A Perfect One-factorization for K_{40}

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Abstract

In this paper, we construct an even starter-induced perfect one-factorization for the complete graph on 40 vertices.

1. Introduction

A *one-factorization* of a complete graph K_{2n} is a partition of the edge-set of K_{2n} into $2n - 1$ *one-factors*, each of which contains n edges that partition the vertex set of K_{2n}. A *perfect* one-factorization (P1F) is a one-factorization in which every pair of distinct one-factors forms a Hamiltonian cycle of the graph.

It has been known for some time that P1Fs on K_{2n} exist when n is prime, when $2n - 1$ is prime, and when $2n \in \{16, 28, 244, 344\}$ (see [1]). In fact, these were the only known examples until recently. A P1F of K_{36} was found by the authors [8]; a non-isomorphic example was also constructed by Kobayashi, Awoki, Nakazaki and Nakamura [5]. A P1F of K_{50} was found by tha authors and E. Ihrig [4]. P1Fs of K_{1332} and K_{6860} were found by Kobayashi and Kiyasu-Zen'iti [6]. Also, Dinitz and Stinson used quotient starters to construct seven new P1Fs, of K_{126}, K_{170}, K_{730}, K_{1370}, K_{1850}, K_{2198}, and K_{3126} [3].

In this paper, we use a combination of hill-climbing and backtracking algorithms to generate even starter-induced one-factorizations, and discover a P1F for K_{40}. The smallest unknown case is now K_{52}.

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2. Even starters

An *even starter* in Z_{2n} is a set of (n – 1) pairs of elements $E = \{\{x_1, y_1\}, \{x_2, y_2\}, ..., \{x_{n-1}, y_{n-1}\}\}$ such that

(1) every non-zero element of Z_{2n} except one, denoted m, occurs as an element in some pair of E, and

(2) every non-zero element of Z_{2n} except n occurs as a difference of some pair of E.

Define $E^* = E \cup \{\{0, \infty_1\}\} \cup \{\{m, \infty_2\}\}$, and $g + \infty_i = \infty_i + g = \infty_i$ for $g \in Z_{2n}$ and i = 1, 2. Also define $Q^* = \{\{g, g+n\} : g \in Z_{2n}\} \cup \{\{\infty_1, \infty_2\}\}$. Then $F = \{E^* + g : g \in Z_{2n}\} \cup \{Q^*\}$ is a one-factorization of K_{2n+2}.

Most known examples of P1F arise from starters (for a definition, see [2] or [8]) or even starters. In [1], Anderson enumerates all P1Fs in K_{2n} arising from starters and even starters, for $2n \le 22$. These empirical results suggest that there exist both a starter-induced P1F and an even starter-induced P1F in K_{2n} for all $2n \ge 12$. Hence, it seems likely that starters or even starters might provide new examples of P1F for larger values of 2n.

We use a modification of the hill-climbing algorithm in [2] to generate in a random manner the first six pairs of elements (having differences one through six) of an even starter. For each such partial even starter, we then use a backtracking algorithm to generate all (complete) even starters extending the given partial even starter, and test the induced one-factorizations for perfection. For those who are interested in hill-climbing algorithms, we suggest [7].

3. A P1F for K_{40}

We implemented the algorithms above using Pascal / VS on the University of Manitoba Amdahl 580 computer. After 150 hours of CPU time, we found the following even starter in Z_{38} which induces a P1F of K_{40}:

{{1, 2},{3, 5},{29, 32},{23, 27},{14, 19},{7, 13},{30, 37},
{28, 36},{8, 17},{15, 25},{20, 31},{4, 16},{9, 22},{21, 35},
{11, 26},{18, 24},{33, 12},{6, 24}}.

We note that the omitted element m = 10. The automorphism group of the induced P1F is Z_{38}.

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