**MATH 135: Randomized Exam Practice Problems**

These are the warm-up exercises and recommended problems take from the extra practice sets presented in random order. The challenge problems have not been included.

1. Let $n \geq 2$ be an integer. Prove that

$$\sum_{k=0}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = 0 = \sum_{k=0}^{n-1} \sin\left(\frac{2k\pi}{n}\right)$$

2. For each of the following polynomials $f(x) \in \mathbb{F}[x]$, factor $f(x)$ into factors with degree as small as possible over $\mathbb{F}[x]$. Cite appropriate propositions to justify each step of your reasoning.

   (a) $x^2 - 2x + 2 \in \mathbb{C}[x]$
   (b) $x^2 + (-3i + 2)x - 6i \in \mathbb{C}[x]$
   (c) $2x^3 - 3x^2 + 2x + 2 \in \mathbb{R}[x]$
   (d) $3x^4 + 13x^3 + 16x^2 + 7x + 1 \in \mathbb{R}[x]$
   (e) $x^4 + 27x \in \mathbb{C}[x]$

3. Disprove the following. Let $a, b, c \in \mathbb{Z}$. Then $\gcd(a, b) = \gcd(a, c) \cdot \gcd(b, c)$.

4. Solve

$$\begin{aligned} 3x - 2 &\equiv 7 \pmod{11} \\ 5 &\equiv 4x - 1 \pmod{9} \end{aligned}$$

5. Prove that a prime $p$ divides $ab^p - ba^p$ for all integers $a$ and $b$.

6. Suppose $r$ is some (unknown) real number, where $r \neq -1$ and $r \neq -2$. Show that

$$\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r\,(2^r)}{(r+1)(r+2)}.$$

7. Prove that if $w$ is an $n^{th}$ root of unity, then $\frac{1}{w}$ is also an $n^{th}$ root of unity.

8. In the proof by contradiction of *Prime Factorization (PF)*, why is it okay to write $r \leq s$?

9. Let $a, b, c \in \mathbb{Z}$. Is the following statement true? Prove that your answer is correct.

   $a \mid b$ if and only if $ac \mid bc$.

10. Find the smallest positive integer $a$ such that $5n^{13} + 13n^5 + a(9n) \equiv 0 \pmod{65}$ for all integers $n$.

11. Assume that it has been established that the following implication is true:

    If I don't see my advisor today, then I will see her tomorrow.

    For each of the statements below, determine if it is true or false, or explain why the truth value of the statement cannot be determined.

    (a) I don't meet my advisor both today and tomorrow. (This is arguably an ambiguous English sentence. Answer the problem using either or both interpretations.)
    (b) I meet my advisor both today and tomorrow.

(c) I meet my advisor either today or tomorrow (but not on both days).

12. Prove that there is a unique real number such that $x^2 - 6x + 9 = 0$.

13. Prove or disprove each of the following statements.

(a) $\forall\, n \in \mathbb{Z}$, $\frac{(5n-6)}{3}$ is an integer.
(b) For every prime number $p$, $p + 7$ is composite.
(c) $\exists k \in \mathbb{Z}, 8 \nmid (4k^2 + 12k + 8)$.
(d) The equation $x^3 + x^2 - 1 = 0$ has a real number solution between $x = 0$ and $x = 1$ (inclusive).

14. Prove the following statements by simple induction.

(a) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n}(2i-1) = n^2$.

(b) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ where $r$ is any real number such that $r \neq 1$. .

(c) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$.

(d) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

(e) For all $n \in \mathbb{N}$ where $n \geq 4$, $n! > n^2$.

15. Find all complex numbers $z$ solutions to $z^2 = \frac{1+i}{1-i}$.

16. Let $n$ be an integer. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

17. Let $a, b, c \in \mathbb{Z}$. Consider the statement $S$: If $\gcd(a, b) = 1$ and $c \mid (a + b)$, then $\gcd(a, c) = 1$. Fill in the blanks to complete a proof of $S$.

(a) Since $\gcd(a, b) = 1$, by __________ there exist integers $x$ and $y$ such that $ax + by = 1$.
(b) Since $c \mid (a + b)$, by __________ there exists an integer $k$ such that $a + b = ck$.
(c) Substituting $a = ck - b$ into the first equation, we get $1 = (ck - b)x + by = b(-x + y) + c(kx)$.
(d) Since 1 is a common divisor of $b$ and $c$ and $-x + y$ and $kx$ are integers, $\gcd(b, c) = 1$ by __________.

18. Given a rational number $r$, prove that there exist coprime integers $p$ and $q$, with $q \neq 0$, so that $r = \dfrac{p}{q}$.

19. Prove that $\forall z, w \in \mathbb{C}$, $|z - w|^2 + |z + w|^2 = 2(|z|^2 + |w|^2)$ (This is the Parallelogram Identity).

20. Prove or disprove the following statements. Let $a, b, c$ be fixed integers.

(a) If there exists an integer solution to $ax^2 + by^2 = c$, then $\gcd(a, b) \mid c$.
(b) If $\gcd(a, b) \mid c$, then there exists an integer solution to $ax^2 + by^2 = c$.

21. Let $x$ and $y$ be integers. Prove or disprove each of the following statements.

(a) If $2 \nmid xy$ then $2 \nmid x$ and $2 \nmid y$.
(b) If $2 \nmid y$ and $2 \nmid x$ then $2 \nmid xy$.
(c) If $10 \nmid xy$ then $10 \nmid x$ and $10 \nmid y$.

(d) If $10 \nmid x$ and $10 \nmid y$ then $10 \nmid xy$.

22. Let $a, b, c \in \mathbb{C}$. Prove: if $|a| = |b| = |c| = 1$, then $\overline{a+b+c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

23. Find all $z \in \mathbb{C}$ satisfying $|z+1|^2 \leq 3$ and shade the corresponding region in the complex plane.

24. Prove the following two quantified statements.

    (a) $\forall\, n \in \mathbb{N},\ n+1 \geq 2$

    (b) $\exists\, n \in \mathbb{Z},\ \frac{(5n-6)}{3} \in \mathbb{Z}$

25. What are the integer solutions to $x^2 \equiv 1 \pmod{15}$?

26. Prove that if $\gcd(a, b) = 1$, then $\gcd(a^m, b^n) = 1$ for all $m, n \in \mathbb{N}$. You may use the result of an example in the notes.

27. Prove that $x^2 + 9 \geq 6x$ for all real numbers $x$.

28. Find all non-negative integer solutions to $12x + 57y = 423$.

29. Let $n \in \mathbb{N}$. Prove by induction that if $n \equiv 1 \pmod 4$, then $i^n = i$.

30. Solve

$$x \equiv 7 \pmod{11}$$
$$x \equiv 5 \pmod{12}$$

31. Each of the following "proofs" by induction incorrectly "proved" a statement that is actually false. State what is wrong with each proof.

    (a) A sequence $\{x_n\}$ is defined by $x_1 = 3$, $x_2 = 20$ and $x_i = 5x_{i-1}$ for $i \geq 3$. Then, for all $n \in \mathbb{N}$, $x_n = 3 \times 5^{n-1}$.

    Let $P(n)$ be the statement: $x_n = 3 \times 5^{n-1}$.
    When $n = 1$ we have $3 \times 5^0 = 3 = x_1$ so $P(1)$ is true. Assume that $P(k)$ is true for some integer $k \geq 1$. That is, $x_k = 3 \times 5^{k-1}$ for some integer $k \geq 1$. We must show that $P(k+1)$ is true, that is, $x_{k+1} = 3 \times 5^k$. Now

$$x_{k+1} = 5x_k = 5(3 \times 5^{k-1}) = 3 \times 5^k$$

    as required. Since the result is true for $n = k+1$, and so holds for all $n$ by the Principle of Mathematical Induction.

    (b) For all $n \in \mathbb{N}$, $1^{n-1} = 2^{n-1}$.

    Let $P(n)$ be the statement: $1^{n-1} = 2^{n-1}$.
    When $n = 1$ we have $1^0 = 1 = 2^0$ so $P(1)$ is true. Assume that $P(i)$ is true for all integers $1 \leq i \leq k$. That is, $1^{i-1} = 2^{i-1}$ for all $1 \leq i \leq k$.
    We must show that $P(k+1)$ is true, that is, $1^{(k+1)-1} = 2^{(k+1)-1}$ or $1^k = 2^k$. By our inductive hypothesis, $P(2)$ is true so $1^1 = 2^1$. Also by our inductive hypothesis, $P(k)$ is true so $1^{k-1} = 2^{k-1}$. Multiplying these two equations together gives $1^k = 2^k$. Since the result is true for $n = k+1$, and so holds for all $n$ by the Principle of Strong Induction.

32. Let $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$. Show that $z = (a+bi)^n + (a-bi)^n$ is real.

33. Consider the following statement.

For all $x \in \mathbb{R}$, if $x^6 + 3x^4 - 3x < 0$, then $0 < x < 1$.

(a) Rewrite the given statement in symbolic form.

(b) State the hypothesis of the implication within the given statement.

(c) State the conclusion of the implication within the given statement.

(d) State the converse of the implication within the given statement.

(e) State the contrapositive of the implication within the given statement.

(f) State the negation of the given statement without using the word "not" or the $\neg$ symbol (but symbols such as $\neq$, $\nmid$, etc. are fine).

(g) Prove or disprove the given statement.

34. Solve $x^3 \equiv 17 \pmod{99}$.

35. Let $a, b, c$ be integers. Prove that if $a \mid b$ then $ac \mid bc$.

36. Let $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $c \mid a$, then $\gcd(b, c) = 1$.

37. Is $27^{129} + 61^{40}$ is divisible by 14? Show and justify your work.

38. Find all $z \in \mathbb{C}$ satisfying $z^2 = |z|^2$.

39. Given the public RSA encryption key $(e, n) = (5, 35)$, find the corresponding decryption key $(d, n)$.

40. Use *De Moivre's Theorem (DMT)* to prove that $\sin(4\theta) = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$.

41. The Fibonacci sequence is defined as the sequence $\{f_n\}$ where $f_1 = 1$, $f_2 = 1$ and $f_i = f_{i-1} + f_{i-2}$ for $i \geq 3$. Use induction to prove the following statements.

(a) For $n \geq 2$,

$$f_1 + f_2 + \cdots + f_{n-1} = f_{n+1} - 1$$

(b) Let $a = \dfrac{1+\sqrt{5}}{2}$ and $b = \dfrac{1-\sqrt{5}}{2}$. For all $n \in \mathbb{N}$, $f_n = \frac{a^n - b^n}{\sqrt{5}}$

42. Let $a$ and $b$ be integers. Prove that $(a \mid b \ \wedge \ b \mid a) \iff a = \pm b$.

43. Compute all the fifth roots of unity and plot them in the complex plane.

44. The Chinese Remainder Theorem deals with the case where the moduli are coprime. We now investigate what happens if the moduli are not coprime.

(a) Consider the following two systems of linear congruences:

$$A : \begin{cases} n \equiv 2 \pmod{12} \\ n \equiv 10 \pmod{18} \end{cases} \qquad B : \begin{cases} n \equiv 5 \pmod{12} \\ n \equiv 11 \pmod{18} \end{cases}$$

Determine which one has solutions and which one has no solutions. For the one with solutions, give the complete solutions to the system. For the one with no solutions, explain why no solutions exist.

(b) Let $a_1, a_2$ be integers, and let $m_1, m_2$ be positive integers. Consider the following system of linear congruences

$$S : \begin{cases} n \equiv & a_1 \pmod{m_1} \\ n \equiv & a_2 \pmod{m_2} \end{cases}$$

Using your observations in (a), complete the following two statements. The system $S$ has a solution if and only if ________________________. If $n_0$ is a solution to $S$, then the complete solution is

$$n \equiv \underline{\hspace{4cm}}.$$

(c) Prove the first statement.

45. Complete a multiplication table for $\mathbb{Z}_5$.

46. Solve $49x^{177} + 37x^{26} + 3x^2 + x + 1 \equiv 0 \pmod{7}$.

47. Prove or disprove: A prime number can be formed using each of the digits from 0 to 9 exactly once.

48. Let $a, b, c \in \mathbb{Z}$. Disprove the statement: If $a \mid (bc)$, then $a \mid b$ or $a \mid c$.

49. Prove that if $\gcd(a, b) = 1$, then $\gcd(2a + b, a + 2b) \in \{1, 3\}$.

50. Let $u$ and $v$ be fixed complex numbers. Let $\omega$ be a non-real cube root of unity. For each $k \in \mathbb{Z}$, define $y_k \in \mathbb{C}$ by the formula

$$y_k = \omega^k u + \omega^{-k} v.$$

(a) Compute $y_1$, $y_2$ and $y_3$ in terms of $u, v$ and $\omega$.

(b) Show that $y_k = y_{k+3}$ for any $k \in \mathbb{Z}$.

(c) Show for any $k \in \mathbb{Z}$,

$$y_k - y_{k+1} = \omega^k (1 - \omega)(u - \omega^{k-1} v).$$

51. Find the complete solution to $28x + 60y = 10$.

52. A complex number $z$ is called a *primitive* $n$-th root of unity if $z^n = 1$ and $z^k \neq 1$ for all $1 \leq k \leq n-1$.

(a) For each $n = 1, 2, 3, 6$, list all the primitive $n$-th roots of unity.

(b) Let $z$ be a primitive $n$-th root of unity. Prove the following statements.

i. For any $k \in \mathbb{Z}$, $z^k = 1$ if and only if $n \mid k$.

ii. For any $m \in \mathbb{Z}$, if $\gcd(m, n) = 1$, then $z^m$ is a primitive $n$-th root of unity.

53. Solve $x^3 - 29x^2 + 35x + 38 \equiv 0 \pmod{195}$.

54. Prove that the converse of *Divisibility of Integer Combinations (DIC)* is true.

55. Suppose $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{C}[x]$. We say $f(x)$ is *palindromic* if the coefficients $a_j$ satisfy

$$a_{n-j} = a_j \quad \text{for all} \quad 0 \leq j \leq n.$$

Prove that

(a) If $f(x)$ is a palindromic polynomial and $c \in \mathbb{C}$ is a root of $f(x)$, then $c$ must be non-zero, and $\frac{1}{c}$ is also a root of $f(x)$.

(b) If $f(x)$ is a palindromic polynomial of odd degree, then $f(-1) = 0$.

(c) If $\deg(f) = 1$ and $f(x)$ is a monic, palindromic polynomial, then $f(x) = x + 1$.

56. Suppose $a$ and $b$ are integers. Prove that $\{ax + by \mid x, y \in \mathbb{Z}\} = \{n \cdot \gcd(a, b) \mid n \in \mathbb{Z}\}$.

57. Prove that for every integer $k$, $\gcd(a, b) \leq \gcd(ak, b)$.

58. Prove that there exists a polynomial in $\mathbb{Q}[x]$ with the root $2 - \sqrt{7}$.

59. For each of the following statements, identify the four parts of the quantified statement (quantifier, variable, domain, and open sentence). Next, express the statement in symbolic form and then write down the negation of the statement (when possible, without using any negative words such as "not" or the $\neg$ symbol, but negative math symbols like $\neq, \nmid$ are okay).

    (a) For all real numbers $x$ and $y$, $x \neq y$ implies that $x^2 + y^2 > 0$.
    (b) For every even integer $a$ and odd integer $b$, a rational number $c$ can always be found such that either $a < c < b$ or $b < c < a$.
    (c) There is some $x \in \mathbb{N}$ such that for all $y \in \mathbb{N}$, $y \mid x$.
    (d) There exist sets of integers $X, Y$ such that for all sets of integers $Z$, $X \subseteq Z \subseteq Y$. (You may use $\mathcal{P}(\mathbb{Z})$ to denote the set of all sets of integers. This is called *power set notation.*)

60. What is the smallest non-negative integer $x$ such that $2000 \equiv x \pmod{37}$?

61. Prove the properties of complex arithmetic given in Proposition 1 in Chapter 30 of the course notes. Only one of the nine results is proved in the notes. A few others may have been proved in class.

62. Prove that the product of any four consecutive integers is one less than a perfect square.

63. Prove that an integer is even if and only if its square is an even integer.

64. Let $x$ and $y$ be integers. Prove that if $xy = 0$ then $x = 0$ or $y = 0$.

65. Determine all $k \in \mathbb{N}$ such that $n^k \equiv n \pmod{7}$ for all integers $n$. Prove that your answer is correct

66. What are the last two digits of $43^{201}$?

67. Suppose $a$, $b$ and $n$ are integers. Prove that $n \mid \gcd(a, n) \cdot \gcd(b, n)$ if and only if $n \mid ab$.

68. What is the remainder when $-98$ is divided by 7?

69. Suppose $p$ is a prime greater than five. Prove that the positive integer consisting of $p - 1$ digits all equal to one $(111 \ldots 1)$ is divisible by p. (Hint: $111111 = \frac{10^6 - 1}{9}$.)

70. Prove that for distinct primes $p$ and $q$, $\left(p^{q-1} + q^{p-1}\right) \equiv 1 \pmod{pq}$.

71. Prove that if $p$ is prime and $p \leq n$, then $p$ does not divide $n! + 1$.

72. Which elements of $\mathbb{Z}_6$ have multiplicative inverses?

73. Four friends: Alex, Ben, Gina and Dana are having a discussion about going to the movies. Ben says that he will go to the movies if Alex goes as well. Gina says that if Ben goes to the movies, then she will join. Dana says that she will go to the movies if Gina does. That afternoon, exactly two of the four friends watch a movie at the theatre. Deduce which two people went to the movies.

74. Let $A = \{1, \{1, \{1\}\}\}$. List all the elements of $A \times A$.

75. Let $A = \{n \in \mathbb{Z} \; : \; 2 \mid n\}$ and $B = \{n \in \mathbb{Z} \; : \; 4 \mid n\}$. Prove that $n \in (A - B)$ if and only if $n = 2k$ for some odd integer $k$.

76. Prove that if $|z| = 1$ or $|w| = 1$ and $\overline{z}w \neq 1$, then $\left| \dfrac{z - w}{1 - \overline{z}w} \right| = 1$.

77. Write $(\sqrt{3}+i)^4$ in standard form.

78. Suppose that $p$ is a prime and $a \in \mathbb{Z}$. Prove using induction that $a^{(p^n)} \equiv a (\text{mod } p)$ for all $n \in \mathbb{N}$.

79. Prove or disprove each of the following statements involving nested quantifiers.

 (a) For all $n \in \mathbb{Z}$, there exists an integer $k > 2$ such that $k \mid (n^3 - n)$.
 (b) For every positive integer $a$, there exists an integer $b$ with $|b| < a$ such that $b$ divides $a$.
 (c) There exists an integer $n$ such that $m(n-3) < 1$ for every integer $m$.
 (d) $\exists n \in \mathbb{N}, \forall m \in \mathbb{Z}, -nm < 0$.

80. Prove that: if $a \mid c$ and $b \mid c$ and $\gcd(a,b) = 1$, then $ab \mid c$.

81. Set up an RSA scheme using two-digit prime numbers. Select values for the other variables and test encrypting and decrypting messages.

82. Prove the following statement using a chain of logical equivalences as in Chapter 3 of the notes.

$$(A \wedge C) \vee (B \wedge C) \equiv \neg((A \vee B) \implies \neg C)$$

83. Prove the following statements.

 (a) There is no smallest positive real number.
 (b) For every even integer $n$, $n$ cannot be expressed as the sum of three odd integers.
 (c) If $a$ is an even integer and $b$ is an odd integer, then $4 \nmid (a^2 + 2b^2)$.
 (d) For every integer $m$ with $2 \mid m$ and $4 \nmid m$, there are no integers $x$ and $y$ that satisfy $x^2 + 3y^2 = m$.
 (e) The sum of a rational number and an irrational number is irrational.
 (f) Let $x$ be a non-zero real number. If $x + \frac{1}{x} < 2$, then $x < 0$.

84. Prove or disprove: If $7a^2 = b^2$ where $a, b \in \mathbb{Z}$, then 7 is a common divisor of $a$ and $b$.

85. State whether the given statement is true or false and prove or disprove accordingly.

 (a) For all $a, b, c, x \in \mathbb{Z}$ such that $c, x > 0$, if $a \equiv b \pmod{c}$ then $a + x \equiv b + x \pmod{c + x}$.
 (b) For all $m \in \mathbb{N}$ and for all $[a] \in \mathbb{Z}_m$ there exists a $[b] \in \mathbb{Z}_m$ such that $[b]^2 = [a]$.

86. Express $\dfrac{2-i}{3+4i}$ in standard form.

87. Let $a, b, c$ and $d$ be integers. Prove that if $a \mid b$ and $b \mid c$ and $c \mid d$, then $a \mid d$.

88. If $a$ and $b$ are integers, $3 \nmid a$, $3 \nmid b$, $5 \nmid a$, and $5 \nmid b$, prove that $a^4 \equiv b^4 \pmod{15}$.

89. How many integers $x$ where $0 \le x < 1000$ satisfy $42x \equiv 105 \pmod{56}$?

90. Let $a, b, c$ and $d$ be positive integers. Suppose $\frac{a}{b} < \frac{c}{d}$. Prove that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

91. How many positive divisors does 33480 have?

92. Is 738645899999992324343123 divisible by 11? Show and justify your work.

93. Give an example of three sets $A$, $B$, and $C$ such that $B \neq C$ and $B - A = C - A$.

94. (a) Use the Extended Euclidean Algorithm to find three integers $x, y$ and $d = \gcd(1112, 768)$ such that $1112x + 768y = d$.

(b) Determine integers $s$ and $t$ such that $768s - 1112t = \gcd(768, -1112)$.

95. Are the following functions onto? Are they 1-1? Justify your answer with a proof.

    (a) $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(n) = 2n + 1$.

    (b) $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2 + 4x + 9$.

    (c) $f : (\mathbb{R} - \{2\}) \to (\mathbb{R} - \{5\})$, defined by $f(x) = \frac{5x+1}{x-2}$.

96. Determine whether $A \implies \neg B$ is logically equivalent to $\neg(A \implies B)$.

97. Evaluate $\sum_{i=3}^{8} 2^i$ and $\prod_{j=1}^{5} \frac{j}{3}$.

98. Prove that $(\neg A) \vee B$ is logically equivalent to $\neg(A \wedge \neg B)$.

99. For what values of $c$ does $8x + 5y = c$ have exactly one solution where both $x$ and $y$ are strictly positive?

100. A basket contains a number of eggs and, when the eggs are removed 2, 3, 4, 5 and 6 at a time, there are 1, 2, 3, 4 and 5 respectively, left over. When the eggs are removed 7 at a time there are none left over. Assuming none of the eggs broke during the preceding operations, determine the minimum number of eggs that were in the basket.

101. Let $z, w \in \mathbb{C}$. Prove that if $zw = 0$ then $z = 0$ or $w = 0$.

102. What is the remainder when $3141^{2001}$ is divided by 17?

103. Let $x$ be a real number. Prove that if $x^3 - 5x^2 + 3x \neq 15$ then $x \neq 5$.

104. Consider the following statement:

     Let $a, b, c \in \mathbb{Z}$. For every integer $x_0$, there exists an integer $y_0$ such that $ax_0 + by_0 = c$.

     (a) Determine conditions on $a, b, c$ such that the statement is true if and only if these conditions hold. State and prove this if and only if statement.

     (b) Carefully write down the negation of the given statement and prove that this negation is true.

105. What is the remainder when $14^{43}$ is divided by 41?

106. Find the complete solution to $7x + 11y = 3$.

107. Let $\gcd(x, y) = d$. Express $\gcd(18x + 3y, 3x)$ in terms of $d$ and prove that you are correct.

108. Let $z \in \mathbb{C}$. Prove that $(x - z)(x - \overline{z}) \in \mathbb{R}[x]$.

109. Prove that if $k$ is an odd integer, then $4k + 7$ is an odd integer.

110. Show that $|\mathrm{Re}(z)| + |\mathrm{Im}(z)| \leq \sqrt{2}\,|z|$.

111. Consider the following proposition about integers $a$ and $b$.

     If $a^3 \mid b^3$, then $a \mid b$.

     We now give three erroneous proofs of this proposition. Identify the major error in each proof, and explain why it is an error.

     (a) Consider $a = 2, b = 4$. Then $a^3 = 8$ and $b^3 = 64$. We see that $a^3 \mid b^3$ since $8 \mid 64$. Since $2 \mid 4$, we have $a \mid b$.

(b) Since $a \mid b$, there exists $k \in \mathbb{Z}$ such that $b = ka$. By cubing both sides, we get $b^3 = k^3a^3$. Since $k^3 \in \mathbb{Z}$, $a^3 \mid b^3$.

(c) Since $a^3 \mid b^3$, there exists $k \in \mathbb{Z}$ such that $b^3 = ka^3$. Then $b = (ka^2/b^2)a$, hence $a \mid b$.

112. Let $n$, $a$ and $b$ be positive integers. Negate the following implication without using the word "not" or the $\neg$ symbol (but symbols such as $\neq$, $\nmid$, etc. are fine). *Implication:* If $a^3 \mid b^3$, then $a \mid b$.

113. Let $a$ and $b$ be two integers. Prove each of the following statements about $a$ and $b$.

(a) If $ab = 4$, then $(a-b)^3 - 9(a-b) = 0$.

(b) If $a$ and $b$ are positive, then $a^2(b+1) + b^2(a+1) \geq 4ab$.

114. Let $n$ be an integer. Prove that $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$.

115. Suppose $S$ and $T$ are two sets. Prove that if $S \cap T = S$, then $S \subseteq T$. Is the converse true?

116. Prove that there is a unique minimum value of $x^2 - 4x + 11$.

117. In each of the following cases, find all values of $[x] \in \mathbb{Z}_m$, $0 \leq x < m$, that satisfy the equation.

(a) $[4][3] + [5] = [x] \in \mathbb{Z}_{10}$

(b) $[7]^{-1} - [2] = [x] \in \mathbb{Z}_{10}$

(c) $[2][x] = [4] \in \mathbb{Z}_8$

(d) $[3][x] = [9] \in \mathbb{Z}_{11}$

118. Let $S$ and $T$ be any two sets in universe $\mathcal{U}$. Prove that $(S \cup T) - (S \cap T) = (S - T) \cup (T - S)$.

119. Divide $f(x) = x^3 + x^2 + x + 1$ by $g(x) = x^2 + 4x + 3$ to find the quotient $q(x)$ and remainder $r(x)$ that satisfy the requirements of the *Division Algorithm for Polynomials (DAP)*.

120. Find all $z \in \mathbb{C}$ which satisfy

(a) $z^2 + 2z + 1 = 0$,

(b) $z^2 + 2\overline{z} + 1 = 0$,

(c) $z^2 = \dfrac{1+i}{1-i}$.

121. For each linear congruence, determine the complete solution, if a solution exists.

(a) $3x \equiv 11 \pmod{18}$

(b) $4x \equiv 5 \pmod{21}$

122. Let $g(x) = x^3 + bx^2 + cx + d \in \mathbb{C}[x]$ be a cubic polynomial whose leading coefficient is 1 (such polynomials are called *monic*). Let $z_1, z_2, z_3$ be three roots of $g(x)$, such that

$$g(x) = (x - z_1)(x - z_2)(x - z_3).$$

Prove that

$$\begin{aligned} z_1 + z_2 + z_3 &= -b, \\ z_1z_2 + z_2z_3 + z_3z_1 &= c, \\ z_1z_2z_3 &= -d. \end{aligned}$$

123. (a) Find all $w \in \mathbb{C}$ satisfying $w^2 = -15 + 8i$,

(b) Find all $z \in \mathbb{C}$ satisfying $z^2 - (3 + 2i)z + 5 + i = 0$.

124. Prove that for all $a \in \mathbb{Z}$, $\gcd(9a + 4, 2a + 1) = 1$

125. Prove the following statements by strong induction.

(a) A sequence $\{x_n\}$ is defined recursively by $x_1 = 8$, $x_2 = 32$ and $x_i = 2x_{i-1} + 3x_{i-2}$ for $i \geq 3$. For all $n \in \mathbb{N}$, $x_n = 2 \times (-1)^n + 10 \times 3^{n-1}$.

(b) A sequence $\{t_n\}$ is defined recursively by $t_n = 2t_{n-1} + n$ for all integers $n > 1$. The first term is $t_1 = 2$. For all $n \in \mathbb{N}$, $t_n = 5 \times 2^{n-1} - 2 - n$.

126. Write $z = \dfrac{9 + i}{5 - 4i}$ in the form $r(\cos\theta + i\sin\theta)$ with $r \geq 0$ and $0 \leq \theta < 2\pi$.

127. Express the following complex numbers in standard form.

(a) $\dfrac{(\sqrt{2} - i)^2}{(\sqrt{2} + i)(1 - \sqrt{2}i)}$

(b) $(\sqrt{5} - i\sqrt{3})^4$

128. Find a real cubic polynomial whose roots are 1 and $i$.