Combinatorial Batch Codes

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combinatorial batch codes

Batch codes were introduced by Ishai, Kushilevitz, Ostrovsky and Sahai at STOC 2004. We study a special case of batch codes that we call combinatorial batch codes.

- n items

- m servers

- N:=total number of items stored
- Goal: retrieve any k items by reading at most one from each server
- Here $n = 6$, $m = 3$, $N = 9$ and $k = 2$

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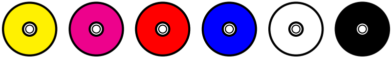

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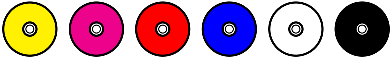

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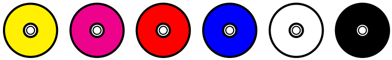

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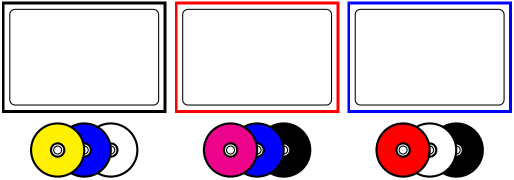

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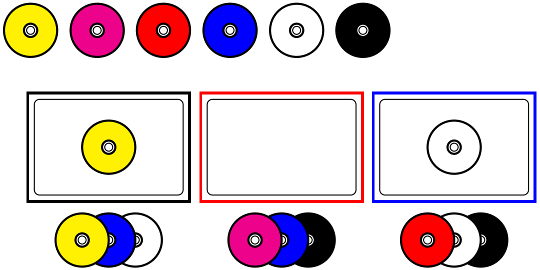

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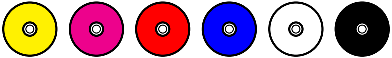

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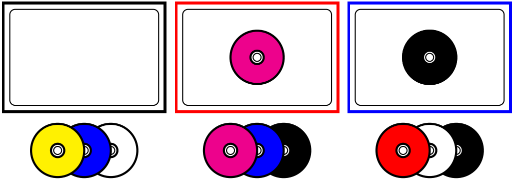

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questions

Notation: $(n, N, k, m) - CBC$
n items, total storage N, m servers, k items read

Given n, m, k, what is the minimum possible value of N? Denote this value by $N(n, k, m)$.

For a fixed rate $\frac{n}{N}$, and fixed k, what is the largest possible value of n (as a function of m)?

incidence matrix representation

Lemma

An $m \times n$ 0-1 matrix containing exactly N ones is an incidence matrix of an $(n, N, k, m) - CBC \Leftrightarrow$ any k columns contain a $k \times k$ submatrix with a transversal containing k ones.

items

servers						
	1	0	0	1	1	0
	0	1	0	1	0	1
	0	0	1	0	1	1

We view the incidence matrix as representing a set system:

point $\leftrightarrow$ server

block $\leftrightarrow$ set of servers containing a particular item

Hall's marriage theorem

Theorem

Suppose $(X, \mathcal{B})$ is a set system. Then any subcollection of k blocks $B_1, B_2, \ldots, B_k$ has a system of distinct representatives $\Leftrightarrow$ for all i, $1 \leq i \leq k$ it holds that

SDR(i) *for any subcollection of i blocks $B_{j_1}, B_{j_2}, \ldots, B_{j_i} \in \mathcal{B}$ it holds that $\left|\bigcup_{l=1}^{i} B_{j_l}\right| \geq i$.*

The previous example corresponds to the set system containing blocks $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$.

trivial examples

- Each server stores all items: $m = k$, $N = kn$
- Each server stores one item: $m = n = N$. Here we have $N(n, k, n) = n$.

Therefore we are only interested in examples with $n < N < kn$. When $k = m$, we have $N(n, k, k) = kn - k(k-1)$.

Example: $k = 4$, $n = 7$

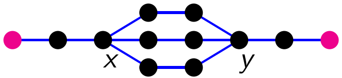
1	0	0	0	1	1	1
0	1	0	0	1	1	1
0	0	1	0	1	1	1
0	0	0	1	1	1	1

Next, we present a construction for batch codes where n is a bit larger than m.

(k,p)-flying saucer

Suppose $k \equiv 2 \pmod 3$

- Two vertices x and y are joined by p (disjoint) paths of length $\frac{k+1}{3}$.
- Paths of length $\frac{k-2}{3}$ are joined to x and y.



$k = 8,\ p = 3$

- number of vertices is $V = \frac{(p+2)(k-2)}{3} + 2$.
- number of edges is $E = \frac{p(k+1)}{3} + \frac{2(k-2)}{3}$.

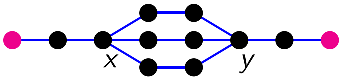
constructing a CBC from a flying-saucer

Theorem

Let k, p be positive integers with $k \equiv 2 \pmod 3$ and suppose $m \geq V$. Then there exists an $(m+p, m+p+E, k, m) - CBC$.

- Construct a (k,p)-FS and add isolated vertices until there are m vertices.
- Add a loop to each isolated vertex, and to each of the vertices of degree 1.
- Construct an incidence matrix whose rows are labelled by the vertices and whose columns are labelled by edges.

constructing a CBC from a flying-saucer



$k = 8, \ p = 3$

1	0	0	0	0	0	0	0	0	0	0	0	0	1	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	1	0	0	1	1	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	1
0	0	0	1	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	1	1	0	0

$(15, 28, 8, 12)$-CBC

the special case $n = m + 1$

Theorem

For any positive integer k, we have $N(m+1, k, m) = m + k$.

- A $(k, 1)$-FS is just a path of length $k - 1$
- The construction produces an $(m+1, m+k, k, m)$-CBC.
- The pigeon-hole principle can be used to show this is the best you can do.

an optimal construction for large n

Theorem
For $n \geq (k-1)\binom{m}{k-1}$, we have

$$N(n,k,m) = kn - (k-1)\binom{m}{k-1}.$$

Example: $n = 23,\ m = 5,\ k = 3$

1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1
1	1	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1	1
0	0	1	1	0	0	0	0	1	1	0	0	0	0	1	1	1	1	0	0	1	1	1
0	0	0	0	1	1	0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0
0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0	0

uniform batch codes with rate $\frac{1}{c}$

A batch code is uniform if every server stores the same number of items.

We would like to determine the maximum n for which there exists a uniform (n, cn, k, m)-CBC; denote this number by $n(m, c, k)$.

Theorem

$n(m, c, k) \leq \frac{(k-1)\binom{m}{c}}{\binom{k-1}{c}}$.

We have equality in the following cases:

- $n(m, c, c+1) = c\binom{m}{c}$
- $n(m, c, c+2) = \binom{m}{c}$

batch codes with rate $\frac{1}{2}$

- Each block has two points $\rightarrow$ we can represent the set system by a multigraph.
- For each $i \leq k$, the graph contains no subgraphs with i edges but fewer than i vertices.

Lemma

If there is a graph G with m vertices, n edges and girth g, then there is a uniform $(n, 2n, k, m)$-CBC with $k = 2g - \lfloor g/2 \rfloor - 1$ and rate $1/2$.

constructions from bipartite graphs

- A bipartite graph has girth at least 4.
- The complete bipartite graph $K_{\lceil \frac{m}{2} \rceil, \lfloor \frac{m}{2} \rfloor}$ yields a uniform $(\left\lceil (m^2-1)/4 \right\rceil, 2\left\lceil (m^2-1/4 \right\rceil, 5, m)$-CBC with rate $1/2$.

Theorem

$$\left\lceil \frac{m^2-1}{4} \right\rceil \leq n(m,2,5) \leq \left\lceil \frac{m^2+2m-3}{4} \right\rceil .$$

The proof of the upper bound uses a result of Dirac on graphs that contain no subgraph isomorphic to $K_4 - e$.

d-regular graphs of large girth

Margulis 1984, Lubotzky *et al.* 1988 constructed d-regular graphs with girth

$$g \geq \frac{4}{3}\frac{\log m}{\log(d-1)} - \frac{\log 4}{\log(d-1)},$$

where $d-1$ is any prime $p \equiv 1 \pmod 4$ and m is the number of vertices.

Theorem

There exists a uniform $(dm/2, dm, 2\log m/\log(d-1), m)$-CBC when $d-1 \equiv 1 \pmod 4$ is prime.

(This CBC has $n = \Omega(m^{(k+2)/k})$.)

probabilistic construction for arbitrary c

Theorem

For integers $c \geq 2$, $k \geq 2$ there exists a constant $a_{c,k}$ such that there exists a uniform (n, cn, k, m)-CBC with $n \geq a_{c,k} m^{ck/(k-1)-1}$, having rate $1/c$.

Here $n = \Omega(m^{ck/(k-1)-1})$. This improves the result in Ishai *et al.* who showed that $n = \Omega(m^{c-1})$.

open problems

1. How close to being optimal are the constructions using flying saucers? In particular, is it true that

$$N(m+p,k,m) - N(m+p-1,k,m) \approx \frac{k}{3}$$

when $p > 1$ and m is sufficiently large as a function of p and k?

2. Are there explicit constructions for "good" uniform batch codes with fixed rate $1/c$, where $c > 2$ is an integer?
3. Can $N(n,k,m)$ be computed for a range of values of n, where $n < (k-1)\binom{m}{k-1}$?

thank you for your attention!

Y. Ishai, E. Kushilevitz, R. Ostrovsky and A. Sahai. *Batch codes and their applications*, in Proceedings of the 36th Annual ACM Symposium on Theory of Computing, ACM Press, New York, 262–271.

M. B. Paterson, D. R. Stinson and R. Wei. *Combinatorial batch codes*, Advances in Mathematics of Communications, 3(1) 13–27, 2009. `http://dx.doi.org/10.3934/amc.2009.3.13`