3) Let $a_1 = 2$, $a_2 = 3$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 3$. Show that $a_n = 2^{n-1} + 1$ for $n \geq 1$.

The proof is by Strong Induction. For $n = 1$ and $n = 2$,

$$a_1 = 2 = 2^{1-1} + 1$$

$$a_2 = 3 = 2 + 1 = 2^{2-1} + 1$$

Induciton Hypothesis: Assume that $a_i = 2^{i-1} + 1$ for all $i \in \{1, 2, ..., k\}$ for some $k \in \mathbb{N}$, $k \geq 2$.

Induction Conclusion: Prove that $a_{k+1} = 2^k + 1$. Since $k + 1 \geq 3$, I can use the recursive definition to see that:

$$\begin{aligned} a_{k+1} &= 3a_k - 2a_{k-1} \\ &= 3(2^{k-1} + 1) - 2(2^{k-2} + 1) && \text{IH} \\ &= 3 \cdot 2^{k-1} + 3 - 2^{k-1} - 2 \\ &= 2 \cdot 2^{k-1} + 1 \\ &= 2^k + 1 \end{aligned}$$

Thus, the statement is true for $n = k + 1$ and the claim is proven by POSI.