Carmen's Core Concepts (Math 135)

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Week 10

Polar Multiplication of Complex Numbers [PMCN]

Theorem: If $z_1 = r_1\text{cis}(\theta_1)$ and $z_2 = r_2\text{cis}(\theta_2)$, then

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Proof: We have

$$\begin{aligned}
z_1 z_2 &= r_1(\cos(\theta_1) + i\sin(\theta_1))r_2(\cos(\theta_2) + i\sin(\theta_2)) \\
&= r_1 r_2(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\
&\qquad\qquad + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2))) \\
&= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\
&= r_1 r_2 \text{cis}(\theta_1 + \theta_2)
\end{aligned}$$

where in line 3 above, we used trig identities. This completes the proof. ■

De Moivre's Theorem [DMT]

Theorem: If $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$, then

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Proof: Exercise $n = 0$. For $n > 0$, use induction and [PMCN]. For $n < 0$, write $n = -m$ for some $m \in \mathbb{N}$. Then

$$\begin{aligned} \text{cis}(\theta)^n &= \text{cis}(\theta)^{-m} \\ &= (\text{cis}(\theta)^m)^{-1} \\ &= \text{cis}(m\theta)^{-1} \\ &= \frac{\cos(m\theta) - i\sin(m\theta)}{\cos^2(m\theta) + \sin^2(m\theta)} \qquad \text{Since } z^{-1} = \overline{z}/|z|^2 \\ &= \cos(m\theta) - i\sin(m\theta) \end{aligned}$$

and $\cos(-m\theta) + i\sin(-m\theta) = \cos(m\theta) - i\sin(m\theta)$ since cosine is even and sine is odd.

Complex Exponential Function

Definition: For a real θ, define

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Why?

- Exponential Laws Work!
- Derivative with respect to θ makes sense.
- Power series agree.

Theorem: Complex nth Roots Theorem (CNRT) Any nonzero complex number has exactly $n \in \mathbb{N}$ distinct nth roots. The roots lie on a circle of radius $|z|$ centred at the origin and spaced out evenly by angles of $2\pi/n$. Concretely, if $a = re^{i\theta}$, then solutions to $z^n = a$ are given by $z = \sqrt[n]{r}e^{i(\theta+2\pi k)/n}$ for $k \in \{0, 1, ..., n-1\}$.

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Definition: An nth root of unity is a complex number z such that $z^n = 1$. These are sometimes denoted by ζ_n.

Roots of Unity Example

Find all eighth roots of unity in standard form.

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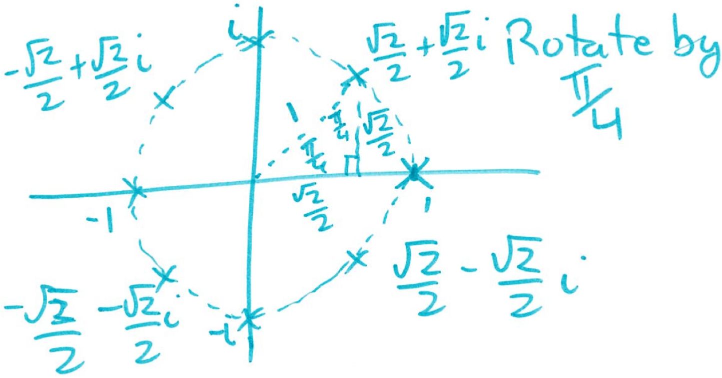
Solution: Since $1^8 = 1$ and $1 = e^0$, we see from the theorem states that $z = e^{2\pi ik/8}$ for $k \in \{0, 1, ..., 7\}$ all gives solutions:

$$
\begin{aligned}
e^{2\pi i(0)/8} &= \cos(0) + i\sin(0) = 1 \\
e^{2\pi i(1)/8} &= \cos(\pi/4) + i\sin(\pi/4) = \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i \\
e^{2\pi i(2)/8} &= \cos(\pi/2) + i\sin(\pi/2) = i \\
e^{2\pi i(3)/8} &= \cos(3\pi/4) + i\sin(3\pi/4) = -\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i \\
e^{2\pi i(4)/8} &= \cos(\pi) + i\sin(\pi) = -1 \\
e^{2\pi i(5)/8} &= \cos(5\pi/4) + i\sin(5\pi/4) = -\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i \\
e^{2\pi i(6)/8} &= \cos(3\pi/2) + i\sin(3\pi/2) = -i \\
e^{2\pi i(7)/8} &= \cos(7\pi/4) + i\sin(7\pi/4) = \tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i
\end{aligned}
$$

Roots of Unity Example Picture

Find all eighth roots of unity in standard form.

...or we could use symmetry and a diagram.



Watch Out! It's a Trap!

Example: Solve $z^5 = -16\overline{z}$.
Might believe this has 5 solutions but it actually has 7 solutions:

$$z \in \{0, \pm 2i, \pm\sqrt{3} \pm i\}$$

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Idea is that this is not a polynomial. Remove the zero solution and then look at the modulus above giving $|z|^4 = 16$ and hence $|z| = 2$ (since modulus is a positive real number). Then multiply the original equation by z on either side and note the right hand side is $|z|^2 = z\bar{z}$.

Polynomial Ring

Definition: A polynomial in x over a ring R is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

where $a_0, a_1, ...a_n \in R$ and $n \geq 0$ is an integer. Denote the set (actually a ring) of all polynomials over R by $R[x]$.

We will usually use the above definition for fields, which for us include $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_p$ where p is a prime number. This makes life easier for us in many of the theorems we have later. We use the notation $\mathbb{F}$ to denote one of these fields.

Assorted Definitions

Definition:

1. The coefficient of $a_n x^n$ is a_n
2. A term of a polynomial is any $a_i x^i$
3. The degree of a polynomial $\sum_{i=0}^{n} a_i x^i$ is n.
4. The degree of the zero polynomial is undefined (also $-\infty$)
5. A root of a polynomial $p(x) \in \mathbb{F}[x]$ is a value $a \in \mathbb{F}$ such that $p(a) = 0$.
6. Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{n} b_i x^i$ be polynomials over $\mathbb{F}[x]$. Then $f(x) = g(x)$ if and only if $a_i = b_i$ for all $i \in \{0, 1, ..., n\}$.
7. x is an indeterminate (or a variable). It has no meaning on it's own but can be replaced by a value whenever it makes sense to do so.
8. Operations on polynomials: Addition, Subtraction, Multiplication

Division Algorithm for Polynomials [DAP]

Theorem: Let $\mathbb{F}$ be a field. If $f(x), g(x) \in \mathbb{F}[x]$ and $g(x) \neq 0$ then there exists unique polynomials $q(x)$ and $r(x)$ in $\mathbb{F}[x]$ such that

$$f(x) = q(x)g(x) + r(x)$$

with $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.