9) Show that for every integer $a$, we have that $a^{561} \equiv a \pmod{561}$ even though 561 is not prime.

**Solution:** Notice that $561 = 3 \cdot 11 \cdot 17$. Our idea is to compute $a^{561}$ modulo these three primes and use CRT to stitch the solutions together. Notice that if $\gcd(a, 3) = 1$, then using FLT gives

$$a^{561} \equiv (a^2)^{280} a \equiv a \pmod{3}$$

and if $3 \mid a$ then $a^{561} \equiv 0 \equiv a \pmod{3}$. Next, if $\gcd(a, 11) = 1$, then using FLT gives

$$a^{561} \equiv (a^{10})^{56} a \equiv a \pmod{11}$$

and if $11 \mid a$ then $a^{561} \equiv 0 \equiv a \pmod{11}$. Lastly, if $\gcd(a, 17) = 1$, then using FLT gives

$$a^{561} \equiv (a^{16})^{35} a \equiv a \pmod{17}$$

and if $17 \mid a$ then $a^{561} \equiv 0 \equiv a \pmod{17}$. In all cases, we have the following simultaneous congruences

$$\begin{aligned} a^{561} &\equiv a \pmod{3} \\ a^{561} &\equiv a \pmod{11} \\ a^{561} &\equiv a \pmod{17} \end{aligned}$$

Hence, by the generalized Chinese Remainder Theorem (or by SM), we must have that

$$a^{561} \equiv a \pmod{3 \cdot 11 \cdot 17}$$

which is exactly what we wanted to show.