Carmen's Core Concepts (Math 135)

Carmen Bruni

University of Waterloo

Week 1

What is Math 135?

- First proofs course
- Proofs differentiate mathematics from science
- Reading, writing and discovering proofs
- Goldilocks. Writing just the right amount with good and proper explanations.

Truth Tables as a Definition

- Basic building block of mathematics.
- Throughout let A and B be statements.
- Saw $\neg A$, $A \wedge B$, $A \vee B$, $A \Rightarrow B$.

A	B	$A \wedge B$	$A \vee B$	$A \Rightarrow B$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

De Morgan's Laws

- Truth Tables as Method of Proof
- $\neg(A \wedge B) \equiv \neg A \vee \neg B$ and $\neg(A \vee B) \equiv \neg A \wedge \neg B$.
- Can prove using truth table:

A	B	$A \vee B$	$\neg(A \vee B)$	$\neg A$	$\neg B$	$\neg A \wedge \neg B$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

- Helps to form chains of equivalences.

Using Chains of Equivalences

- Recall: $A \Rightarrow B \equiv \neg A \vee B$.
- Prove that $\neg(A \Rightarrow B) \equiv A \wedge \neg B$

  Proof:

$$\begin{aligned} \neg(A \Rightarrow B) &\equiv \neg(\neg A \vee B) && \text{By the above proposition} \\ &\equiv \neg(\neg A) \wedge \neg B && \text{De Morgan's Law} \\ &\equiv A \wedge \neg B && \text{By proposition from class} \end{aligned}$$

Direct Proof

- Proving an equality (or inequality).
- $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$

Proof:

$$
\begin{aligned}
\text{LHS} &= \sin(3\theta) \\
&= \sin(2\theta + \theta) \\
&= \sin(2\theta)\cos(\theta) + \sin(\theta)\cos(2\theta) && \text{Trig Identity} \\
&= (2\sin(\theta)\cos(\theta))\cos(\theta) \\
&\quad + \sin(\theta)(\cos^2(\theta) - \sin^2(\theta)) && \text{Trig Identity} \\
&= 3\sin(\theta)\cos^2(\theta) - \sin^3(\theta)) \\
&= 3\sin(\theta)(1 - \sin^2(\theta)) - \sin^3(\theta)) && \text{Pythagorean Identity} \\
&= 3\sin(\theta) - 4\sin^3(\theta) \\
&= \text{RHS}
\end{aligned}
$$

Direct Proof From a True Statement

- Prove $5x^2y - 3y^2 \leq x^4 + x^2y + y^2$, $x, y \in \mathbb{R}$

  Proof: Since $0 \leq (x^2 - 2y)^2$, we have

$$\begin{aligned} 0 &\leq (x^2 - 2y)^2 \\ 0 &\leq x^4 - 4x^2y + 4y^2 \\ 5x^2y - 3y^2 &\leq x^4 - 4x^2y + 4y^2 + 5x^2y - 3y^2 \\ 5x^2y - 3y^2 &\leq x^4 + x^2y + y^2 \end{aligned}$$

- Discovery:

$$\begin{aligned} 5x^2y - 3y^2 &\leq x^4 + x^2y + y^2 \\ 0 &\leq x^4 + x^2y + y^2 - 5x^2y + 3y^2 \\ 0 &\leq x^4 - 4x^2y + 4y^2 \\ 0 &\leq (x^2 - 2y)^2. \end{aligned}$$

Direct Proof Breaking Into Cases

- Let $n \in \mathbb{Z}$. If 2^{2n} is an odd integer, then 2^{-2n} is also an odd integer.

  Proof: Note that the hypothesis is only true when $n = 0$. If $n < 0$, then 2^{2n} is not an integer. If $n > 0$ then $2^{2n} = 2 \cdot 2^{2n-1}$ and since $2n - 1 > 0$, we see that 2^{2n} is even. Hence $n = 0$ and thus $2^{2n} = 1 = 2^{-2n}$. Thus 2^{-2n} is an odd integer.
- Some examples of ways to beak into cases:
  1. Even vs odd
  2. Positive vs negative vs zero
  3. $a \leq b$ and $b \leq a$ for integers a and b

Divisibility

- Let $m, n \in \mathbb{Z}$. We say that m *divides* n and write $m \mid n$ if (and only if) there exists a $k \in \mathbb{Z}$ such that $mk = n$. Otherwise, we write $m \nmid n$, that is, when there is no integer k satisfying $mk = n$.

Bounds By Divisibility (BBD)

- $a \mid b \wedge b \neq 0 \Rightarrow |a| \leq |b|$
- Note: If we don't specify the domain for variables, take it to be maximal.

  Proof: Let $a, b \in \mathbb{Z}$ such that $a \mid b$ and $b \neq 0$. Then $\exists k \in \mathbb{Z}$ such that $ak = b$. Since $b \neq 0$, we know that $k \neq 0$. Thus, $|a| \leq |a||k| = |ak| = |b|$ as required.
- Reminder Symbol (and theorem) cheat sheets can be found on the Math 135 Resources Page.