Triangle Inequality: Let $z, w \in \mathbb{C}$. Then $|z+w| \leq |z| + |w|$.

Proof: It suffices to prove that

$$|z+w|^2 \leq (|z| + |w|)^2 = |z|^2 + 2|zw| + |w|^2$$

since the modulus is a positive real number. Using the Properties of Modulus and the Properties of Conjugates, we have

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z+w}) && \text{PM} \\
&= (z+w)(\bar{z}+\bar{w}) && \text{PCJ} \\
&= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\
&= |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 && \text{PCJ and PM}
\end{aligned}$$

Now, from Properties of Conjugates, we have that

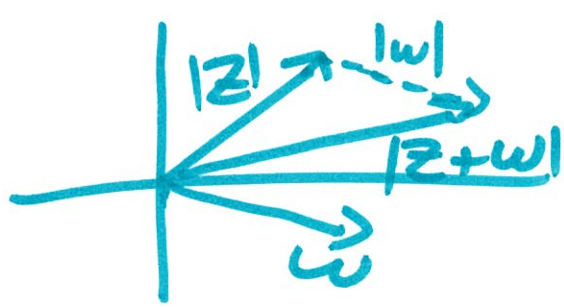
$$z\bar{w} + \overline{z\bar{w}} = 2\Re(z\bar{w}) \leq 2|z\bar{w}| = 2|zw|$$

and hence

$$|z+w|^2 = |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 \leq |z|^2 + 2|zw| + |w|^2$$

completing the proof.

Diagram:

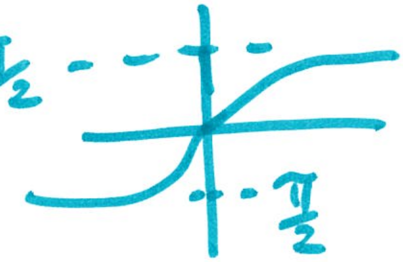


Express the following in terms of polar coordinates:

1. -3
2. $1-i$

1. $z=-3$

$r=|-3|=3$



$\theta=\arctan\left(\frac{0}{-3}\right)=0$

$-3=3(\cos(0)+i\sin(0))$ ×

actually, $\theta=0+\pi$.

$-3=3(\cos\pi+i\sin\pi)$

2. $1-i$ $\qquad r=|1-i|=\sqrt{1^2+1^2}=\sqrt{2}$

$1-i=\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)$

$=\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right)\right)$

$=\sqrt{2}\,\mathrm{cis}\left(7\pi/4\right)$

1. Write cis$(15\pi/6)$ in standard form.

2. Write $-3\sqrt{2} + 3\sqrt{6}i$ in polar form.

1. $\text{cis}\left(\frac{15\pi}{6}\right) = \cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right)$

$= i$

2. $r = |-3\sqrt{2} + 3\sqrt{6}i|$

$= \sqrt{(-3\sqrt{2})^2 + (3\sqrt{6})^2}$

$= \sqrt{18 + 54}$

$= \sqrt{72}$

$= 6\sqrt{2}$

$-3\sqrt{2} + 3\sqrt{6}i = 6\sqrt{2}\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= 6\sqrt{2}\,\text{cis}(2\pi/3)$

(PMCN)

Polar Multiplication of Complex Numbers

If $z_1 = r_1 \operatorname{cis}\theta_1$ & $z_2 = r_2 \operatorname{cis}\theta_2$

Then $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

Pf: $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

$+ i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2$

trig identities. $= r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

Corollary: Multiplication by $i = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})$ gives a rotation by $\pi/2$.

Ex. $(\sqrt{6}+\sqrt{2}\,i)(-3\sqrt{2}+3\sqrt{6}\,i)$

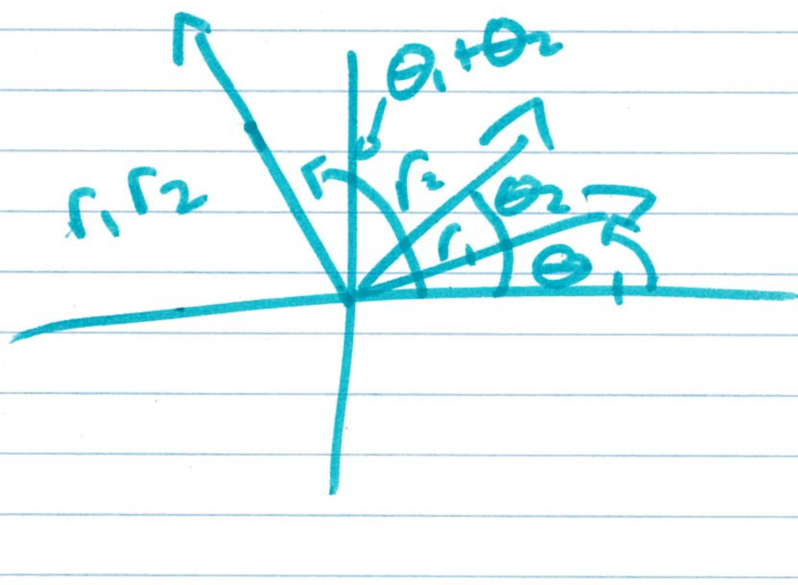
$= 2\sqrt{2}\,\text{CIS}(\pi/6)\cdot 6\sqrt{2}\,\text{CIS}(2\pi/3)$

PMCN $= 24\,\text{CIS}(\pi/6 + 2\pi/3)$

$= 24\,\text{CIS}(5\pi/6)$

$= 24\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$= (-12\sqrt{3} + 12i)$



De Moivre's Theorem

If $\theta \in \mathbb{R}$ & $n \in \mathbb{Z}$ then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Pf: First note that when $n=0$,

$$(\cos\theta + i\sin\theta)^0 = 1$$

$$\cos(0\cdot\theta) + i\sin(0\cdot\theta) = 1$$

Now, if $n<0$, write $n=-m$ for some $m \in \mathbb{N}$

$$\begin{aligned}
(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m} \\
&= \left((\cos\theta + i\sin\theta)^{-1}\right)^m \\
&= \left(\frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}\right)^m \\
&= (\cos\theta - i\sin\theta)^m \\
&= (\cos(-\theta) + i\sin(-\theta))^m
\end{aligned}$$

Thus, it suffices to prove DMT with a positive exponent.