6) Determine all complex numbers (in polar or standard form) of the equation

$$z^6 + 2iz^3 - 4 = 0.$$

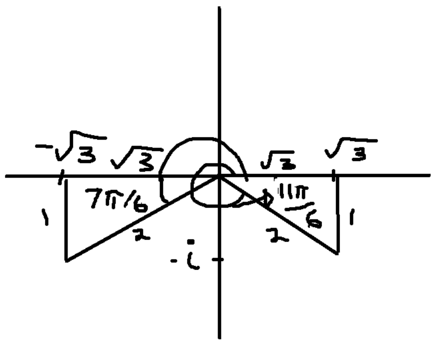
Solution: Let $w = z^3$ so that $w^2 + 2iw - 4 = 0$. Using the quadratic formula yields

$$w = \frac{-2i \pm \sqrt{(2i)^2 - 4(1)(-4)}}{2(1)} = \frac{-2i \pm \sqrt{12}}{2} = \frac{-2i \pm 2\sqrt{3}}{2} = -i \pm \sqrt{3}$$

Since $z^3 = w$, it now suffices to solve $z^3 = -i - \sqrt{3}$ and $z^3 = -i + \sqrt{3}$. Notice that $|-i \pm \sqrt{3}| = \sqrt{(-1)^2 + (\pm\sqrt{3})^2} = 2$ and thus, we are solving the two equations given by

$$z^3 = 2\left(\frac{\pm\sqrt{3}}{2} - \frac{i}{2}\right)$$

By the diagram,



We see that the two equations are equal to

$$z^3 = 2(\cos(7\pi/6) + i\sin(7\pi/6)) \qquad z^3 = 2(\cos(11\pi/6) + i\sin(11\pi/6))$$

By CNRT, the six solutions are given by:

$$\begin{aligned}
z &= \sqrt[3]{2}(\cos(\tfrac{7\pi/6}{3}) + i\sin(\tfrac{7\pi/6}{3}))\\
z &= \sqrt[3]{2}(\cos(\tfrac{7\pi/6+2\pi}{3}) + i\sin(\tfrac{7\pi/6+2\pi}{3}))\\
z &= \sqrt[3]{2}(\cos(\tfrac{7\pi/6+4\pi}{3}) + i\sin(\tfrac{7\pi/6+4\pi}{3}))\\
z &= \sqrt[3]{2}(\cos(\tfrac{11\pi/6}{3}) + i\sin(\tfrac{11\pi/6}{3}))\\
z &= \sqrt[3]{2}(\cos(\tfrac{11\pi/6+2\pi}{3}) + i\sin(\tfrac{11\pi/6+2\pi}{3}))\\
z &= \sqrt[3]{2}(\cos(\tfrac{11\pi/6+4\pi}{3}) + i\sin(\tfrac{11\pi/6+4\pi}{3}))
\end{aligned}$$