Carmen's Core Concepts (Math 135)

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Week 12

1. Real Quadratic Factors (RQF)
2. Show $g(x) = x^2 - 2\Re(c)x + |c|^2$ is a factor of $f(x)$.
3. Real Factors of Real Polynomials (RFPF)
4. An Example
5. Square Roots of Complex Numbers

Real Quadratic Factors (RQF)

Theorem: Let $f(x) \in \mathbb{R}[x]$. If $c \in \mathbb{C} - \mathbb{R}$ and $f(c) = 0$, then there exists a $g(x) \in \mathbb{R}[x]$ such that $g(x)$ is a real quadratic factor of $f(x)$.

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Proof: Take

$$\begin{aligned} g(x) &= (x - c)(x - \overline{c}) \\ &= x^2 - (c + \overline{c})x + c\overline{c} \\ &= x^2 - 2\Re(c)x + |c|^2 \in \mathbb{R}[x] \end{aligned}$$

It suffices to show that $g(x)$ is a factor of $f(x)$.

Show $g(x) = x^2 - 2\Re(c)x + |c|^2$ is a factor of $f(x)$.

By the Division Algorithm for Polynomials, there exists a unique $q(x)$ and $r(x)$ in $\mathbb{R}[x]$ such that

$$f(x) = g(x)q(x) + r(x)$$

with $r(x) = 0$ or $\deg(r(x)) < \deg(g(x)) = 2$. Assume towards a contradiction that $r(x) \neq 0$. Then $\deg(r(x)) = 0$ or 1, that is, $r(x)$ is linear or constant. Substituting $x = c$ into the above gives

$$0 = f(c) = g(c)q(c) + r(c) = r(c)$$

and hence $r(c) = 0$. Now, if $r(x)$ was constant, then $r(x) = 0$ which is a contradiction. If $r(x)$ was linear, say $r(x) = ax + b$, then

$$r(c) = ac + b = 0 \qquad \Longrightarrow \qquad c = \tfrac{-b}{a} \in \mathbb{R}$$

and this too is a contradiction. Therefore, $r(x) = 0$ and $g(x) \mid f(x)$.

Real Factors of Real Polynomials (RFRP)

Theorem: Let $f(x) = a_n x^n + ... + a_1 x + a_0 \in \mathbb{R}[x]$. Then $f(x)$ can be written as a product of real linear and real quadratic factors,

Real Factors of Real Polynomials (RFRP)

Theorem: Let $f(x) = a_n x^n + ... + a_1 x + a_0 \in \mathbb{R}[x]$. Then $f(x)$ can be written as a product of real linear and real quadratic factors,

Proof: By CPN, $f(x)$ has n roots over $\mathbb{C}$. Let $r_1, r_2, ..., r_k$ be the real roots and let $c_1, c_2, ..., c_\ell$ be the strictly complex roots. By CJRT, complex roots come in pairs, say $c_2 = \overline{c_1}$, $c_4 = \overline{c_3}$, ..., $c_\ell = \overline{c_{\ell-1}}$ (hence also ℓ is even). For each pair, by RQF, we have an associated quadratic factor, say $q_1(x), q_2(x), ..., q_{\ell/2}(x)$. By the Factor Theorem, each real root corresponds to a linear factor, say $g_1(x), ..., g_k(x)$. Hence

$$f(x) = cg_1(x)...g_k(x)q_1(x)...q_{\ell/2}(x)$$

where c is the coefficient of the leading term completing the proof.

An Example

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Solution: Assume towards a contradiction that $p(x)$ is a real polynomial of odd degree without a root. By the Factor Theorem, we know that if $p(x)$ cannot have a real linear factor. By Real Factors of Real Polynomials, we see that

$$p(x) = q_1(x)...q_k(x)$$

for some quadratic factors $q_i(x)$. Now, taking degrees shows that

$$\deg(p(x)) = 2k$$

contradicting the fact that the degree was of $p(x)$ is odd. Hence, the polynomial must have a real root.

Square Roots of Complex Numbers

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Solution: We are seeking values z such that $z^2 = 5+2i$. We can either write $5+2i$ in polar form and solve or we can write $z = x+iy$ and solve this way. Using the first method yields $5+2i = \sqrt{29}e^{i\theta}$ where $\theta = \arctan(2/5)$. Hence, $\sqrt[4]{29}e^{i\theta/2}$ and $\sqrt[4]{29}e^{i\theta/2+i\pi}$ are solutions. Simplifying gives $\pm\sqrt[4]{29}e^{i\theta/2}$. (You could convert back to standard form if you wanted but it will not be a nice answer.)