Prove by induction that the number of non-empty subsets of a set with "n" elements is 2^n-1.

<u>Pf</u>: Let $P(n)$ be the statement that the number of non-empty subsets of a set with n elements is 2^n-1.
We prove $P(n)$ is true for all $n \in \mathbb{N}$ by Induction.
BC: $P(1)$ is true since a set S with one element has only S as a nonempty subset and so this is $2^1-1=1$ non empty subset.
<u>IH</u>: Assume $P(K)$ is true for some $K \in \mathbb{N}$.
IC: Suppose that S is a set with $K+1$ elements. Let $s \in S$ and consider the set $T = S - \{s\}$. By IH, as $|T| = K$, we know that

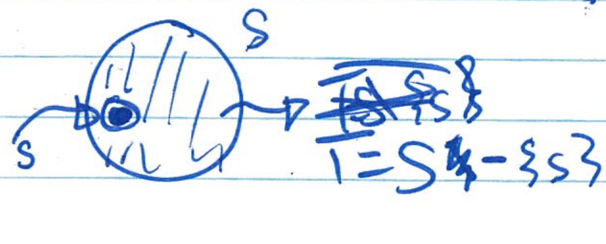
T has 2^k-1 total nonempty subsets. Any subset of S either contains s or doesn't. If it doesn't, then we have that this subset is a subset of T of which we have 2^k-1 such examples: If the subset contains s, then either $\{s\}$ is the subset <u>OR</u> the subset is

$\{s\} \cup W$ where W is a ~~sub~~ nonempty subset of T. The total is

$$\underbrace{2^{k}-1}_{\text{subsets of } T} + \underbrace{1}_{\{s\}} + \underbrace{2^{k}-1}_{\substack{\text{subsets } \{s\}\cup W \\ W \subseteq T}} = 2\cdot 2^{k}-1 = 2^{k+1}-$$

$\therefore$ S has $2^{k+1}-1$ total subsets ie $P(k+1)$ is true

$\therefore$ $P(n)$ is true for all $n \in \mathbb{N}$ by POMI. $\blacksquare$



Solve $x^2 \equiv 30 \bmod 35$

<u>Sol'n</u>: By SM, this is equivalent to solving

$$x^2 \equiv 30 \bmod 5$$
$$x^2 \equiv 30 \bmod 7.$$
valid $\because \gcd(5,7)=1$.

These are equivalent to solving

$$x^2 \equiv 0 \bmod 5$$
$$x^2 \equiv 2 \bmod 7.$$

Solutions to $x^2 \equiv 0 \bmod 5$ are just $x \equiv 0 \bmod 5$.

For the second equation, notice that $x \equiv \pm 3 \bmod 7$ gives 2 unique solutions and since $\mathbb{Z}_7$ is a field, these are the only solutions to the degree 2 equation. Thus, our possibilities are

$$x \equiv 0 \bmod 5 \qquad\qquad x \equiv 0 \bmod 5$$
$$x \equiv 3 \bmod 7 \qquad\qquad x \equiv -3 \bmod 7.$$

For the first system, note that $x \equiv 10$ is a solution and by CRT, $x \equiv 10 \bmod 35$ is ~~an~~ the unique solution in $\mathbb{Z}_{35}$. For the second eqn, $x = 0 + 5k$ for some $k \in \mathbb{Z}$, and $5k \equiv -3 \bmod 7$. Either use LCT1 and state that $5(5) \equiv 25 \equiv -3 \bmod 7$ works so $k \equiv 5 \bmod 7$. OR use EEA to solve the LDE $5k + 7y = -3$

k	y	r	q
0	1	7	
1	0	5	
-1	1	2	1
3	-2	1	2

$\therefore 5(3) + 7(-2) = 1$

and $5(-9) + 7(6) = -3$

ie $k \equiv -9 \mod 7$

$k \equiv 5 \mod 7$.

ie $k = 5 + 7l$ for some $l \in \mathbb{Z}$

Thus, $x = 5k = 5(5 + 7l) = 25 + 35l$.

So $x \equiv 25 \mod 35$.

$\therefore x \equiv 10 \mod 35$ and $x \equiv 25 \mod 35$

are the solutions. Check:

$x \equiv \pm 10 \mod 35$. $(\pm 10)^2 \equiv 100 - 35 \cdot 2 \equiv 30 \mod 35$

so $x \equiv \pm 10 \mod 35$ give all solutions.

For all $z \in \mathbb{C}$, prove that

$$|\mathrm{Re}(z)| + |\mathrm{Im}(z)| \leq \sqrt{2}\,|z|.$$

This is equivalent to proving that

$$(|\mathrm{Re}(z)| + |\mathrm{Im}(z)|)^2 \leq 2|z|^2.$$

valid since all the terms in the inequality are positive. Expanding and setting $z = x + yi$ gives

$$(|x| + |y|)^2 \leq 2(x^2 + y^2)$$

$$\iff |x|^2 + 2|x||y| + |y|^2 \leq 2(x^2 + y^2)$$

$$\iff x^2 + 2|xy| + y^2 \leq 2x^2 + 2y^2$$

$$\iff 2|xy| \leq x^2 + y^2.$$

Now, if $xy \leq 0$ then $2xy \leq x^2 + y^2$ is true

if $xy > 0$ then $2|xy| = 2xy$.

and $2xy \leq x^2 + y^2$ is true since

$$0 \leq (x-y)^2 \iff 0 \leq x^2 - 2xy + y^2$$

$$\iff 2xy \leq x^2 + y^2.$$

OR $2|xy| \leq x^2+y^2$

$\Longleftrightarrow \quad 0 \leq |x|^2 - 2|x||y| + |y|^2$

$\Longleftrightarrow \quad 0 \leq (|x| - |y|)^2$ ← is true $\because x, y \in \mathbb{R}$.

Hence the claim is true. $\square$

If a and a' are coprime, prove forall $b \in \mathbb{Z}$ that

$$\gcd(aa', b) = \gcd(a,b)\gcd(a',b).$$

Pf: Let $d = \gcd(aa', b)$, $e = \gcd(a,b)$ and $f = \gcd(a', b)$. By Bézout's Lemma, $\exists\, x, y, x', y' \in \mathbb{Z}$ s.t.

$$ax + by = e$$

$$a'x' + by' = f$$

Multiplying: $aa'xx' + abxy' + ba'x'y + b^2yy' = ef$

$$aa'(xx') + b(axy' + a'x'y + byy') = ef \quad (*)$$

Now $d \mid aa'$ and $d \mid b$ and by DIC and $(*)$ $d \mid ef$. Now, it suffices to show $ef \mid d$. Since $\gcd(a, a') = 1$ and so by Bézout's Lemma, $\exists\, \hat{x}, \hat{y} \in \mathbb{Z}$ s.t. $a\hat{x} + a'\hat{y} = 1$ $(**)$. Since $f \mid b$, $\exists\, l \in \mathbb{Z}$ s.t. $fl = b$. Now, $e \mid b$ so $e \mid fl$. also, $e \mid a$ and $f \mid a'$. Let $g = \gcd(e, f)$.

Note: $g|e$ and $g|ea \Rightarrow g|a$ by TD.
also $g|f$ and $f|a' \Rightarrow g|a'$ by TD. By (**)
and DIC, $g|1$ and $g>0$ so $g=1$.
Recap: $e|fl$ and $\gcd(e,f)=1$ $\therefore$ by CAD, $e|l$
ie $\exists\, m \in \mathbb{Z}$ s.t. $em=l$. Thus, $b=efm$ and $\because d|b$,
$d|efm$. Also, $ef|b$.

POSTPONE.

Since $e|a$ and $f|a'$, $ef|aa'$ $\left(\begin{matrix} e\alpha=a \\ f\beta=a' \end{matrix}\right\} \Rightarrow ef\alpha\beta=aa'$
Also, By BL, $\exists\, X,Y \in \mathbb{Z}$ s.t. $aa'X+bY=d$. (***)
Since $ef|aa'$ and $ef|b$, then $ef|d$ by DIC and (***)

As $ef \mid d$ and $d \mid ef$ and $d > 0$ and $ef > 0$, we have that $d = ef$. $\square$

Let $n \geq 2$ be an integer. Prove that

$$S = \sum_{k=0}^{n-1} \cos\left(\frac{2\pi k}{n}\right) = 0 = \sum_{k=0}^{n-1} \sin\left(\frac{2\pi k}{n}\right) = T$$

PF: $e^{i\frac{2\pi k}{n}} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$ for any $k \in \mathbb{Z}$

Then, $\displaystyle\sum_{k=0}^{n-1}\left(e^{i2\pi/n}\right)^k = \sum_{k=0}^{n-1}\cos\left(\frac{2\pi k}{n}\right) + i\sum_{k=0}^{n-1}\sin\left(\frac{2\pi k}{n}\right).$

Recall $\displaystyle\sum_{k=0}^{n-1} r^k = \frac{(r^n - 1)}{(r-1)}$ Sum of a geometric series.

In our case, when $r = e^{2\pi i/n}$, we get.

$$\sum_{k=0}^{n-1}\left(e^{2\pi i/n}\right)^k = \frac{\left(e^{2\pi i/n}\right)^n - 1}{e^{2\pi i/n} - 1} = \frac{e^{2\pi i} - 1}{e^{2\pi i/n} - 1} = \frac{1-1}{e^{2\pi i/n} - 1} = 0$$

Then $0 = S + iT \iff S = 0$ and $T = 0$

For what values of c does $8x+5y=c$ have exactly one solution where both x & y are strictly positive?

Sol'n: Let (x_0, y_0) be the only strictly positive solution to $8x+5y=c$. By LDET2, we have that $x=x_0-5n$ and $y=y_0+8n$ gives all solutions for any $n \in \mathbb{Z}$. If $x_0>5$, then when $n=0$ and $n=1$, the solutions ~~are both~~ (x_0, y_0) and (x_0-5, y_0+8) are both positive. $\therefore x_0 \leq 5$. Similarly, if $y_0>8$, then when $n=0$ and $n=-1$, the solutions (x_0, y_0) and (x_0+5, y_0-8) are both positive. $\therefore$ $0<x_0 \leq 5$ and $0<y_0 \leq 8$.

$\therefore$ the values of c that satisfy our conditions are $c \in \{8x_0+5y_0 : 0<x_0 \leq 5 \wedge 0<y_0 \leq 8\}$