Carmen's Core Concepts (Math 135)

Carmen Bruni

University of Waterloo

Week 11 Part 2

Definition of Irreducible

Let $\mathbb{F}$ be a field. We say a polynomial of positive degree in $\mathbb{F}[x]$ is reducible in $\mathbb{F}[x]$ when it can be written as the product of two polynomials in $\mathbb{F}[x]$ of positive degree. Otherwise, we say that the polynomial is irreducible in $\mathbb{F}[x]$. For example, $x^2 + 1$ is irreducible in $\mathbb{R}[x]$ but reducible in $\mathbb{C}[x]$.

The Field Matters

The factorization depends on the field! For example, factoring $z^5 - z^4 - z^3 + z^2 - 2z + 2$...

1. ... over $\mathbb{C}$, $(z-i)(z+i)(z-\sqrt{2})(z+\sqrt{2})(z-1)$
2. ... over $\mathbb{R}$, $(z^2+1)(z-\sqrt{2})(z+\sqrt{2})(z-1)$
3. ... over $\mathbb{Q}$, $(z^2+1)(z^2-2)(z-1)$

Using Long Division

Example: Factor $f(x) = x^4 - 2x^3 + 3x^2 - 4x + 2$ over $\mathbb{Z}_7$.

Using Long Division

Example: Factor $f(x) = x^4 - 2x^3 + 3x^2 - 4x + 2$ over $\mathbb{Z}_7$.

Proof: Note that $f(1) = 0$ and thus, by the Factor Theorem, $x - 1$ is a factor. By long division, we have that

$$f(x) = (x-1)(x^3 - x^2 + 2x - 2)$$

Now, the sum of the coefficients of the cubic is still 0 hence $x - 1$ is another factor of $f(x)$! By a second application of long division, we see that

$$f(x) = (x-1)^2(x^2 + 2)$$

Now, the Factor Theorem says that if $x^2 + 2$ could be factored, it must have a root since the factors must be linear. Checking the 7 possible roots, the corresponding polynomial values when $x \in \{0, 1, 2, 3, 4, 5, 6\}$ are $x^2 + 2 \in \{2, 3, 6, 4, 4, 6, 3\}$ modulo 7. Therefore, $x^2 + 2$ has no root in $\mathbb{Z}_7$ and the above form was completely factorized. ■

Multiplicity of Roots

Definition: The multiplicity of a root $c \in \mathbb{F}$ of $f(x) \in \mathbb{F}[x]$ is the largest $k \in \mathbb{N}$ such that $(x - c)^k$ is a factor of $f(x)$.

Example: The multiplicity of the root 1 in the last example is 2.

Techniques for Finding Roots

- Using the Rational Roots Theorem to guess a rational root.
- Trial and error (guessing roots)
- Using the Conjugate Roots Theorem
- Factoring and grouping
- Long division
- Quadratic formula

A Rational Roots Example

Factor $x^3 - \frac{32}{15}x^2 + \frac{1}{5}x + \frac{2}{15}$ as a product of irreducible polynomials over $\mathbb{R}$.

A Rational Roots Example

Factor $x^3 - \frac{32}{15}x^2 + \frac{1}{5}x + \frac{2}{15}$ as a product of irreducible polynomials over $\mathbb{R}$.

Solution: The above polynomial is equal to

$$\tfrac{1}{15}(15x^3 - 32x^2 + 3x + 2) = f(x)$$

By the Rational Roots Theorem, possible roots are

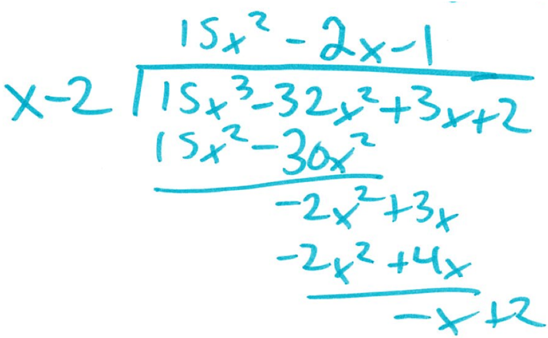
$$\pm 1, \pm\tfrac{1}{3}, \pm\tfrac{1}{5}, \pm\tfrac{1}{15}, \pm 2, \pm\tfrac{2}{3}, \pm\tfrac{2}{5}, \pm\tfrac{2}{15},$$

Note that $x = 2$ is a root. Hence by the Factor Theorem, $x - 2$ is a factor. By long division...

A Rational Roots Example Pt. 2

Factor $x^3 - \frac{32}{15}x^2 + \frac{1}{5}x + \frac{2}{15}$ as a product of irreducible polynomials over $\mathbb{R}$.

By long division...



we have that
$f(x) = \frac{1}{15}(x-2)(15x^2 - 2x - 1) = \frac{1}{15}(x-2)(5x+1)(3x-1)$
completing the question.

A Conjugate Roots Example

Factor $f(z) = z^4 - 5z^3 + 16z^2 - 9z - 13$ over $\mathbb{C}$ into a product of irreducible polynomials given that $2 - 3i$ is a root.

A Conjugate Roots Example

Factor $f(z) = z^4 - 5z^3 + 16z^2 - 9z - 13$ over $\mathbb{C}$ into a product of irreducible polynomials given that $2 - 3i$ is a root.

Solution: Factors are (using the Factor Theorem and CJRT)

$$(z - (2 - 3i))(z - (2 + 3i)) = z^2 - 4z + 13$$

After long division,

$$f(z) = (z^2 - 4z + 13)(z^2 - z - 1)$$

By the quadratic formula on the last quadratic,

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

Hence,
$f(z) = (z-(2-3i))(z-(2+3i))(z-(1+\sqrt{5})/2)(z-(1-\sqrt{5})/2)$.

Rationality of Numbers

Prove that $\sqrt{5} + \sqrt{3}$ is irrational.

Rationality of Numbers

Prove that $\sqrt{5}+\sqrt{3}$ is irrational.

Solution: Assume towards a contradiction that $\sqrt{5}+\sqrt{3}=x\in\mathbb{Q}$. Squaring gives

$$5+2\sqrt{15}+3=x^2 \qquad \Longrightarrow \qquad 2\sqrt{15}=x^2-8$$

Squaring again gives

$$60=x^4-16x^2+64 \qquad \Longrightarrow \qquad 0=x^4-16x^2+4x$$

By the Rational Roots Theorem, the only possible roots are

$$\pm 1, \pm 2, \pm 4$$

A quick check shows that none of these work.