + 1	173	865 = 19(47-1) + 1
89	445 = 37(13-1) + 1	179	895 = 37(31-7)+7, 31 = 5(7-1) + 1
97	485 = 11(45-1) + 1	191	955 = 53(19-1) + 1
107	535 = 56.9 + 6.5 + 1	193	965 = 56.17 + 6.2 + 1
109	545 = 5(121-15)+15, 121 = 15(9-1) + 1		
113	565 = 47(13-1) + 1	199	995 = 71(15-1) + 1
127	635 = 56.11 + 6.3 + 1	211	1055 = 31(35-1) + 1
131	655 = 27(31-7)+7, 31 = 5(7-1) + 1		
		227	1135 = 81(15-1) + 1
		233	1165 = 97(13-1) + 1

This establishes the lemma. □

Lemma 3.3. Let v be an odd integer such that $v \equiv 0 \mod 5$, $(v,3) = 1$, and $7 \le v \le 1200$. If $v \notin SS$, then $v \in \{95,115,215,335,355,395,515,695,1115,1195\}$.

Proof. Clearly if $5n$ is odd and satisfies (i) $(n,3) = 1$ and (ii) $7 \le v \le 1200$, then n is a product of primes in the set Q of the previous lemma and $n \le 240$. The above values correspond to the members of Q itself, therefore we need only consider proper products of these. However $19^2 = 361 > 240$. □

Lemma 3.4. Suppose $v \equiv 5 \mod 6$, and $1195 < v \leq 4625$. If $v \notin SS$, then $v = 1565$.

Proof. Consider the following tables.

m	56m+1	62m+1
23	1289	1400
29	1625	1799
32	1793	1985
41	2297	2543
47	2633	2915
53	2969	3287
71	3977	4403

m	112m+1	118m+1
13	1457	1535
31	3473	3659

In view of the above we need only consider the following values (multiples of 25 are omitted) in view of lemma 2.4 (^{o} denotes a value $\equiv 1 \mod 8$).

$1205 = 43(29-1) + 1$

$1235 = 19.65$

$1265 = 23.55$

$1415 = 101(15-1) + 1$

$1445 = 17.85 = 7(13-1) + 1$

$1595 = 55.29$

$2015 = 31.65$

$2045 = 73(29-1) + 1$

2105^{o}

$2135 = 35.61$

$2165 =$ lemma 2.16

$2195 = 5(451-15)+15$, $451 = 15(13-1) + 1$

$2255 = 41.55$

$2285 = 17(141-7)+7$, $141 = 7(21-1) + 1$

$2555 = 35.73$

$2585 = 55.47$

$2615 = 43.59 + 6.13$

2945^{o}

3305^{o}

$3335 = 23.145$, $145 = 9(17-1) + 1$

$3365 = 29(117-1) + 1$, $117 = 9.13$

$3395 = 15.97$

$3455 = 157(23-1) + 1$

3665^{o}

$3695 =$ lemma 2.16

$3755 = 83.43 + 6.31$

3785^{o}

$3815 = 15.109$

$3845 = 31(125-1) + 1$

$3905 = 55.71$

$3935 = 281(15-1) + 1$

$3965 = 61.65$

$4415 = 19(239-7) + 7$, $239 = 7(35-1) + 1$

$4445 = 35.127$

$4505 = 53.85$, $85 = 7(13-1) + 1$

$4535 = 101.43 + 6.32$

$4565 = 101.43 + 6.39$

$4595 = 101.43 + 6.42$

This completes the lemma. □

Lemma 3.5. If $v \equiv 5 \bmod 6$ and $4625 < v \leq 9353$, then $v \in SS$.

Proof. Consider the following tables:

m	56m+1	62m+1
83	4649	5147
89	4985	5519
101	5657	6263
107	5993	6335
113	6329	7007
125	7001	7751
137	7673	8495
149	8345	9239

In view of the above we need only consider the following cases. (Again multiples of 25 are omitted).

$5555 = 55.101$	$5645 = 17(333-1) + 1, \ 333 = 13.27$
$5585 = 349(17-1) + 1$	$7115 = 112.61 + 6.47 + 1$
$5615 = 401(15-1) + 1$	$7355 = 56.131 + 6.3 + 1$

For $9245 \le v \le 9335$, and $v \equiv 5 \bmod 6$, write $v = 191.43 + 6w$, where $w = 172, 177, 182, 187$. □

<u>Lemma 3.6</u>. If $v \equiv 5 \bmod 6$ and $9353 \le v \le 17423$, then $v \in SS$.

<u>Proof</u>. Consider the following table:

m	56m+1	62m+1
167	9353	10355
179	10025	10799
191	10697	11843
197	11033	12215
209	11705	12703
221	12377	13703
239	13385	14819
263	14729	16307
281	15737	17403

This completes the lemma. □

The foregoing can be summarized as follows.

<u>Lemma 3.7</u>. If $v > 5$ and $v \equiv 5 \bmod 6$, then $v \in SS$ with the possible exception of $v \in \{95,215,335,395,515,1115,1565\}$.

4. Squares of side $v \equiv 0 \bmod 3$.

In this section we investigate the remaining case of v odd and $v \equiv 0 \bmod 3$, that is, $v \equiv 3 \bmod 6$.

Lemma 4.1. Suppose that there exists an increasing sequence of integers $M^* = (m_1, m_2, \ldots, m_n)$ such that

(i) $m_i \in oa(10)$, $i = 1,2,\ldots,n$;

(ii) $m_i \notin \{16,19,199\}$, $i = 1,2,\ldots,n$;

(iii) $56m_i + 1 \leq 62m_{i-1} + 1$ for $i = 2,3,\ldots,n$;

(iv) $m_i \equiv 1 \bmod 3$, $i = 1,2,\ldots,n$.

If $v \equiv 3 \bmod 6$ **and** v **satisfies** $56m_1 + 1 \leq v \leq 62m_n + 1$, then $v \in SS$, **with the possible exception of** $v \in \{56m + w : m \in M^*, w \in \{115,355,1195\}, w \leq 6m + 1, m \not\equiv 1 \bmod 10\}$.

Proof. **See theorem 2.18,** mutatis mutandis. (The condition $m \not\equiv 1 \bmod 10$ arises from lemma 2.8). □

Lemma 4.2. Suppose there exists a sequence of increasing odd integers $M^* = (m_1, m_2, \ldots, m_n)$ such that

(i) $m \in oa(9)$, $i = 1,2,\ldots,n$;

(ii) $57m_i \leq 63m_{i-1}$ for $i = 2,3,\ldots,n$;

(iii) $m_i \not\equiv 0 \bmod 3$, $i = 1,2,\ldots,n$.

If $v \equiv 3 \bmod 6$ and v satisfies $57m_1 \leq v \leq 63m_n$ then $v \in SS$, with the possible exception of $v \in \{57m + 6t : m + 6t \in \{95,115,215,335,355,395,515,1115,1195,1565\}$.

Proof. See theorem 2.18, mutatis mutandis. □

Lemma 4.3. Suppose that there exists a sequence of odd integers $M^* = (m_1, m_2, \ldots, m_n)$ such that

(i) $m_i \in oa(10)$, $i = 1,2,\ldots,n$;

(ii) $m_i \notin \{16,19,199\}$;

(iii) $57m_i \leq 62m_{i-1} + 1$ for $i = 2,3,\ldots,n$;

(iv) $m_i \equiv 1 \bmod 3$, $i = 1,2,\ldots,n$.

If $v \equiv 3 \bmod 6$ and v satisfies $57m \leq v \leq 62m_n + 1$, then $v \in SS$.

Proof. As above, with the following extension. The constructions of lemmata 4.1 and 4.2 can only fail in the simultaneous application above if $6t + 1$ and $6t + m$ are simultaneously in $\{115,1195\}$. This requires that $m = 1801$ and $t = 19$, that is, $v = 60651$. However $60651 \in SS$ by theorem 1.1. □

Lemma 4.4. If $v \equiv 3 \bmod 6$ and $v \geq 6777$ then $v \in SS$.

Proof. Consider the following table.

m	56m+1	57m	62m+1	possible exceptions
121	6777	6897	7503	
127	7113	7239	7875	7227
139	7785	7923	8619	7899
151		8607	9363	
163		9291	10107	
169		9633	10479	
181		10317	11223	
193		11001	11967	
211	11817		13083	
229		13053	14119	
253	14169	14421	15687	14283
277	15513	15789	17175	15627
283		16131	17751	

Thus in view of theorem 1.1, once we show that $\{7227, 7899, 14283, 15627\} \subset SS$, the theorem follows. But $7227 = 9.803$, $7899 = 359(23-1) + 1$, $14823 = 243.61$, $15627 = 601(27-1) + 1$. □

Lemma 4.1 can be strengthened as follows.

Lemma 4.5. Suppose there exists a sequence of odd integers M^* satisfying the hypothesis of lemma 4.1. If $m_n \le 115$, $v \equiv 3 \bmod 6$ and v satisfies the inequalities $56m_1 + 1 \le v \le 62m_n + 1$, then $v \in SS$.

Proof. Since $m_n \le 115$ any exceptions arising to the lack of skew Room squares of sides $\equiv 1 \bmod 6$ are compensated for by lemma 4.3. □

Lemma 4.6. If $v \equiv 3 \bmod 6$ and if $v > 2073$, then $v \in SS$.

Proof. We must treat appropriate v satisfying $2073 \le v \le 6771$. For v satisfying $2073 \le v \le 2103$, $v \in \{5637 + 6t + 1, 0 \le t \le 6\}$.

Now consider the following table.

m	57m	63m	possible exceptions		exceptions covered
37*	2109	2331			
41	2337	2583	2391	2511	
43*	2451	2709			2511
47	2679	2961	2727	2847	
49*	2793	3087			2847
53	3621	3329	3063,3183,3303		

(* Since these numbers are congruent to 1 mod 6, lemma 4.5 applies).

This shows that the result is valid for $2109 \le v \le 3339$ with the possible exception of $v \in \{2391, 2727, 3063, 3183, 3303\}$. But $2391 \equiv 1 \bmod 10$ and

2727 and 3303 are multiplies of 9, hence lemmata 2.4, 2.8 apply. Moreover, 3063 = 176.17 + 10.7 + 1 and 3183 = 51.61 + 6.12 (cf lemma 2.14).

For $3299 < v < 3417$, $v \neq 3363$, v is covered by $\{112.29 + 6t + 1, 16 \leq t \leq 29, t \neq 19\}$. Moreover 3363 = 57.59.

m	56m+1	62m+1	possible exceptions
61	3417	3783	
67	3753	4155	
73	4089	4527	
79	4425	4899	
82*	4593	5085	4707, 4947.

(* Since m is even, lemma 4.5 does not apply).

Since $4707 \equiv 0 \bmod 9$ and 4947 = 51.97, we need only consider $v \geq 5091$. However for $5091 \leq v \leq 5229$, consider $\{83.57 + 6t : 60 \leq t \leq 83\}$. This covers all cases except 5133 = 29.177, but 177 = 11(17-1) + 1. Note that 5235 = 15.349, 5241 = 131(41-1) + 1, and $5247 \equiv 0 \bmod 9$ and 5235 = 51.103. For $v \equiv 3 \bmod 6$ and $5256 \leq v \leq 5447$, $v \neq 5379$, note that $v \in \{112.47 + 6t + 1, 0 \leq t \leq 47, t = 19\}$. Moreover 5379 = 33.163. Returning to tables we have the following.

m	56m+1	62m+1
97	5433	6051
103	5769	6387
109	6105	6759

Since 6765 = 15.451 and 6771 = 61.111 where 111 can be written as 11(11-1) + 1, the result follows.

The foregoing can be summarized as follows. If v is an odd integer and $v \geq 2069$, then there exists a skew Room square of side v.

5. More special cases.

Lemma 5.1. If $v \equiv 0 \mod 15$ and v is an odd positive integer, then $v \in SS$ with the possible exception of $v = 75$.

Proof. It suffices to prove that $75p \in SS$ for all primes satisfying $7 \le p \le 23$ in view of the foregoing. But if $5p \in SS$, then $15 \times 5p \in SS$. Hence we need only consider those $p \le 23$ such that $5p$ has not been shown to belong to SS, i.e., $p \in \{19,23\}$. But $75.19 = 57.25$ and $75.23 = 57.26 + 6.12$. □

Lemma 5.2. Suppose $v \equiv 3 \mod 6$ and $v \ge 1311$, then $v \in SS$.

Proof. For $2015 \le v \le 2067$, we consider all cases. (Multiples of 5 and 9 are omitted, as are $v \equiv 1 \mod 10$).

$2013 = 33.61$ $\qquad$ $2049 \equiv 1 \mod 8$

$2019 = 17(131-13) + 13$, $131 = 13(11-1) + 1$

$2037 = 21.97$ $\qquad$ $2067 = 39.53$

For $1737 \le v \le 2007$, consider the following tables

m	56m+1	62m+1
31	1737	1923

m	112m+1	118m+1
17	1905	2007

For $1695 \le v \le 1731$, there is only one case not ruled out by the criteria of divisibility by 5 and 9 and $v \equiv 1 \mod 10$, namely $1713 \equiv 1 \mod 8$.
For $1653 \le v \le 1689$, $v \in \{57.29 + 6t, 0 \le t \le 6\}$. For $1581 \le v \le 1647$, we consider all cases, with the usual exceptions.

1587 = 61(27-1) + 1, 1599 = 39.41, 1617 = 33.49. For $1311 \le v \le 1575$, consider the following.

m	56m+1	57m	62m+1	63m	possible exceptions
25	1401		1551	1575	
23		1311		1449	1383

Also 1383 = 5(287-13) + 13, 287 = 13(23-1) + 1. □

<u>Lemma 5.3</u>. Suppose there exists a group divisible design of order v with block sizes from SS and with group sizes from L. Suppose also that for each $\ell \in L$ there exists a skew Room square of order $\ell + k$ which contains a skew subsquare of side k. Then $v + k \in SS$.

<u>Proof</u>. This is obtained by using the standard PBD construction of Lawless, on which this work is based, with a straightforward modification. □

<u>Lemma 5.4</u>. Suppose $7 \le p < 461$. If $3p \notin SS$, then $p \in \{23,29,31,41,43,53,71,73,79,101,151,199,233,239,293,311,349,359,409,421\}$.

<u>Proof</u>. We consider all cases. Clearly $p \equiv 7 \bmod 10$ and $p \equiv 3 \bmod 8$ can be omitted, and by lemma 2.5, we need not consider primes $p < 23$. □

<u>Corollary</u>. $\{447,843\} \subset SS$.

<u>Proof</u>. By deleting a point from EG(7,2), one obtains a group divisible design with blocks of size 7 and group type 8{6}. Moreover since $8 \in oa(7)$ there is a PBD of blocks of size 9 and 7 from which one can obtain a group divisible design with blocks of size 7 and 9 and group type {8} + 8{6}. Since $8 \in oa(9)$, one can construct a group divisible design of order 440 with block sizes 7 and 9 and group type 8{48} + {56}. By lemma 2.9 there is a skew Room square of side 48+7 with a subsquare of side 7. Also 56 + 7 = 9.7.

Hence $440 + 7 \in SS$. Similarly there are group divisible designs from PG(8,2) and EG(9,2) with block sizes 9 and group types 9{8} and 10{8} respectively. Since $11 \in oa(10)$, there exists a group divisible design with blocks of size 9 and group type 9{88} + {40}. But $88 + 11 = 9.11$ and $40 + 11 = 5(11-1) + 1$. □

p	3p	p	3p
61	183 = 13(15-1) + 1	269	807 = 31(27-1) + 1
89	267 = 19(15-1) + 1	271	813 = 29(29-1) + 1
103	309 = 11(29-1) + 1	281	843 = lemma 5.3
109	327 = 5(71-7)+7,71=7(11-1)+1	313	939 = 67(15-1) + 1
113	339 = 13(27-1) + 1	353	1059 = 23(47-1) + 1
149	447 = lemma 5.3	373	1119 = 43(27-1) + 1
173	519 = 37(15-1) + 1	379	1137 = 71(17-1) + 1
181	543 = 57.9 + 6.5	383	1149 = 41(29-1) + 1
191	573 = 13(45-1) + 1	389	1167 = 53(23-1) + 1
193	579 = 17(35-1) + 1	401	1203 = 13(99-7)+7,99=7(15-1)+1
223	669 = 57.11 + 6.7	431	1293 = 17(77-1) + 1
229	687 = 49(15-1) + 1	433	1299 = 59(23-1) + 1
241	723 = 19(39-1) + 1	439	1317 = 47(29-1) + 1

This completes the lemma. □

Lemma 5.5. Suppose that $v \equiv 3 \bmod 6$, $(v,5) = 1$ and $7 \le v \le 1305$. If $v \in SS$, then $v \in \{69,87,93,123,129,159,213,219,237,303,453,597,699,717,879,933,1047,1077,1227,1263\}$.

Proof. This is immediate from lemma 5.4 and the fact that $23^2 > 461$. □

We conclude with our main result, which summarizes the foregoing.

Theorem 5.6. If v is an odd integer ≥ 7, then there exists a skew Room square of side v with the possible exception of those given in the following table.

69	75	87	93	95
115	123	129	159	213
215	219	237	303	335
355	395	453	515	597
699	717	879	933	1047
1077	1115	1195	1227	1263
1565				

Corollary. There are at most 31 odd $v \geq 7$ such that no skew Room square of side v exists.

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