Let a, b, c be integers such that $c \mid ab$. Show that $c \mid \gcd(a,c)\gcd(b,c)$.

Solution: By Bezout's Lemma (EEA) there exists integers x and y such that

$$ax + cy = \gcd(a,c).$$

Using Bezout's Lemma (EEA) again, there exist integers u and v such that

$$bu + cv = \gcd(b,c).$$

Multiply the previous two equations together to see that

$$abxu + acxv + bcuy + c^2yv = \gcd(a,c)\gcd(b,c).$$

Write $ck = ab$ for some integer k. Substituting above and factoring yields

$$c(kxu + axv + buy + cyv) = \gcd(a,c)\gcd(b,c).$$

Thus $c \mid \gcd(a,c)\gcd(b,c)$.