Let p be a prime. Prove that in $\mathbb{Z}_p[x]$, there exist a non-constant polynomial with no root in $\mathbb{Z}_p$.

Solution: Let $0, 1, 2, ..., p-1$ be all the elements of $\mathbb{Z}_p$. Consider the following:

$$p(x) = x(x-1)(x-2)\cdots(x-(p-1)) + 1 = \prod_{j=0}^{p-1}(x-j) + 1$$

Notice that $p(x) \in \mathbb{Z}_p[x]$. Further, this polynomial has no root in $\mathbb{Z}_p$ since

$$p(k) = 1$$

for any $k \in \mathbb{Z}_p$. Hence this polynomial has no root in $\mathbb{Z}_p$.