MATH 135: Randomized Exam Practice Problems

These are the warm-up exercises and recommended problems taken from all the extra practice sets presented in random order. The challenge problems have not been included.

1. Prove the properties of complex arithmetic given in Proposition 1 in Chapter 30 of the course notes. Only one of the nine results is proved in the notes. A few others may have been proved in class.

2. Determine whether $A \implies B$ is logically equivalent to $(\neg A) \vee B$.

3. Suppose p is a prime greater than five. Prove that the positive integer consisting of $p-1$ digits all equal to one $(111\ldots1)$ is divisible by p. (Hint: $111111 = \frac{10^6-1}{9}$.)

4. Find all non-negative integer solutions to $12x + 57y = 423$.

5. Prove that for distinct primes p and q, $\left(p^{q-1} + q^{p-1}\right) \equiv 1 \pmod{pq}$.

6. Let $a, b, c \in \mathbb{C}$. Prove: if $|a| = |b| = |c| = 1$, then $\overline{a+b+c} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$.

7. What are the integer solutions to $x^2 \equiv 1 \pmod{15}$?

8. Given a rational number r, prove that there exist coprime integers p and q, with $q \neq 0$, so that $r = \dfrac{p}{q}$.

9. Find the complete solution to $7x + 11y = 3$.

10. Is $27^{129} + 61^{40}$ is divisible by 14? Show and justify your work.

11. Let $a, b, c \in \mathbb{Z}$. Disprove the statement: If $a \mid (bc)$, then $a \mid b$ or $a \mid c$.

12. The Fibonacci sequence is defined as the sequence $\{f_n\}$ where $f_1 = 1$, $f_2 = 1$ and $f_i = f_{i-1} + f_{i-2}$ for $i \geq 3$. Use induction to prove the following statements.

    (a) For $n \geq 2$,
    $$f_1 + f_2 + \cdots + f_{n-1} = f_{n+1} - 1$$

    (b) Let $a = \dfrac{1+\sqrt{5}}{2}$ and $b = \dfrac{1-\sqrt{5}}{2}$. For all $n \in \mathbb{N}$, $f_n = \frac{a^n - b^n}{\sqrt{5}}$

13. Prove that $x^2 + 9 \geq 6x$ for all real numbers x.

14. What is the remainder when -98 is divided by 7?

15. Suppose a, b and n are integers. Prove that $n \mid \gcd(a, n) \cdot \gcd(b, n)$ if and only if $n \mid ab$.

16. Prove the following statements.

    (a) There is no smallest positive real number.
    (b) For every even integer n, n cannot be expressed as the sum of three odd integers.
    (c) If a is an even integer and b is an odd integer, then $4 \nmid (a^2 + 2b^2)$.
    (d) For every integer m with $2 \mid m$ and $4 \nmid m$, there are no integers x and y that satisfy $x^2 + 3y^2 = m$.
    (e) The sum of a rational number and an irrational number is irrational.
    (f) Let x be a non-zero real number. If $x + \frac{1}{x} < 2$, then $x < 0$.

17. Let x and y be integers. Prove that if $xy = 0$ then $x = 0$ or $y = 0$.

18. Prove that $(\neg A) \vee B$ is logically equivalent to $\neg(A \wedge \neg B)$.

19. Let $n \in \mathbb{N}$. Prove by induction that if $n \equiv 1 \pmod 4$, then $i^n = i$.

20. Prove there is a unique set T such that for every set S, $S \cup T = S$.

21. In the proof of *Prime Factorization (PF)* given in the course notes, why is it okay to write $r \leq s$?

22. Write $(\sqrt{3} + i)^4$ in standard form.

23. Solve

$$\begin{aligned} x &\equiv 7 \pmod{11} \\ x &\equiv 5 \pmod{12} \end{aligned}$$

24. Given the public RSA encryption key $(e, n) = (5, 35)$, find the corresponding decryption key (d, n).

25. Prove that if k is an odd integer, then $4k + 7$ is an odd integer.

26. The *floor function* assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The *ceiling function* assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$. Prove that there is a unique real number x such that $\lfloor x \rfloor = \lceil x \rceil = 7$.

27. Consider the following statement:

    Let $a, b, c \in \mathbb{Z}$. For every integer x_0, there exists an integer y_0 such that $ax_0 + by_0 = c$.

    (a) Determine conditions on a, b, c such that the statement is true if and only if these conditions hold. State and prove this if and only if statement.
    (b) Carefully write down the negation of the given statement and prove that this negation is true.

28. Prove that if p is prime and $p \leq n$, then p does not divide $n! + 1$.

29. Are the following functions onto? Are they 1-1? Justify your answer with a proof.

    (a) $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(n) = 2n + 1$.
    (b) $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2 + 4x + 9$.
    (c) $f : (\mathbb{R} - \{2\}) \to (\mathbb{R} - \{5\})$, defined by $f(x) = \frac{5x+1}{x-2}$.

30. Prove the following statement using a chain of logical equivalences as in Chapter 3 of the notes.

$$(A \wedge C) \vee (B \wedge C) \equiv \neg((A \vee B) \implies \neg C)$$

31. Prove that the product of any four consecutive integers is one less than a perfect square.

32. Express the following complex numbers in standard form.

    (a) $\dfrac{(\sqrt{2} - i)^2}{(\sqrt{2} + i)(1 - \sqrt{2}i)}$
    (b) $(\sqrt{5} - i\sqrt{3})^4$

33. Write $z = \dfrac{9 + i}{5 - 4i}$ in the form $r(\cos\theta + i \sin\theta)$ with $r \geq 0$ and $0 \leq \theta < 2\pi$.

34. Prove that: if $a \mid c$ and $b \mid c$ and $\gcd(a, b) = 1$, then $ab \mid c$.

35. A basket contains a number of eggs and, when the eggs are removed 2, 3, 4, 5 and 6 at a time, there are 1, 2, 3, 4 and 5 respectively, left over. When the eggs are removed 7 at a time there are none left over. Assuming none of the eggs broke during the preceding operations, determine the minimum number of eggs that were in the basket.

36. Solve $49x^{177} + 37x^{26} + 3x^2 + x + 1 \equiv 0 \pmod 7$.

37. Suppose r is some (unknown) real number, where $r \neq -1$ and $r \neq -2$. Show that

$$\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r\,(2^r)}{(r+1)(r+2)}.$$

38. Find the smallest positive integer a such that $5n^{13} + 13n^5 + a(9n) \equiv 0 \pmod{65}$ for all integers n.

39. Let n, a and b be positive integers. Negate the following implication without using the word "not" or the $\neg$ symbol (but symbols such as $\neq$, $\nmid$, etc. are fine). *Implication:* If $a^3 \mid b^3$, then $a \mid b$.

40. Let $z, w \in \mathbb{C}$. Prove that if $zw = 0$ then $z = 0$ or $w = 0$.

41. Evaluate $\displaystyle\sum_{i=3}^{8} 2^i$ and $\displaystyle\prod_{j=1}^{5} \frac{j}{3}$.

42. Let $a, b, c \in \mathbb{Z}$. Is the following statement true? Prove that your answer is correct.

    $a \mid b$ if and only if $ac \mid bc$.

43. Prove or disprove each of the following statements involving nested quantifiers.

    (a) For all $n \in \mathbb{Z}$, there exists an integer $k > 2$ such that $k \mid (n^3 - n)$.
    (b) For every positive integer a, there exists an integer b with $|b| < a$ such that b divides a.
    (c) There exists an integer n such that $m(n-3) < 1$ for every integer m.
    (d) $\exists n \in \mathbb{N}, \forall m \in \mathbb{Z}, -nm < 0$.

44. Prove the following statement using a chain of logical equivalences as in Chapter 3 of the notes.

$$(A \wedge C) \vee (B \wedge C) \equiv \neg((A \vee B) \implies \neg C)$$

45. In each of the following cases, find all values of $[x] \in \mathbb{Z}_m$, $0 \leq x < m$, that satisfy the equation.

    (a) $[4][3] + [5] = [x] \in \mathbb{Z}_{10}$

    (b) $[7]^{-1} - [2] = [x] \in \mathbb{Z}_{10}$

    (c) $[2][x] = [4] \in \mathbb{Z}_8$

    (d) $[3][x] = [9] \in \mathbb{Z}_{11}$

46. Let a, b, c be integers. Prove that if $a \mid b$ then $ac \mid bc$.

47. Let n, a and b be positive integers. Negate the following implication without using the word "not" or the $\neg$ symbol (but symbols such as $\neq$, $\nmid$, etc. are fine). *Implication:* If $a^3 \mid b^3$, then $a \mid b$.

48. Let S and T be any two sets in universe $\mathcal{U}$. Prove that $(S \cup T) - (S \cap T) = (S - T) \cup (T - S)$.

49. Determine all $k \in \mathbb{N}$ such that $n^k \equiv n \pmod 7$ for all integers n. Prove that your answer is correct

50. In a strange country, there are only 4 cent and 7 cent coins. Prove that any integer amount of currency greater than 17 cents can always be formed.

51. For each of the following statements, identify the four parts of the quantified statement (quantifier, variable, domain, and open sentence). Next, express the statement in symbolic form and then write down the negation of the statement (when possible, without using any negative words such as "not" or the $\neg$ symbol, but negative math symbols like $\neq, \nmid$ are okay).

    (a) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.

    (b) For every even integer a and odd integer b, a rational number c can always be found such that either $a < c < b$ or $b < c < a$.

    (c) There is some $x \in \mathbb{N}$ such that for all $y \in \mathbb{N}$, $y \mid x$.

    (d) There exist sets of integers X, Y such that for all sets of integers Z, $X \subseteq Z \subseteq Y$.
    (You may use $\mathcal{P}(\mathbb{Z})$ to denote the set of all sets of integers. This is called *power set notation*.)

52. Prove that an integer is even if and only if its square is an even integer.

53. Let a, b, c and d be integers. Prove that if $a \mid b$ and $b \mid c$ and $c \mid d$, then $a \mid d$.

54. Let a, b, c be integers. Prove that if $a \mid b$ then $ac \mid bc$.

55. Suppose r is some (unknown) real number, where $r \neq -1$ and $r \neq -2$. Show that

$$\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r\left(2^r\right)}{(r+1)(r+2)}.$$

56. Prove that the product of any four consecutive integers is one less than a perfect square.

57. Prove that if $\gcd(a, b) = 1$, then $\gcd(2a + b, a + 2b) \in \{1, 3\}$.

58. If a and b are integers, $3 \nmid a$, $3 \nmid b$, $5 \nmid a$, and $5 \nmid b$, prove that $a^4 \equiv b^4 \pmod{15}$.

59. Suppose that p is a prime and $a \in \mathbb{Z}$. Prove using induction that $a^{(p^n)} \equiv a (\text{mod } p)$ for all $n \in \mathbb{N}$.

60. Prove that if $|z| = 1$ or $|w| = 1$ and $\overline{z}w \neq 1$, then $\left|\dfrac{z - w}{1 - \overline{z}w}\right| = 1$.

61. Each of the following "proofs" by induction incorrectly "proved" a statement that is actually false. State what is wrong with each proof.

    (a) A sequence $\{x_n\}$ is defined by $x_1 = 3$, $x_2 = 20$ and $x_i = 5x_{i-1}$ for $i \geq 3$. Then, for all $n \in \mathbb{N}$, $x_n = 3 \times 5^{n-1}$.

    Let $P(n)$ be the statement: $x_n = 3 \times 5^{n-1}$.
    When $n = 1$ we have $3 \times 5^0 = 3 = x_1$ so $P(1)$ is true. Assume that $P(k)$ is true for some integer $k \geq 1$. That is, $x_k = 3 \times 5^{k-1}$ for some integer $k \geq 1$. We must show that $P(k+1)$ is true, that is, $x_{k+1} = 3 \times 5^k$. Now

$$x_{k+1} = 5x_k = 5(3 \times 5^{k-1}) = 3 \times 5^k$$

    as required. Since the result is true for $n = k + 1$, and so holds for all n by the Principle of Mathematical Induction.

(b) For all $n \in \mathbb{N}$, $1^{n-1} = 2^{n-1}$.

Let $P(n)$ be the statement: $1^{n-1} = 2^{n-1}$.
When $n = 1$ we have $1^0 = 1 = 2^0$ so $P(1)$ is true. Assume that $P(i)$ is true for all integers $1 \le i \le k$ where $k \ge 1$ is an integer. That is, $1^{i-1} = 2^{i-1}$ for all $1 \le i \le k$.
We must show that $P(k+1)$ is true, that is, $1^{(k+1)-1} = 2^{(k+1)-1}$ or $1^k = 2^k$. By our inductive hypothesis, $P(2)$ is true so $1^1 = 2^1$. Also by our inductive hypothesis, $P(k)$ is true so $1^{k-1} = 2^{k-1}$. Multiplying these two equations together gives $1^k = 2^k$. Since the result is true for $n = k+1$, and so holds for all n by the Principle of Strong Induction.

62. Let a and b be integers. Prove that $(a \mid b \ \wedge \ b \mid a) \iff a = \pm b$.

63. Solve $x^3 \equiv 17 \pmod{99}$.

64. Let n be an integer. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

65. Let $a, b, c \in \mathbb{Z}$. Consider the statement S: If $\gcd(a, b) = 1$ and $c \mid (a + b)$, then $\gcd(a, c) = 1$. Fill in the blanks to complete a proof of S.

    (a) Since $\gcd(a, b) = 1$, by __________ there exist integers x and y such that $ax + by = 1$.
    (b) Since $c \mid (a + b)$, by __________ there exists an integer k such that $a + b = ck$.
    (c) Substituting $a = ck - b$ into the first equation, we get $1 = (ck - b)x + by = b(-x + y) + c(kx)$.
    (d) Since 1 is a common divisor of b and c and $-x + y$ and kx are integers, $\gcd(b, c) = 1$ by __________.

66. Find all $z \in \mathbb{C}$ which satisfy

    (a) $z^2 + 2z + 1 = 0$,
    (b) $z^2 + 2\overline{z} + 1 = 0$,
    (c) $z^2 = \dfrac{1+i}{1-i}$.

67. What are the last two digits of 43^{201}?

68. Assume that it has been established that the following implication is true:

    If I don't see my advisor today, then I will see her tomorrow.

    For each of the statements below, determine if it is true or false, or explain why the truth value of the statement cannot be determined.

    (a) I don't meet my advisor both today and tomorrow. (This is arguably an ambiguous English sentence. Answer the problem using either or both interpretations.)
    (b) I meet my advisor both today and tomorrow.
    (c) I meet my advisor either today or tomorrow (but not on both days).

69. Solve $x^3 - 29x^2 + 35x + 38 \equiv 0 \pmod{195}$.

70. Consider the following statement.

    For all $x \in \mathbb{R}$, if $x^6 + 3x^4 - 3x < 0$, then $0 < x < 1$.

    (a) Rewrite the given statement in symbolic form.
    (b) State the hypothesis of the implication within the given statement.

(c) State the conclusion of the implication within the given statement.
(d) State the converse of the implication within the given statement.
(e) State the contrapositive of the implication within the given statement.
(f) State the negation of the given statement without using the word "not" or the $\neg$ symbol (but symbols such as $\neq$, $\nmid$, etc. are fine).
(g) Prove or disprove the given statement.

71. Find the complete solution to $28x + 60y = 10$.

72. Is 7386458999999992324343123 divisible by 11? Show and justify your work.

73. Prove that if $\gcd(a, b) = 1$, then $\gcd(a^m, b^n) = 1$ for all $m, n \in \mathbb{N}$. You may use the result of an example in the notes.

74. Let $A = \{n \in \mathbb{Z} : 2 \mid n\}$ and $B = \{n \in \mathbb{Z} : 4 \mid n\}$. Prove that $n \in (A - B)$ if and only if $n = 2k$ for some odd integer k.

75. Find all $z \in \mathbb{C}$ satisfying $z^2 = |z|^2$.

76. Prove that a prime p divides $ab^p - ba^p$ for all integers a and b.

77. Assume that it has been established that the following implication is true:

    If I don't see my advisor today, then I will see her tomorrow.

    For each of the statements below, determine if it is true or false, or explain why the truth value of the statement cannot be determined.

    (a) I don't meet my advisor both today and tomorrow. (This is arguably an ambiguous English sentence. Answer the problem using either or both interpretations.)
    (b) I meet my advisor both today and tomorrow.
    (c) I meet my advisor either today or tomorrow (but not on both days).

78. Let a, b, c and d be positive integers. Suppose $\frac{a}{b} < \frac{c}{d}$. Prove that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

79. Which elements of $\mathbb{Z}_6$ have multiplicative inverses?

80. Prove that the converse of *Divisibility of Integer Combinations (DIC)* is true.

81. Let a and b be two integers. Prove each of the following statements about a and b.

    (a) If $ab = 4$, then $(a - b)^3 - 9(a - b) = 0$.
    (b) If a and b are positive, then $a^2(b + 1) + b^2(a + 1) \geq 4ab$.

82. Prove the following statements by strong induction.

    (a) A sequence $\{x_n\}$ is defined recursively by $x_1 = 8$, $x_2 = 32$ and $x_i = 2x_{i-1} + 3x_{i-2}$ for $i \geq 3$. For all $n \in \mathbb{N}$, $x_n = 2 \times (-1)^n + 10 \times 3^{n-1}$.
    (b) A sequence $\{t_n\}$ is defined recursively by $t_n = 2t_{n-1} + n$ for all integers $n > 1$. The first term is $t_1 = 2$. For all $n \in \mathbb{N}$, $t_n = 5 \times 2^{n-1} - 2 - n$.

83. State whether the given statement is true or false and prove or disprove accordingly.

    (a) For all $a, b, c, x \in \mathbb{Z}$ such that $c, x > 0$, if $a \equiv b \pmod{c}$ then $a + x \equiv b + x \pmod{c + x}$.
    (b) For all $m \in \mathbb{N}$ and for all $[a] \in \mathbb{Z}_m$ there exists a $[b] \in \mathbb{Z}_m$ such that $[b]^2 = [a]$.

84. Prove or disprove each of the following statements.

   (a) $\forall\, n \in \mathbb{Z}$, $\frac{(5n-6)}{3}$ is an integer.
   (b) For every prime number p, $p+7$ is composite.
   (c) There exists an integer $m < 123456$ such that 123456m is a perfect square.
   (d) $\exists k \in \mathbb{Z}, 8 \nmid (4k^2 + 12k + 8)$.

85. The Chinese Remainder Theorem deals with the case where the moduli are coprime. We now investigate what happens if the moduli are not coprime.

   (a) Consider the following two systems of linear congruences:

$$A : \begin{cases} n \equiv & 2 \pmod{12} \\ n \equiv & 10 \pmod{18} \end{cases} \qquad\qquad B : \begin{cases} n \equiv & 5 \pmod{12} \\ n \equiv & 11 \pmod{18} \end{cases}$$

   Determine which one has solutions and which one has no solutions. For the one with solutions, give the complete solutions to the system. For the one with no solutions, explain why no solutions exist.

   (b) Let a_1, a_2 be integers, and let m_1, m_2 be positive integers. Consider the following system of linear congruences

$$S : \begin{cases} n \equiv & a_1 \pmod{m_1} \\ n \equiv & a_2 \pmod{m_2} \end{cases}$$

   Using your observations in (a), complete the following two statements. The system S has a solution if and only if ____________________________. If n_0 is a solution to S, then the complete solution is

$$n \equiv \underline{\hspace{4cm}}.$$

   (c) Prove the first statement.

86. Let a and b be two integers. Prove each of the following statements about a and b.

   (a) If $ab = 4$, then $(a-b)^3 - 9(a-b) = 0$.
   (b) If a and b are positive, then $a^2(b+1) + b^2(a+1) \geq 4ab$.

87. Suppose S and T are two sets. Prove that if $S \cap T = S$, then $S \subseteq T$. Is the converse true?

88. Consider the following proposition about integers a and b.

   If $a^3 \mid b^3$, then $a \mid b$.

   We now give three erroneous proofs of this proposition. Identify the major error in each proof, and explain why it is an error.

   (a) Consider $a = 2, b = 4$. Then $a^3 = 8$ and $b^3 = 64$. We see that $a^3 \mid b^3$ since $8 \mid 64$. Since $2 \mid 4$, we have $a \mid b$.
   (b) Since $a \mid b$, there exists $k \in \mathbb{Z}$ such that $b = ka$. By cubing both sides, we get $b^3 = k^3a^3$. Since $k^3 \in \mathbb{Z}$, $a^3 \mid b^3$.
   (c) Since $a^3 \mid b^3$, there exists $k \in \mathbb{Z}$ such that $b^3 = ka^3$. Then $b = (ka^2/b^2)a$, hence $a \mid b$.

89. Determine whether $A \implies B$ is logically equivalent to $(\neg A) \vee B$.

90. How many integers x where $0 \leq x < 1000$ satisfy $42x \equiv 105 \pmod{56}$?

91. Let x be a real number. Prove that if $x^3 - 5x^2 + 3x \neq 15$ then $x \neq 5$.

92. Let $A = \{1, \{1, \{1\}\}\}$. List all the elements of $A \times A$.

93. Prove the following two quantified statements.

    (a) $\forall\, n \in \mathbb{N},\, n + 1 \geq 2$

    (b) $\exists\, n \in \mathbb{Z},\, \frac{(5n-6)}{3} \in \mathbb{Z}$

94. Prove or disprove: A prime number can be formed using each of the digits from 0 to 9 exactly once.

95. Let n be an integer. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

96. Let a, b, c and d be positive integers. Suppose $\frac{a}{b} < \frac{c}{d}$. Prove that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

97. Let n be an integer. Prove that $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$.

98. Four friends: Alex, Ben, Gina and Dana are having a discussion about going to the movies. Ben says that he will go to the movies if Alex goes as well. Gina says that if Ben goes to the movies, then she will join. Dana says that she will go to the movies if Gina does. That afternoon, exactly two of the four friends watch a movie at the theatre. Deduce which two people went to the movies.

99. Find all $z \in \mathbb{C}$ satisfying $|z + 1|^2 \leq 3$ and shade the corresponding region in the complex plane.

100. Suppose a and b are integers. Prove that $\{ax + by \mid x, y \in \mathbb{Z}\} = \{n \cdot \gcd(a, b) \mid n \in \mathbb{Z}\}$.

101. Let x and y be integers. Prove or disprove each of the following statements.

    (a) If $2 \nmid xy$ then $2 \nmid x$ and $2 \nmid y$.

    (b) If $2 \nmid y$ and $2 \nmid x$ then $2 \nmid xy$.

    (c) If $10 \nmid xy$ then $10 \nmid x$ and $10 \nmid y$.

    (d) If $10 \nmid x$ and $10 \nmid y$ then $10 \nmid xy$.

102. Prove that $\forall z, w \in \mathbb{C}$, $|z - w|^2 + |z + w|^2 = 2(|z|^2 + |w|^2)$ (This is the Parallelogram Identity).

103. Prove that for every integer k, $\gcd(a, b) \leq \gcd(ak, b)$.

104. Use *De Moivre's Theorem (DMT)* to prove that $\sin(4\theta) = 4 \sin\theta \cos^3\theta - 4 \sin^3\theta \cos\theta$.

105. Consider the following proposition about integers a and b.

    If $a^3 \mid b^3$, then $a \mid b$.

    We now give three erroneous proofs of this proposition. Identify the major error in each proof, and explain why it is an error.

    (a) Consider $a = 2, b = 4$. Then $a^3 = 8$ and $b^3 = 64$. We see that $a^3 \mid b^3$ since $8 \mid 64$. Since $2 \mid 4$, we have $a \mid b$.

    (b) Since $a \mid b$, there exists $k \in \mathbb{Z}$ such that $b = ka$. By cubing both sides, we get $b^3 = k^3 a^3$. Since $k^3 \in \mathbb{Z}$, $a^3 \mid b^3$.

    (c) Since $a^3 \mid b^3$, there exists $k \in \mathbb{Z}$ such that $b^3 = ka^3$. Then $b = (ka^2/b^2)a$, hence $a \mid b$.

106. What is the smallest non-negative integer x such that $2000 \equiv x \pmod{37}$?

107. Let a, b, c and d be integers. Prove that if $a \mid b$ and $b \mid c$ and $c \mid d$, then $a \mid d$.

108. (a) Use the Extended Euclidean Algorithm to find three integers x, y and $d = \gcd(1112, 768)$ such that $1112x + 768y = d$.

(b) Determine integers s and t such that $768s - 1112t = \gcd(768, -1112)$.

109. Disprove the following. Let $a, b, c \in \mathbb{Z}$. Then $\gcd(a, b) = \gcd(a, c) \cdot \gcd(b, c)$.

110. Four friends: Alex, Ben, Gina and Dana are having a discussion about going to the movies. Ben says that he will go to the movies if Alex goes as well. Gina says that if Ben goes to the movies, then she will join. Dana says that she will go to the movies if Gina does. That afternoon, exactly two of the four friends watch a movie at the theatre. Deduce which two people went to the movies.

111. Solve

$$\begin{aligned} 3x - 2 &\equiv 7 \pmod{11} \\ 5 &\equiv 4x - 1 \pmod{9} \end{aligned}$$

112. For each linear congruence, determine the complete solution, if a solution exists.

    (a) $3x \equiv 11 \pmod{18}$

    (b) $4x \equiv 5 \pmod{21}$

113. What is the remainder when 14^{43} is divided by 41?

114. How many positive divisors does 33480 have?

115. Prove or disprove: If $7a^2 = b^2$ where $a, b \in \mathbb{Z}$, then 7 is a common divisor of a and b.

116. (a) Find all $w \in \mathbb{C}$ satisfying $w^2 = -15 + 8i$,

    (b) Find all $z \in \mathbb{C}$ satisfying $z^2 - (3 + 2i)z + 5 + i = 0$.

117. Prove that if k is an odd integer, then $4k + 7$ is an odd integer.

118. Set up an RSA scheme using two-digit prime numbers. Select values for the other variables and test encrypting and decrypting messages.

119. What is the remainder when 3141^{2001} is divided by 17?

120. Prove that for all $a \in \mathbb{Z}$, $\gcd(9a + 4, 2a + 1) = 1$

121. Prove that $(\neg A) \vee B$ is logically equivalent to $\neg(A \wedge \neg B)$.

122. For what values of c does $8x + 5y = c$ have exactly one solution where both x and y are strictly positive?

123. Prove the following statements by simple induction.

    (a) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} (2i - 1) = n^2$.

    (b) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$ where r is any real number such that $r \neq 1$. .

    (c) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$.

    (d) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

    (e) For all $n \in \mathbb{N}$ where $n \geq 4$, $n! > n^2$.

124. Show that $|\mathrm{Re}(z)| + |\mathrm{Im}(z)| \leq \sqrt{2}\,|z|$.

125. Prove that $x^2 + 9 \geq 6x$ for all real numbers x.

126. Let $\gcd(x, y) = d$. Express $\gcd(18x + 3y, 3x)$ in terms of d and prove that you are correct.

127. Prove or disprove the following statements. Let a, b, c be fixed integers.

    (a) If there exists an integer solution to $ax^2 + by^2 = c$, then $\gcd(a, b) \mid c$.

    (b) If $\gcd(a, b) \mid c$, then there exists an integer solution to $ax^2 + by^2 = c$.

128. Let $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$. Show that $z = (a + bi)^n + (a - bi)^n$ is real.

129. Express $\dfrac{2-i}{3+4i}$ in standard form.

130. Complete a multiplication table for $\mathbb{Z}_5$.

131. Give an example of three sets A, B, and C such that $B \neq C$ and $B - A = C - A$.

132. Let $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $c \mid a$, then $\gcd(b, c) = 1$.