All or Nothing at All

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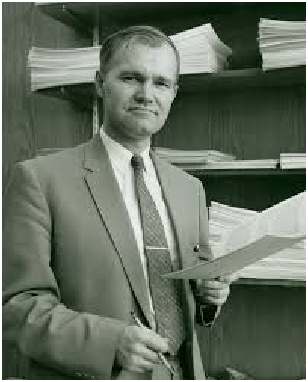
In Memory of Ralph Stanton, 1923–2010



All-or-nothing Transforms

- X is a finite set
- s is a positive integer, and $\phi : X^s \to X^s$.
- ϕ is an unconditionally secure all-or-nothing transform provided that the following properties are satisfied:
    1. ϕ is a bijection.
    2. If any $s-1$ of the s output values $y_1, \ldots, y_s$ are fixed, then the value of any one input value x_i $(1 \leq i \leq s)$ is completely undetermined.
- We will denote such a function as an (s, v)-*AONT*, where $v = |X|$.
- The desired property can be expressed as

$$\mathsf{H}(X_i \mid Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_s) = \mathsf{H}(X_i),$$

for all i and j such that $1 \leq i \leq s$ and $1 \leq j \leq s$.

Cryptographic Motivation

- Rivest defined *AONT* in 1997 to provide a mode of operation for block ciphers that would require the decryption of all blocks of an encrypted message in order to determine any specific single block of plaintext.
- Suppose we are given s blocks of plaintext, $(x_1, \ldots, x_s)$.
- First, we apply an *AONT*, computing

$$(y_1, \ldots, y_s) = \phi(x_1, \ldots, x_s).$$

- Then we encrypt $(y_1, \ldots, y_s)$ using a block cipher.
- At the receiver's end, the ciphertext is decrypted, and then the inverse transform ϕ^{-1} is applied to restore the s plaintext blocks.
- Note that the transform ϕ is not secret.

Linear AONT

- Let $\mathbb{F}_q$ be a finite field of order q.
- An (s, q)-*AONT* defined on $\mathbb{F}_q$ is linear if each y_i is an $\mathbb{F}_q$-linear function of $x_1, \ldots, x_s$.

Theorem 1 (Stinson, 2000)

Suppose that q is a prime power and M is an invertible s by s matrix with entries from $\mathbb{F}_q$, such that no entry of M is equal to 0. Then the function $\phi : (\mathbb{F}_q)^s \to (\mathbb{F}_q)^s$ defined by

$$\phi(\mathbf{x}) = \mathbf{x}M^{-1}$$

is a linear (s, q)-AONT.

Example: Hadamard Matrices

- Suppose $p > 2$ is prime, $s \equiv 0 \bmod 4$, and H is a Hadamard matrix of order s.
- H has entries ± 1 and $HH^T = sI_s$.
- Construct M by reducing the entries of H modulo p.
- Then M yields a linear (s, p)-*AONT*.
- If $s = 4$ and $p = 3$, we have

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \rightarrow M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

Example: Cauchy Matrices

- An s by s Cauchy matrix can be defined over $\mathbb{F}_q$ if $q \geq 2s$.
- Let $a_1, \dots, a_s, b_1, \dots, b_s$ be distinct elements of $\mathbb{F}_q$.
- Let

$$c_{ij} = \frac{1}{a_i - b_j},$$

  for $1 \leq i \leq s$ and $1 \leq j \leq s$.
- Then $C = (c_{ij})$ is the Cauchy matrix defined by the sequence $a_1, \dots, a_s, b_1, \dots, b_s$.
- A Cauchy matrix C is invertible, and all of its entries are non-zero, so C yields an (s, q)-*AONT*.

Example: The Bierbrauer Construction

- Let $q = p^k$ where $q > 2$, p is prime and k is a positive integer.
- Let $\lambda \in \mathbb{F}_q$ be such that $\lambda \notin \{s - 1 \bmod p, s - 2 \bmod p\}$.
- Define $\gamma = (s - 1 - \lambda)^{-1}$; note that $\gamma \neq 0, 1$.
- Let M be the following (symmetric) matrix:

$$M = \begin{pmatrix} 1-\gamma & -\gamma & -\gamma & \dots & -\gamma & \gamma \\ -\gamma & 1-\gamma & -\gamma & \dots & -\gamma & \gamma \\ -\gamma & -\gamma & 1-\gamma & \dots & -\gamma & \gamma \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\gamma & -\gamma & -\gamma & \dots & 1-\gamma & \gamma \\ \gamma & \gamma & \gamma & \dots & \gamma & -\gamma \end{pmatrix}.$$

Example: The Bierbrauer Construction (cont.)

- It is straightforward to verify that M is invertible; indeed, we have

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & \lambda \end{pmatrix}.$$

- Therefore M yields an (s, q)-*AONT*.
- This *AONT* is also very efficient computationally, since it is sparse (it contains mostly 0 entries).

Binary Transforms

- A transform defined over $\mathbb{F}_2$ is termed a binary transform.
- A binary transform automatically yields a transform over any field $\mathbb{F}_{2^n}$, in which the only computations are exclusive-ors of bitstrings.
- Unfortunately, there is no (linear or nonlinear) $(s, 2)$-*AONT* for any $s \geq 2$!
- This suggests looking for (binary, linear) transforms that are "close to" *AONT*.
- Suppose that s is even, and let $M = J_s - I_s$ (where J_s is the s by s all-1's matrix).

Binary Transforms (cont.)

- For example, when $s = 4$, we have

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

- Then $M^{-1} = M$, where M is considered as a matrix over $\mathbb{F}_2$.
- In this resulting transform, each x_j will depend on all the y_i's except for y_j.
- The density of 1's in the example above is $12/16 = 3/4$.

Generalized AONT

- Let $|X| = v$ and let $1 \leq t \leq s$.
- $\phi : X^s \to X^s$ is a t-all-or-nothing transform provided that the following properties are satisfied:
  1. ϕ is a bijection.
  2. If any $s - t$ of the s output values $y_1, \ldots, y_s$ are fixed, then any t of the input values x_i $(1 \leq i \leq s)$ are completely undetermined.
- We will denote such a function ϕ as a (t, s, v)-*AONT*.
- The original definition corresponds to a 1-*AONT*.
- Property 2 can be rephrased as follows: for all $\mathcal{X} \subseteq \{X_1, \ldots, X_s\}$ with $|\mathcal{X}| = t$, and for all $\mathcal{Y} \subseteq \{Y_1, \ldots, Y_s\}$ with $|\mathcal{Y}| = t$, it holds that

$$\mathsf{H}(\mathcal{X} \mid \{Y_1, \ldots, Y_s\} \setminus \mathcal{Y}) = \mathsf{H}(\mathcal{X}). \tag{1}$$

Linear t-AONT

For an s by s matrix M with entries from $\mathbb{F}_q$, and for $I, J \subseteq \{1, \ldots, s\}$, define $M(I, J)$ to be the $|I|$ by $|J|$ submatrix of M induced by the columns in I and the rows in J.

Theorem 2

Suppose that q is a prime power and M is an invertible s by s matrix with entries from $\mathbb{F}_q$. Let

$$\mathcal{X} \subseteq \{X_1, \ldots, X_s\}, |\mathcal{X}| = t,$$

and let

$$\mathcal{Y} \subseteq \{Y_1, \ldots, Y_s\}, |\mathcal{Y}| = t.$$

Then the function $\phi(\mathbf{x}) = \mathbf{x}M^{-1}$ satisfies (1) with respect to $\mathcal{X}$ and $\mathcal{Y}$ if and only if the t by t submatrix $M(I, J)$ is invertible, where $I = \{i : X_i \in \mathcal{X}\}$ and $J = \{j : Y_j \in \mathcal{Y}\}$.

Cauchy Matrices, Again

Any square submatrix of a Cauchy matrix is again a Cauchy matrix, and therefore it (the submatrix) is invertible. So we have the following result.

Theorem 3

Suppose q is a prime power and $q \geq 2s$. Then there is a linear transform that is simultaneously a (t, s, q)-AONT for all t such that $1 \leq t \leq s$.

Binary t-AONT

- We quantify the "closeness" of M to a *t-AONT* by considering the ratio of the number of invertible t by t submatrices to the total number of t by t submatrices.
- For an s by s invertible $0-1$ matrix M and for $1 \leq t \leq s$, we define

$$N_t(M) = \text{number of invertible } t \text{ by } t \text{ submatrices of } M$$

and

$$R_t(M) = \frac{N_t(M)}{\binom{s}{t}^2}.$$

- We refer to $R_t(M)$ as the t-density of the matrix M.
- We also define

$$R_t(s) = \max\{R_t(M) : M \text{ is an } s \text{ by } s \text{ invertible } 0-1 \text{ matrix}\}.$$

- $R_t(s)$ denotes the maximum t-density of any s by s invertible $0-1$ matrix.

Invertible 2 by 2 $0-1$ Matrices

A 2 by 2 $0-1$ matrix is invertible if and only if it is one of the following six matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Example

- Define a 3 by 3 matrix:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

- Seven of the nine 2 by 2 submatrices of M are invertible.
- The only non-invertible 2 by 2 submatrices are $M(\{1,3\},\{1,2\})$ and $M(\{1,2\},\{1,3\})$.
- Both of these submatrices are equal to

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

- Finally, M itself is invertible.
- Therefore, $R_2(M) = 7/9$.
- In fact, this is optimal, so $R_2(3) = 7/9$.

Another Example

- Consider the 4 by 4 matrix $J_4 - I_4$:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

- 30 of the 36 2 by 2 submatrices of M are invertible.
- Also, M itself is invertible.
- Therefore, $R_2(M) = 5/6$.
- In fact, this is optimal, so $R_2(4) = 5/6$.

An Upper Bound on $R_2(s)$

- Let N be a 2 by s $0-1$ matrix and consider its 2 by 1 submatrices.
- Suppose there are:
  - a_0 occurrences of $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$,
  - a_1 occurrences of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,
  - a_2 occurrences of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and
  - a_3 occurrences of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- Of course $a_0 + a_1 + a_2 + a_3 = s$.
- The number of invertible 2 by 2 submatrices in N is

$$a_1 a_2 + a_1 a_3 + a_2 a_3.$$

An Upper Bound on $R_2(s)$ (cont.)

- This expression is maximized when

$$a_0 = 0, \quad a_1 = a_2 = a_3 = s/3.$$

- Therefore, the maximum number of invertible 2 by 2 submatrices is

$$3\left(\frac{s}{3}\right)^2 = \frac{s^2}{3}.$$

- We have proven the following result.

Lemma 4

A 2 by s $0-1$ matrix contains $\leq s^2/3$ invertible 2 by 2 submatrices.

An Upper Bound on $R_2(s)$ (cont.)

Theorem 5

For any $s \geq 2$, it holds that

$$R_2(s) \leq \frac{2s}{3(s-1)}.$$

Proof.

From Lemma 4, in any two rows of M there are at most $s^2/3$ invertible 2 by 2 submatrices. In the entire matrix M, there are $\binom{s}{2}$ ways to choose two rows, and there are $\binom{s}{2}^2$ submatrices of order 2. This immediately yields

$$R_2(s) \leq \frac{\binom{s}{2}(s^2/3)}{\binom{s}{2}^2} = \frac{2s}{3(s-1)}.$$

$\square$

An Improved Upper Bound

- We begin by establishing upper bound on the number of invertible 2 by 2 submatrices in any 4 by s $0-1$ matrix.
- Label the non-zero vectors in $\{0,1\}^4$ in lexicographic order as follows:

$$\begin{array}{ccc} b_0 = (0,0,0,0) & b_1 = (0,0,0,1) & b_2 = (0,0,1,0) \\ b_3 = (0,0,1,1) & \dots & b_{15} = (1,1,1,1). \end{array}$$

- For $1 \leq i, j \leq 15$, define c_{ij} to be the number of invertible 2 by 2 submatrices in the 4 by 2 matrix $\left(\begin{array}{c|c} b_i^T & b_j^T \end{array} \right)$.
- Let $C = (c_{ij})$.
- C is a 15 by 15 symmetric matrix with zero diagonal such that every off-diagonal element is a positive integer.

The Matrix C

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 3 & 2 & 3 \\ 1 & 1 & 0 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 4 & 5 & 5 & 4 \\ 1 & 1 & 2 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 1 & 0 & 3 & 2 & 2 & 3 & 4 & 5 & 3 & 2 & 5 & 4 \\ 2 & 1 & 3 & 1 & 3 & 0 & 2 & 2 & 4 & 3 & 5 & 3 & 5 & 2 & 4 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 3 & 5 & 5 & 5 & 5 & 5 & 5 & 3 \\ 1 & 1 & 2 & 1 & 2 & 2 & 3 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 2 & 3 & 4 & 5 & 1 & 0 & 3 & 2 & 3 & 2 & 5 & 4 \\ 2 & 1 & 3 & 2 & 4 & 3 & 5 & 1 & 3 & 0 & 2 & 3 & 5 & 2 & 4 \\ 2 & 2 & 2 & 3 & 5 & 5 & 5 & 2 & 2 & 2 & 0 & 5 & 5 & 5 & 3 \\ 2 & 2 & 4 & 1 & 3 & 3 & 5 & 1 & 3 & 3 & 5 & 0 & 2 & 2 & 4 \\ 2 & 3 & 5 & 2 & 2 & 5 & 5 & 2 & 2 & 5 & 5 & 2 & 0 & 5 & 3 \\ 3 & 2 & 5 & 2 & 5 & 2 & 5 & 2 & 5 & 2 & 5 & 2 & 5 & 0 & 3 \\ 3 & 3 & 4 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 4 & 3 & 3 & 0 \end{pmatrix}.$$

A Quadratic Program

Define $\mathbf{z} = (z_1, \ldots, z_{15})$ and consider the following quadratic program $\mathcal{Q}$:

Maximize	$\gamma = \frac{1}{2}\mathbf{z}C\mathbf{z}^T$
subject to	$\sum_{i=1}^{15} z_i \leq 1$ and $z_i \geq 0$, for all $i, 1 \leq i \leq 15$.

We were able to solve the quadratic program $\mathcal{Q}$ using the BARON software on the NEOS server

`http://www.neos-server.org/neos/`.

The optimal solution to $\mathcal{Q}$ is $\gamma = 15/8$.

The Improved Bound

- It follows that the number of invertible 2 by 2 submatrices in a 4 by s matrix is at most $15s^2/8$.
- The number of invertible 2 by 2 submatrices in an s by s matrix is at most

$$\frac{\binom{s}{4}}{\binom{s-2}{2}} \times \frac{15s^2}{8} = \frac{5s^3(s-1)}{32}.$$

- Hence,

$$R_2(s) \leq \frac{5s^3(s-1)}{32} \times \frac{1}{\binom{s}{2}^2} = \frac{5s}{8(s-1)}.$$

- Asymptotically, the upper bound on $R_2(s)$ has been improved from $2/3$ to $5/8$.

Symmetric BIBDs

- A (v, k, λ)-balanced incomplete block design (*BIBD*) is a pair $(X, \mathcal{A})$, where X is a set of v points and $\mathcal{A}$ is a collection of k-subsets of X called blocks, such that every pair of points occurs in exactly λ blocks.
- Denote $b = |\mathcal{A}|$; then $b = \lambda v(v-1)/(k(k-1))$.
- Every point occurs in exactly $r = bk/v = \lambda(v-1)/(k-1)$ blocks.
- A *BIBD* is symmetric if $v = b$.
- Suppose $(X, \mathcal{A})$ is a (v, k, λ)-*BIBD*.
- Denote $X = \{x_i : 1 \leq i \leq v\}$ and $\mathcal{A} = \{A_j : 1 \leq j \leq b\}$.
- The incidence matrix of $(X, \mathcal{A})$ is the v by b $0-1$ matrix $M = (m_{ij})$ where $m_{ij} = 1$ if $x_i \in A_j$, and $m_{ij} = 0$ if $x_i \notin A_j$.

Invertibility of Incidence Matrices of Symmetric BIBDs

Lemma 6

Suppose M is the incidence matrix of a symmetric (v, k, λ)-BIBD. Then M is invertible over $\mathbb{F}_2$ if and only if k is odd and λ is even.

Proof.

It is well-known that $\det(M)$ is an integer and

$$(\det(M))^2 = k^2(k - \lambda)^{v-1}.$$

Reducing modulo 2, we see that $\det(M) \equiv 1 \bmod 2$ if and only if k is odd and λ is even. $\square$

Invertibility of Incidence Matrices of Symmetric BIBDs

Theorem 7

Suppose M is the incidence matrix of a (v, k, λ)-BIBD where k is odd and λ is even. Then

$$R_2(M) = \frac{k^2 - \lambda^2}{\binom{v}{2}}. \tag{2}$$

Proof.

Given any two rows of M, we have $a_3 = \lambda$, $a_1 = a_2 = k - \lambda$. Hence,

$$a_1a_2 + a_1a_3 + a_2a_3 = (k - \lambda)^2 + 2\lambda(k - \lambda) = k^2 - \lambda^2.$$

□

The expression (2) is maximized when $k \approx \frac{v}{\sqrt{2}}$, in which case $R_2(M) \approx 1/2$.

An Infinite Class of Examples from SBIBDs

- The points and hyperplanes of the m-dimensional projective geometry over $\mathbb{F}_3$ yield a $\left(\frac{3^{m+1}-1}{2}, \frac{3^m-1}{2}, \frac{3^{m-1}-1}{2}\right)$-*SBIBD*.
- Complement it to get a $\left(\frac{3^{m+1}-1}{2}, 3^m, 2 \times 3^{m-1}\right)$-*SBIBD*.
- Since k odd and λ even, we can apply Theorem 7.
- Then
$$R_2\left(\frac{3^{m+1}-1}{2}\right) \geq \frac{40 \times 3^{2m-3}}{(3^{m+1}-1)(3^m-1)}.$$

Example

- If we take $m = 2$, then we are starting with a $(13, 4, 1)$-*SBIBD*.
- After complementing, we have a $(13, 9, 6)$-*SBIBD*.
- This yields
$$R_2\left(13\right) \geq \frac{15}{26}.$$
- Asymptotically, this class of examples has
$$R_2(M) \approx \frac{40}{81} \approx .494.$$
- This is the best asymptotic result we have at present.

A Possibly Infinite Class of Examples from SBIBDs

- Suppose $q = 4t^2 + 9$ is prime and t is odd.
- Then the quartic residues modulo q, together with 0, form a difference set which generates a $\left(q, \frac{q+3}{4}, \frac{q+3}{16}\right)$-*SBIBD*.
- Complement this design to get a $\left(q, \frac{3(q-1)}{4}, \frac{3(3q-7)}{16}\right)$-*SBIBD*.
- Since k is odd and λ is even, the incidence matrix M is invertible.
- Unfortunately, it is not known if an infinite number of primes of the desired form exist.
- If there are arbitrarily large primes of this type, we obtain

$$R_2(M) \approx \frac{63}{128} \approx .492.$$

Examples from Cyclotomy

- Let $p = 4f + 1$ be prime, where f is even, and let $\nu \in {\mathbb{F}_p}^*$ be a primitive element.
- Let $C_0 = \{\nu^{4i} : 0 \leq i \leq f - 1\}$; this is the unique subgroup of ${\mathbb{F}_p}^*$ having order f.
- The multiplicative cosets of C_0 are $C_j = \nu^j C_0$, for $j = 0, 1, 2, 3$.
- These cosets are often called cyclotomic classes.
- Construct a p by p $0 - 1$ matrix $M' = (m_{ij})$ from C_0.
- The rows and columns of M' are indexed by $\mathbb{F}_p$, and

$$m_{ij} = 1 \text{ if and only if } j - i \in C_0.$$

- The ith row of M' is the incidence vector of $i + C_0$.
- Finally, define M to be the complement of M'.

Cyclotomic Numbers

Theorem 8

Suppose $p = 4f + 1$ is prime and f is even. Let $\nu \in \mathbb{F}_q$ be a primitive element. Let $p = \alpha^2 + \beta^2$, where $\alpha \equiv 1 \bmod 4$ and $\nu^f \equiv \alpha/\beta \bmod p$. Then the cyclotomic numbers denoted (j, j), where $(j, j) = |C_j \cap (1 + C_j)|$ for $0 \leq j \leq 3$, are as follows:

$$
\begin{aligned}
(0,0) &= \frac{p - 11 - 6\alpha}{16} \\
(1,1) &= \frac{p - 3 + 2\alpha - 4\beta}{16} \\
(2,2) &= \frac{p - 3 + 2\alpha}{16} \\
(3,3) &= \frac{p - 3 + 2\alpha + 4\beta}{16}.
\end{aligned}
$$

Invertible 2 by 2 Submatrices

- Using these results on cyclotomic numbers, we can show that the total number of invertible 2 by 2 submatrices in M is

$$\begin{aligned}\binom{p}{2}\sum_{i=0}^{3}\left(\frac{5f^2+2f-A_i(4f+2+A_i)}{4}\right) \\ = \quad \binom{p}{2}\frac{252f^2+168f+25-3\alpha^2-2\beta^2-6\alpha}{64},\end{aligned}$$

- Asymptotically, we have that the density of these examples approaches $63/128 \approx .492$.
- But are the matrices invertible?
- We can check invertibility by a simple $\gcd$ computation.
- Up to order 500, we get invertible matrices when $p = 17, 97, 193, 241, 401, 433, 449$.

Future Work and Open Problems

- Can we improve the upper bounds on $R_2(s)$ by using appropriate software to solve larger quadratic programs?
- Is there a theoretical criterion to determine the invertibility of the matrices obtained from cyclotomy-based constructions?
- It is easy to show that the expected density of invertible 2 by 2 submatrices in an s by s matrix is 0.5, if every entry is chosen randomly to be a "1" with probability $1/\sqrt{2}$. But what about the invertibility of the s by s matrices?
- Does $\lim_{s\to\infty} R_2(s)$ exist? If so, is $\lim_{s\to\infty} R_2(s) = 0.5$?
- We have determined the optimal density $R_2(s)$ for $s \leq 8$ by exhaustive search. Can we extend the exhaustive search to compute $R_2(9)$?

References

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Thank You For Your Attention!