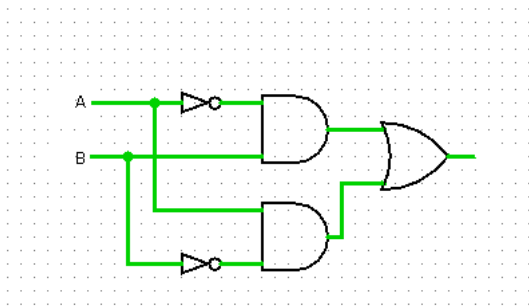# Warm-Up Problem

In the given circuit diagram, which values for A and B will give a true (on) output?

Propositional Logic: Entailment

Carmen Bruni

Lecture 6

Based on work by J. Buss, A. Gao, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Learning goals

Semantic entailment

- Define semantic entailment.
- Explain subtleties of semantic entailment.
- Determine whether a semantic entailment holds by using truth tables, valuation trees, and/or logical identities.
- Prove semantic entailment using truth tables and/or valuation trees.

Natural deduction in propositional logic

- Describe rules of inference for natural deduction.
- Prove a conclusion from given premises using natural deduction inference rules.
- Describe strategies for applying each inference rule when proving a conclusion formula using natural deduction.

Satisfiability of Sets of Formulas

The notion of satisfiability extends to sets of formulas.

Let Σ denote a set of well-formed formulas and t a valuation. Define

$$\Sigma^t = \begin{cases} \text{T} & \text{if for each } \beta \in \Sigma, \ \beta^t = \text{T} \\ \text{F} & \text{otherwise} \end{cases}$$

that is, $\Sigma^t = \text{T}$ if and only if for all $\beta \in \Sigma$, $\beta^t = \text{T}$.

When $\Sigma^t = \text{T}$, we say that t *satisfies* Σ.

A set Σ is *satisfiable* iff there is some valuation t such that $\Sigma^t = \text{T}$.

Example. The set $\big\{((p \to q) \vee r), \ (p \vee (q \vee s))\big\}$ is satisfiable.

Let V be the set of all truth valuations. Write down using only symbols what it means for a set of formulas Σ to be satisfiable. Do the same for what it means for $\Sigma^t = \mathtt{T}$ for some arbitrary $t \in V$.

Exercise

Let V be the set of all truth valuations. Write down using only symbols what it means for a set of formulas Σ to be satisfiable. Do the same for what it means for $\Sigma^t = \mathtt{T}$ for some arbitrary $t \in V$.

1. $\exists t \in V, \ \Sigma^t = \mathtt{T}$
2. $\Sigma^t = \mathtt{T} \leftrightarrow (\forall \beta \in \Sigma, \ \beta^t = \mathtt{T})$

Proving arguments valid

Recall that logic is the science of reasoning.

One important goal of logic is to infer that a conclusion is true based on a set of premises.

A logical argument:

Premise 1

Premise 2

...

Premise n

———

Conclusion

A common problem is to prove that an argument is valid, that is the set of premises semantically entails the conclusion.

Logical Consequence, a.k.a. Entailment

Let Σ be a set of formulas, and let α be a formula. We say that

- α is a *logical consequence* of Σ, or
- Σ *(semantically) entails* α, or
- $\Sigma \vDash \alpha$,

if and only if

> for all truth valuations t, if $\Sigma^t = \mathtt{T}$ then also $\alpha^t = \mathtt{T}$.

We write $\Sigma \nvDash \alpha$ for "not $\Sigma \vDash \alpha$". That is,

> there exists a truth valuation t such that $\Sigma^t = \mathtt{T}$ and $\alpha^t = \mathtt{F}$.

For the previous slide, $\Sigma = \{p_1, p_2, ..., p_n\}$ could be a set of premises and let α could be the conclusion that we want to derive.

Exercise

Write down using only symbols what $\Sigma \vDash \alpha$ means where Σ is a set of well-formed formulas and α is well-formed.

Exercise

Write down using only symbols what $\Sigma \vDash \alpha$ means where Σ is a set of well-formed formulas and α is well-formed.

$$\Sigma \vDash \alpha \leftrightarrow (\forall t \in V, (\Sigma^t \rightarrow \alpha^t = T))$$

Proving or disproving entailment

Proving that Σ entails α, denoted $\Sigma \vDash \alpha$:

- Using a truth table: Consider all rows of the truth table in which all of the formulas in Σ are true. Verify that α is true in all of these rows.
- Direct proof: For every truth valuation under which all of the premises are true, show that the conclusion is also true under this valuation.
- Proof by contradiction: Assume that the entailment does not hold, which means that there is a truth valuation under which all of the premises are true and the conclusion is false. Derive a contradiction.

Proving that Σ does not entail α, denoted $\Sigma \nvDash \alpha$:

- Find one truth valuation t under which all of the premises in Σ are true and the conclusion α is false.

Equivalence and Entailment

Equivalence can be expressed using the notion of entailment.

Lemma. $\alpha \equiv \beta$ if and only if both $\{\alpha\} \vDash \beta$ and $\{\beta\} \vDash \alpha$.

Proofs in Propositional Logic:

Natural Deduction

Learning goals

Natural deduction in propositional logic

- Describe rules of inference for natural deduction.
- Prove a conclusion from given premises using natural deduction inference rules.
- Describe strategies for applying each inference rule when proving a conclusion formula using natural deduction.

The Natural Deduction Proof System

We will consider a proof system called Natural Deduction.

- It closely follows how people (mathematicians, at least) normally make formal arguments.
- It extends easily to more-powerful forms of logic.

A proof is syntactic

First, we think about proofs in a purely syntactic way.

A proof

- starts with a set of premises,
- transforms the premises based on a set of inference rules (which form a proof system),
- and reaches a conclusion.

We write

$$\Sigma \vdash_{ND} \varphi \quad \text{or simply} \quad \Sigma \vdash \varphi$$

if we can find such a proof in our proof system that starts with a set of premises Σ and ends with the conclusion φ.

Goal is to show semantic entailment

Next, we think about connecting proofs to semantic entailment.

We will answer these questions:

- (Soundness)
    1. Does $\Sigma \vdash \varphi$ imply $\Sigma \vDash \varphi$?
    2. Does every proof establish a semantic entailment?
    3. If a proof of Σ to φ exists, is it true that Σ semantically entails φ?

- (Completeness)
    1. Does $\Sigma \vDash \varphi$ imply $\Sigma \vdash \varphi$?
    2. For every semantic entailment, can I find a proof for it?
    3. If Σ semantically entails φ, does a proof of Σ to φ exist?

Reflexivity / Premise

If you want to write down a previous formula in the proof again, you can do it by *reflexivity*.

Name	$\vdash$-notation	inference notation
Reflexivity, or Premise	$\Sigma, \alpha \vdash \alpha$	$\dfrac{\alpha}{\alpha}$

The notation on the right: Given the formulas above the line, we can infer the formula below the line.

The version in the center reminds us of the role of assumptions in Natural Deduction. Other rules will make more use of it.

An example using reflexivity

Here is a proof of $\{p, q\} \vdash p$.

> 1. p Premise
> 2. q Premise
> 3. p Reflexivity: 1

Alternatively, we could simply write

> 1. p Premise

and be done.

For each symbol, the rules come in pairs.

- An "introduction rule" adds the symbol to the formula.
- An "elimination rule" removes the symbol from the formula.

Rules for Conjunction

Name	$\vdash$-**notation**	inference notation
$\wedge$-introduction ($\wedge$i)	If $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$, then $\Sigma \vdash (\alpha \wedge \beta)$	$\dfrac{\alpha \quad \beta}{(\alpha \wedge \beta)}$

Name	$\vdash$-**notation**	inference notation
$\wedge$-elimination ($\wedge$e)	If $\Sigma \vdash (\alpha \wedge \beta)$, then $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$	$\dfrac{(\alpha \wedge \beta)}{\alpha} \quad \dfrac{(\alpha \wedge \beta)}{\beta}$

Example: Conjunction Rules

Example. Show that $\{(p \wedge q)\} \vdash (q \wedge p)$.

Example: Conjunction Rules

Example. Show that $\{(p \wedge q)\} \vdash (q \wedge p)$.

1.	$(p \wedge q)$	Premise
2.	q	$\wedge$e: 1
3.	p	$\wedge$e: 1
4.	$(q \wedge p)$	$\wedge$i: 2, 3

Example. Show that $\{(p \wedge q),\, r\} \vdash (q \wedge r)$.

Example: Conjunction Rules (2)

Example. Show that $\{(p \wedge q),\, r\} \vdash (q \wedge r)$.

1.	$(p \wedge q)$	Premise
2.	r	Premise
3.	q	$\wedge$e: 1
4.	$(q \wedge r)$	$\wedge$i: 3, 2