Carmen's Core Concepts (Math 135)

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Week 9 Part 2

Complex Numbers

Definition: A complex numbers (in standard form) is an expression of the form $x + yi$ where $x, y \in \mathbb{R}$ and i is the imaginary unit. Denote the set of complex numbers by

$$\mathbb{C} := \{x + yi : x, y \in \mathbb{R}\}$$

Example: $1 + 2i$, $3i$, $\sqrt{13} + \pi i$, 2 (or $2 + 0i$).

Definition: Two complex numbers $z = x + yi$ and $w = u + vi$ are equal if and only if $x = u$ and $y = v$.

$\mathbb{C}$ is a field

We turn $\mathbb{C}$ into a commutative ring by defining

1. $(x + yi) \pm (u + vi) := (x \pm u) + (y \pm v)i$
2. $(x + yi)(u + vi) := (xu - vy) + (xv + uy)i$

The multiplication operation above gives us that $i^2 = -1$ and with this, we can remember the multiplication operation by expanding the product as we would before.

$$\begin{aligned}(x + yi)(u + vi) &= xu + xvi + yiu + yivi \\ &= xu + (xv + yu)i + yvi^2 \\ &= xu - yv + (xv + uy)i\end{aligned}$$

We note that $\mathbb{C}$ is a field by observing that the multiplicative inverse of a nonzero complex numbers is

$$(x + yi)^{-1} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i$$

Complex Conjugation

Definition: The complex conjugate of a complex number $z = x + yi$ is $\overline{z} := x - yi$.

Proposition: (Properties of Conjugates (PCJ)) Let $z, w \in \mathbb{C}$. Then

1. $\overline{z + w} = \overline{z} + \overline{w}$
2. $\overline{zw} = \overline{z} \cdot \overline{w}$
3. $\overline{\overline{z}} = z$
4. $z + \overline{z} = 2\Re(z)$
5. $z - \overline{z} = 2i\Im(z)$.

Proof:

3. $\overline{\overline{z}} = \overline{\overline{x + yi}} = \overline{x - yi} = x + yi = z$
4. $z + \overline{z} = x + yi + x - yi = 2x = 2\Re(z)$
5. $z - \overline{z} = x + yi - (x - yi) = 2yi = 2i\Im(z)$

Properties of Modulus

Definition: The modulus of $z = x + yi$ is the nonnegative real number

$$|z| = |x + yi| := \sqrt{x^2 + y^2}$$

Proposition: (Properties of Modulus (PM))

1. $|\overline{z}| = |z|$
2. $z\overline{z} = |z|^2$
3. $|z| = 0 \Leftrightarrow z = 0$
4. $|zw| = |z||w|$
5. $|z + w| \leq |z| + |w|$ (This is called the triangle inequality)

Proof of the Triangle Inequality

To prove $|z+w| \le |z| + |w|$, it suffices to prove that

$$|z+w|^2 \le (|z|+|w|)^2 = |z|^2 + 2|zw| + |w|^2$$

since the modulus is a positive real number. Using the Properties of Modulus and the Properties of Conjugates, we have

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) && \text{PM} \\ &= (z+w)(\bar{z}+\bar{w}) && \text{PCJ} \\ &= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 && \text{PCJ and PM} \end{aligned}$$

Now, from Properties of Conjugates, we have that

$$z\bar{w} + \overline{z\bar{w}} = 2\Re(z\bar{w}) \le 2|z\bar{w}| = 2|zw|$$

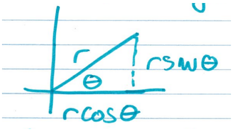
and hence

$$|z+w|^2 = |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 \le |z|^2 + 2|zw| + |w|^2$$

completing the proof.

Polar Coordinates

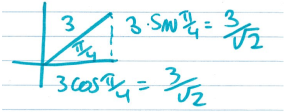
A point in the plane corresponds to a length and an angle:



Example: $(r, \theta) = (3, \frac{\pi}{4})$ corresponds to

$$3\cos(\pi/4) + i(3\sin(\pi/4)) = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$$

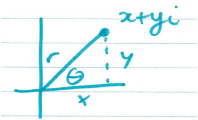
via the picture



Converting From Standard to Polar Form

Given $z = x + yi$, we see that

$$r = |z| = \sqrt{x^2 + y^2}$$
$$\theta = \arccos(x/r) = \arcsin(y/r) = \arctan(y/x)$$



Note: The angle θ might be $\arctan(y/x)$ OR $\pi + \arctan(y/x)$ depending on which quadrant we are in.

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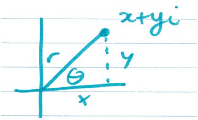

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Example: Write $z = \sqrt{6} + \sqrt{2}i$ using polar coordinates. What about $-z$?

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Solution: Note that $r = \sqrt{\sqrt{6}^2 + \sqrt{2}^2} = \sqrt{8} = 2\sqrt{2}$. Further,

$$\arctan(\sqrt{2}/\sqrt{6}) = \arctan 1/\sqrt{3} = \pi/6$$

Thus $(r, \theta) = (2\sqrt{2}, \pi/6)$. For $-z$, the angle and radius we get above is the same however we are now in the third quadrant so the coordinates are $(r, \theta) = (2\sqrt{2}, 7\pi/6)$