A Hands-on Approach to the Great Theorems

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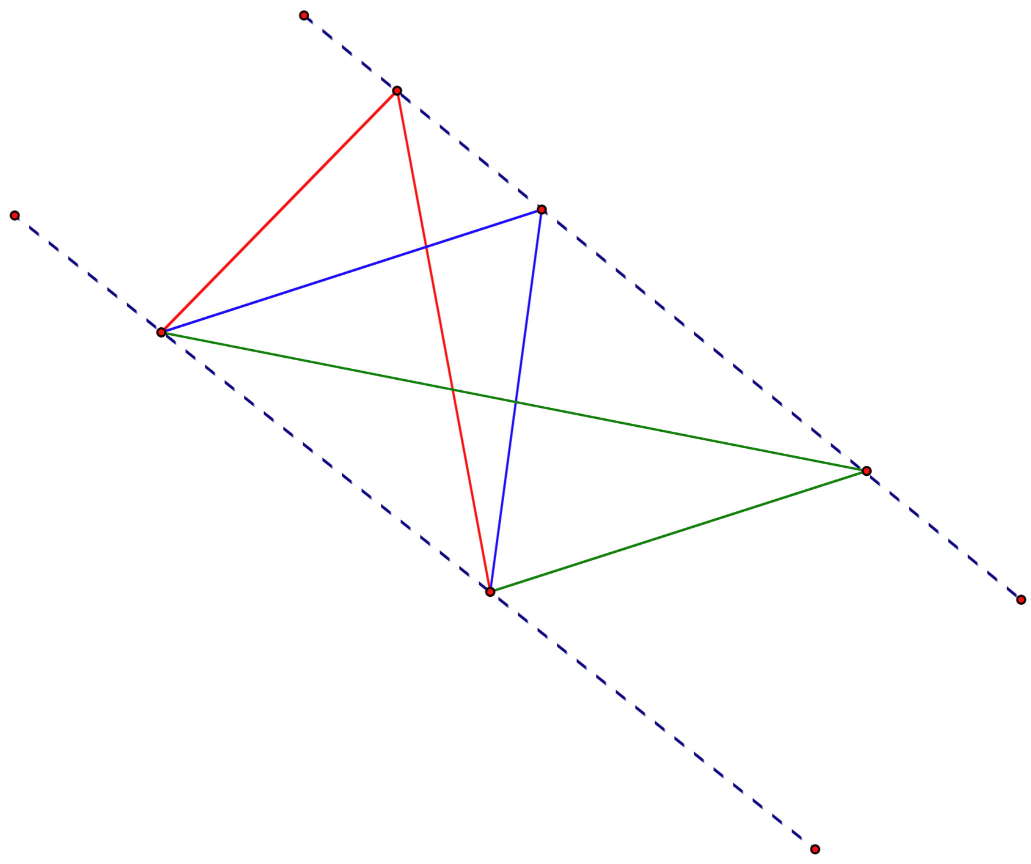
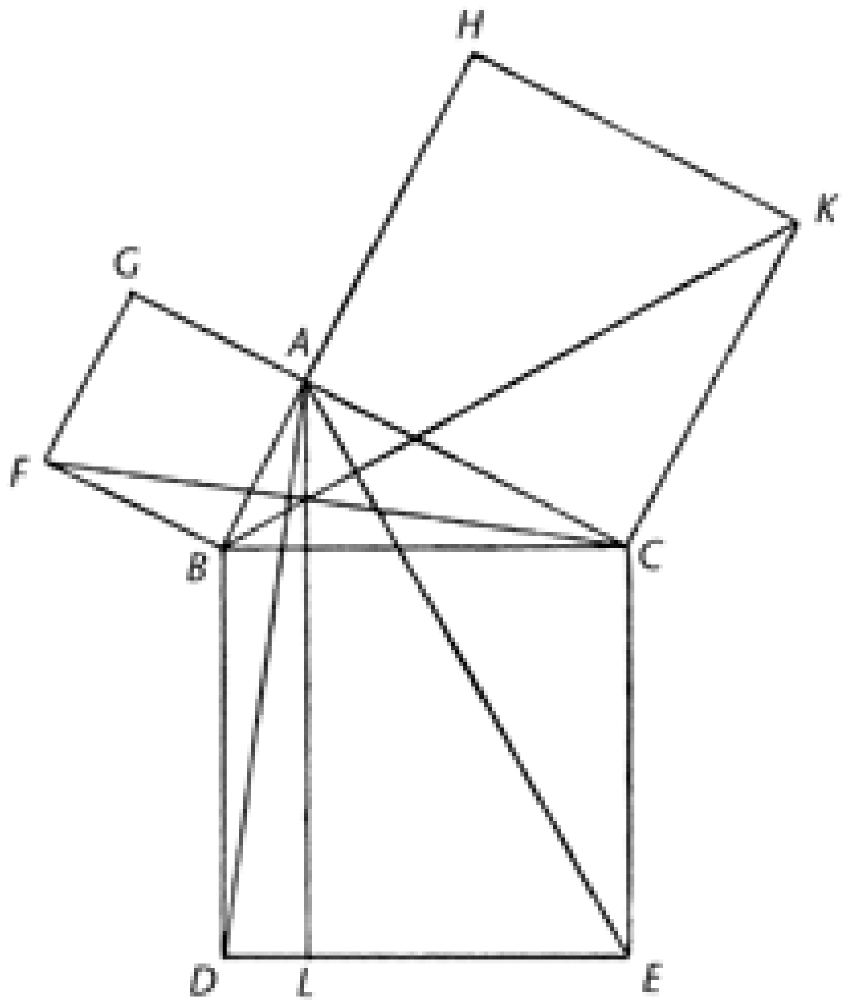
Euclid Book I, Proposition 37

Triangles which are on the same base and be in the same parallels are equal to one another.



Euclid Book I, Proposition 47

In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.



Nicolo Fontana da Brescio, aka Tartaglia, the Stammerer

Tartaglia, 1500-1557

Nicolo of Brescia, known as Tartaglia, meaning 'the stammerer,' managed to solve equations of the form $x^3 + mx^2 = n$ and made no secret of his discovery.

Antonio Fior, a student of a colleague of Paciola who had worked on cubics, challenged Tartaglia to a public contest: Each person gave the other 30 problems with 40 or 50 days in which to solve them. The winner was the one to solve the most problems, but a small prize was also offered for each problem.

Excerpts from article by: *J J O'Connor* and *E F Robertson* [http://www-history.mcs.st-andrews.ac.uk/HistTopics/Tartaglia_v_Cardan.html] This article consists of quotes, mainly from Tartaglia, concerning the 'cubic dispute'. The following exchange took place in 1539.

Ferrari: These are the thirty problems proposed by me Antonio Maria Fior to you Master Niccolo Tartaglia.

1. Find me a number that when its cube root is added to it, the result is six, that is, 6.

2. Find me two numbers in double proportion such that when the square of the larger number is multiplied by the smaller, and this product is added to the two original numbers, the result is forty, that is, 40.

15. A man sells a sapphire for 500 ducats, making a profit of the cube root of his capital. How much is the profit?

Tartaglia’s Discovery

Tartaglia solved all of Fior's problems in the space of two hours. All the problems Fior had sent were of the form

$$x^3 + mx = n,$$

since he believed Tartaglia would be unable to solve this type. Eight days before the problems were to be collected, Tartaglia had found the general method for solving all types of cubics.

Tartaglia’s Solution of the Cubic

Thm: Rule to solve $x^3 + mx = n$, or,

Cube and cosa equals number.

Cube one-third the coefficient of x; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same…. Then, subtracting the cube root of the first from the cube root of the second, the remainder which is left is the value of x.

In Modern Notation

$$x=\sqrt[3]{\frac{n}{2}+\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}-\sqrt[3]{-\frac{n}{2}+\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}$$

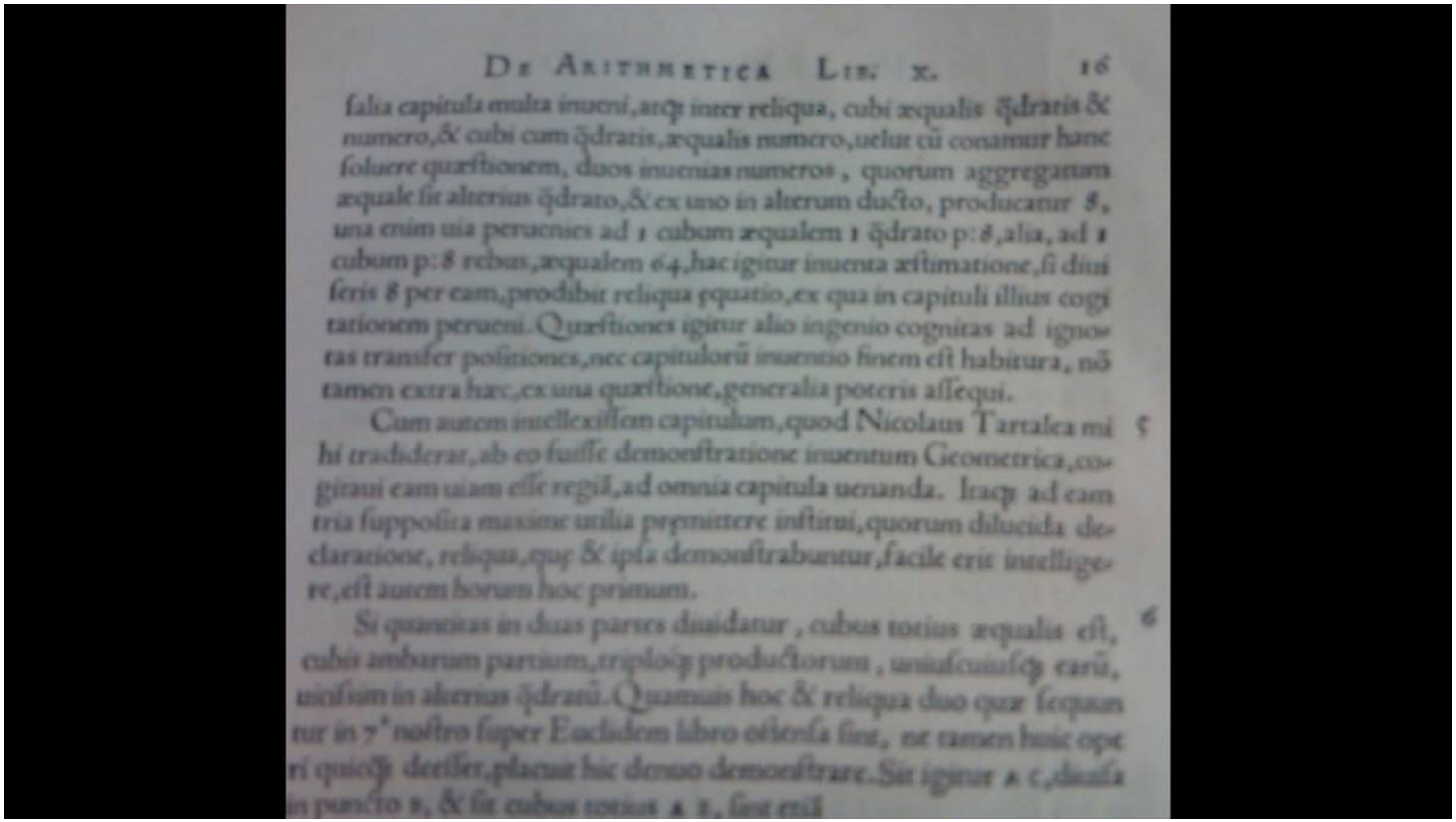
Cardan

News of Tartaglia's victory reached Girolamo Cardan in Milan where he was preparing to publish *Practica Arithmeticae* (1539). Cardan invited Tartaglia to visit him and, after much persuasion, made him divulge the secret of his solution of the cubic equation. This Tartaglia did, having made Cardan promise to keep it secret until Tartaglia had published it himself. Cardan did not keep his promise. In 1545 he published *Ars Magna*, “The Great Art.” This was the first Latin treatise on algebra and contained Tartaglia’s proof for the cubic.

Cardan's *Ars Magna*, 1545
Tartaglia's *Nova Scientia*, 1537

Huntington Library, Oct 23, 2010

Cardan gives Tartaglia "credit" *Ars Magna*, p. 16



ſalia capitula multa inueni, atq; inter reliqua, cubi æqualis q̃dratis &
numero, & cubi cum q̃dratis, æqualis numero, uelut cũ conamur hanc
ſoluere quæſtionem, duos inuenias numeros, quorum aggregatum
æquale ſit alterius q̃drato, & ex uno in alterum ducto, producatur 8,
una enim uia peruenies ad 1 cubum æqualem 1 q̃drato p: 8, alia, ad 1
cubum p: 8 rebus, æqualem 64, hac igitur inuenta æſtimatione, ſi diui
ſeris 8 per eam, prodibit reliqua æquatio, ex qua in capituli illius cogi
tationem perueni. Quæſtiones igitur alio ingenio cognitas ad igno-
tas transfer poſitiones, nec capitulorũ inuentio finem eſt habitura, nõ
tamen extra hæc, ex una quæſtione, generalia poteris aſſequi.

Cum autem intellexiſſem capitulum, quod Nicolaus Tartalea mi 5
hi tradiderat, ab eo fuiſſe demonſtratione inuentum Geometrica, co-
gitaui eam uiam eſſe regiã, ad omnia capitula uenanda. Itaq; ad eam
tria ſuppoſita maxime utilia præmittere inſtitui, quorum dilucida de-
claratione, reliqua, quę & ipſa demonſtrabuntur, facile erit intellige-
re, eſt autem horum hoc primum.

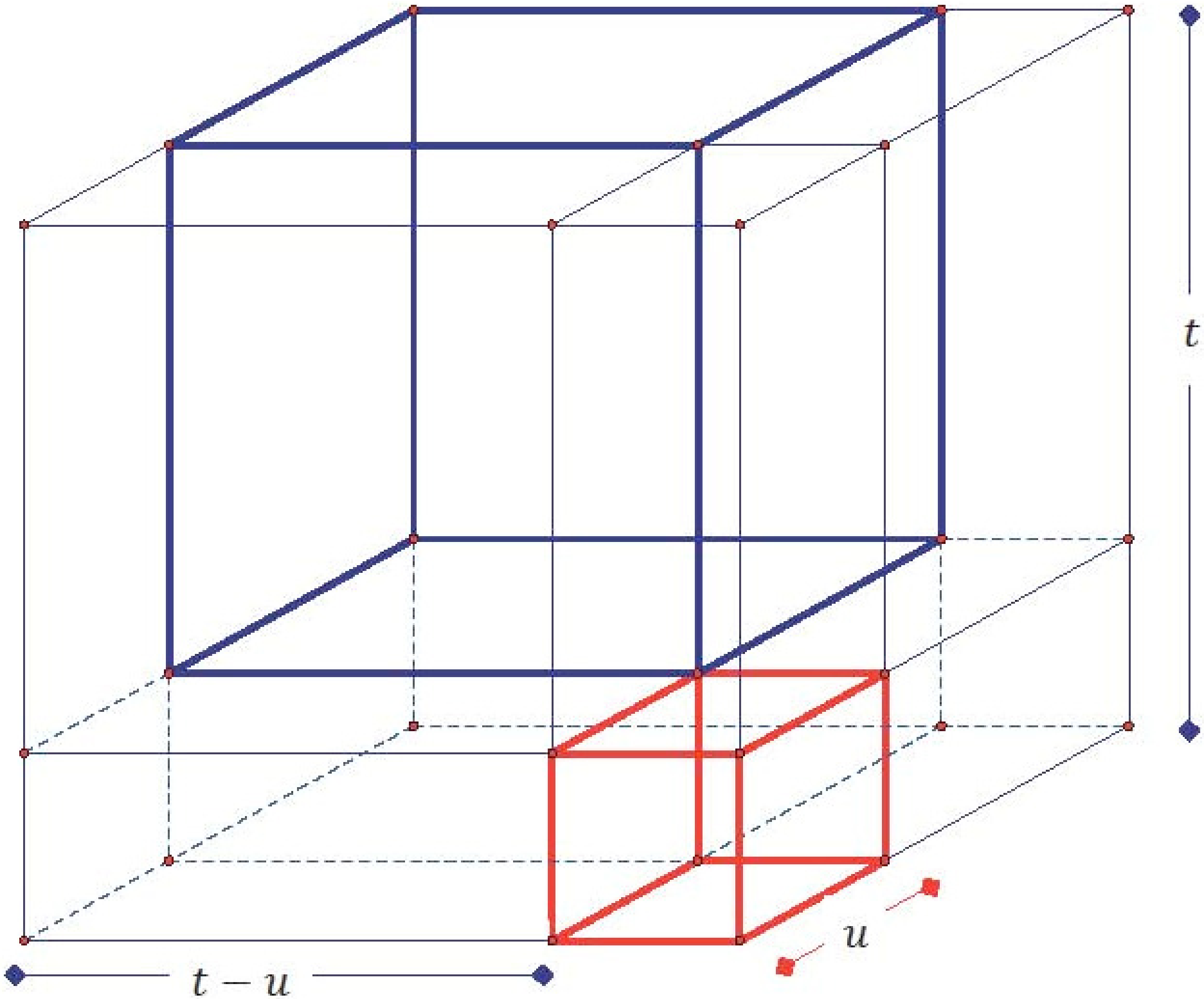
Si quantitas in duas partes diuidatur, cubus totius æqualis eſt, 6
cubis ambarum partium, triploq; productorum, uniuſcuiuſq; earũ,
uiciſſim in alterius q̃dratũ. Quamuis hoc & reliqua duo quæ ſequun
tur in 7° noſtro ſuper Euclidem libro oſtenſa ſint, ne tamen huic ope
ri quicq̃ deeſſet, placuit hic denuo demonſtrare. Sit igitur A C, diuiſa
in puncto B, & ſit cubus totius A B, ſint etiã

Cardano Remembered　　Tartaglia Forgotten

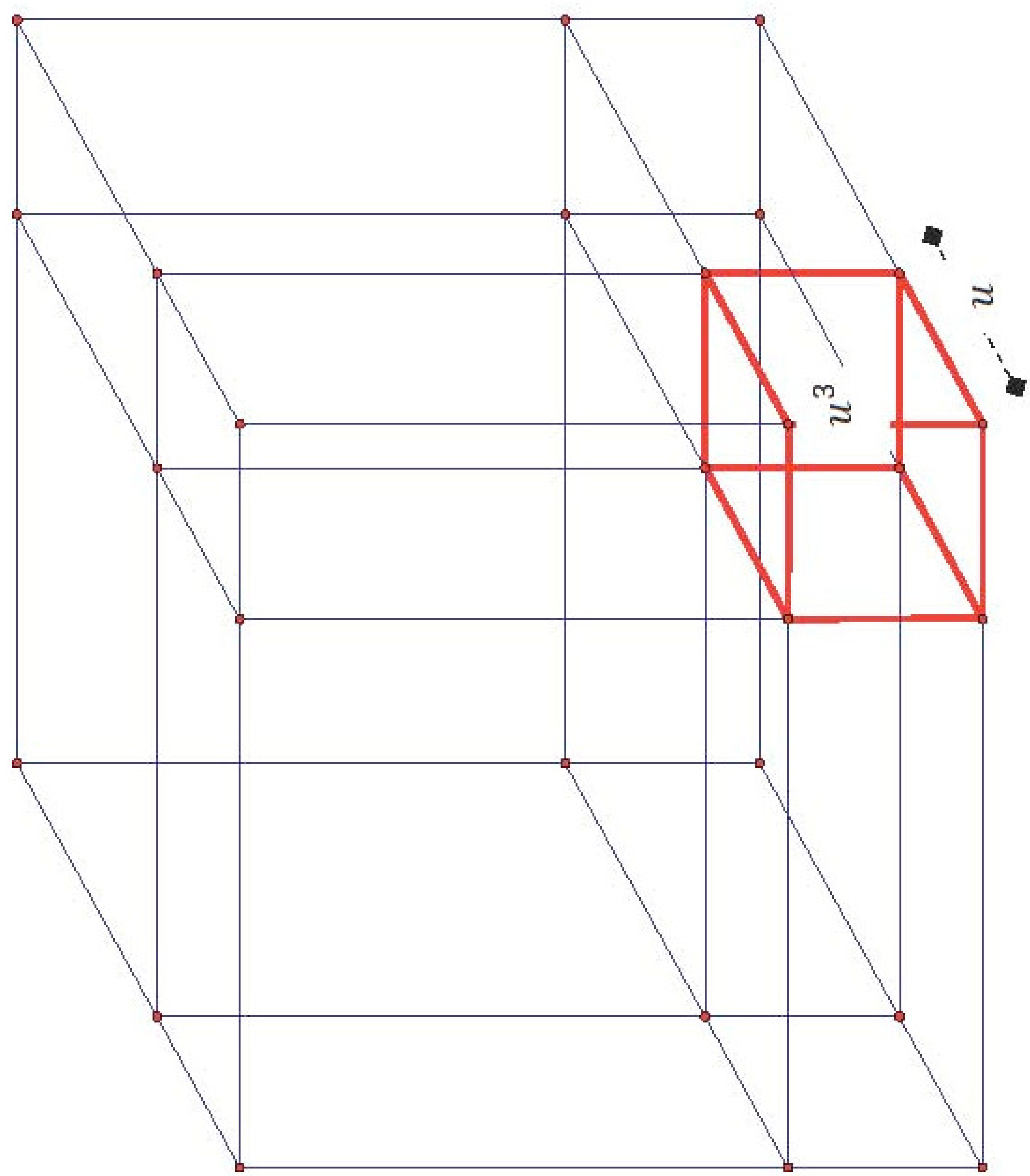
Tartaglia's/Cardano's Dissection of The Cube

Small Cube, Tower, Two Slabs, Base, Big Cube

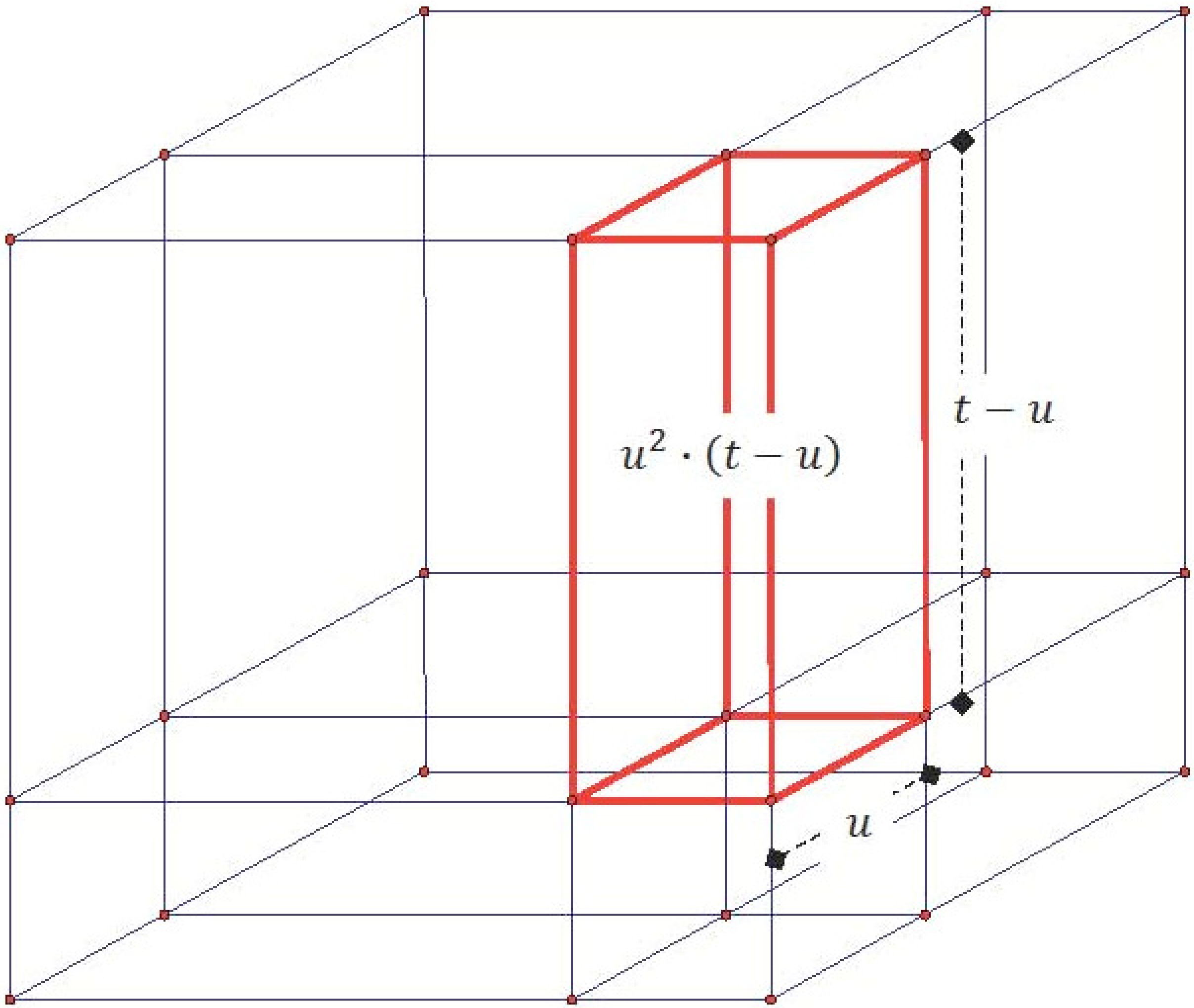
Tartaglia’s Cube



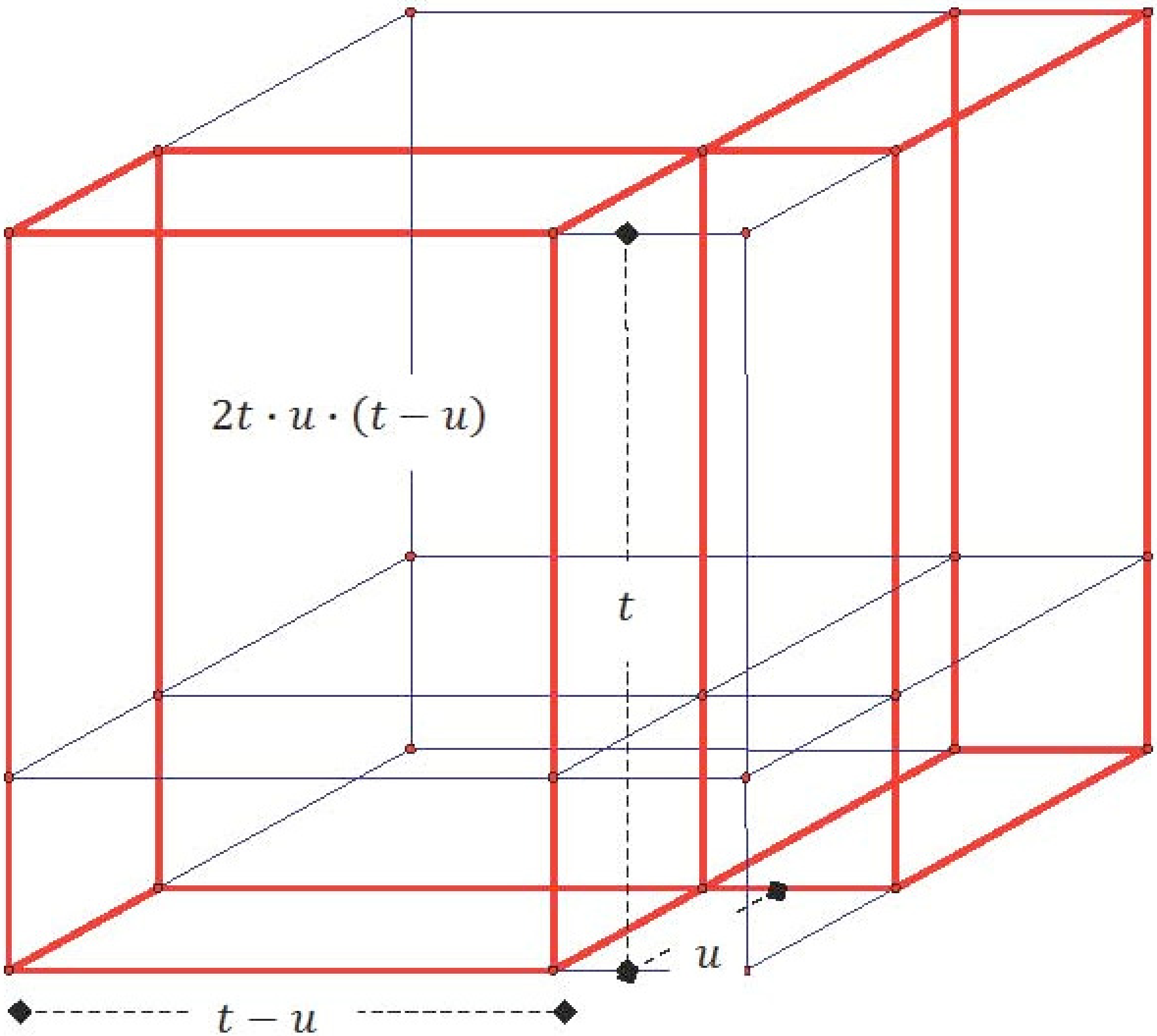
Small Cube



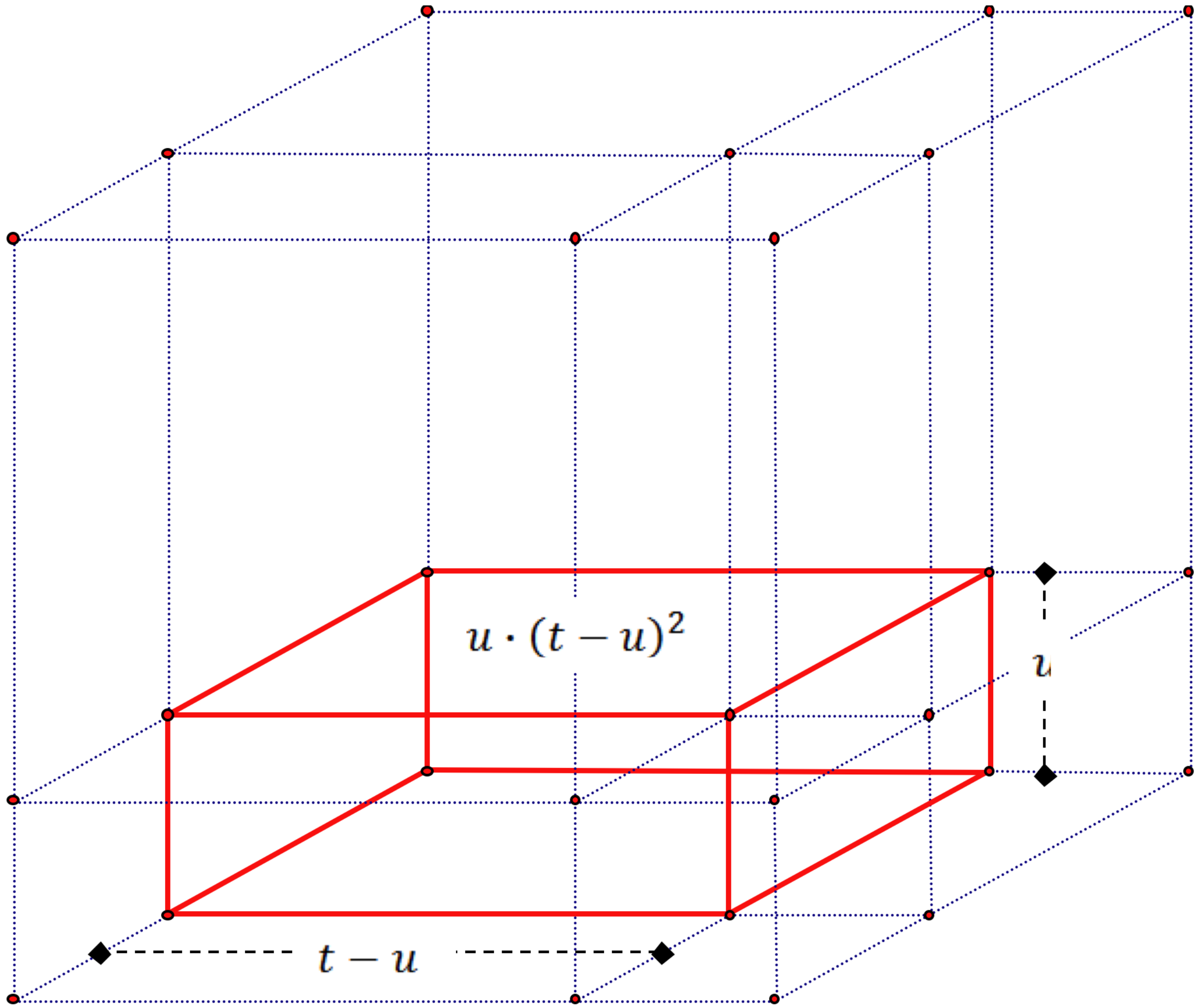
Tower



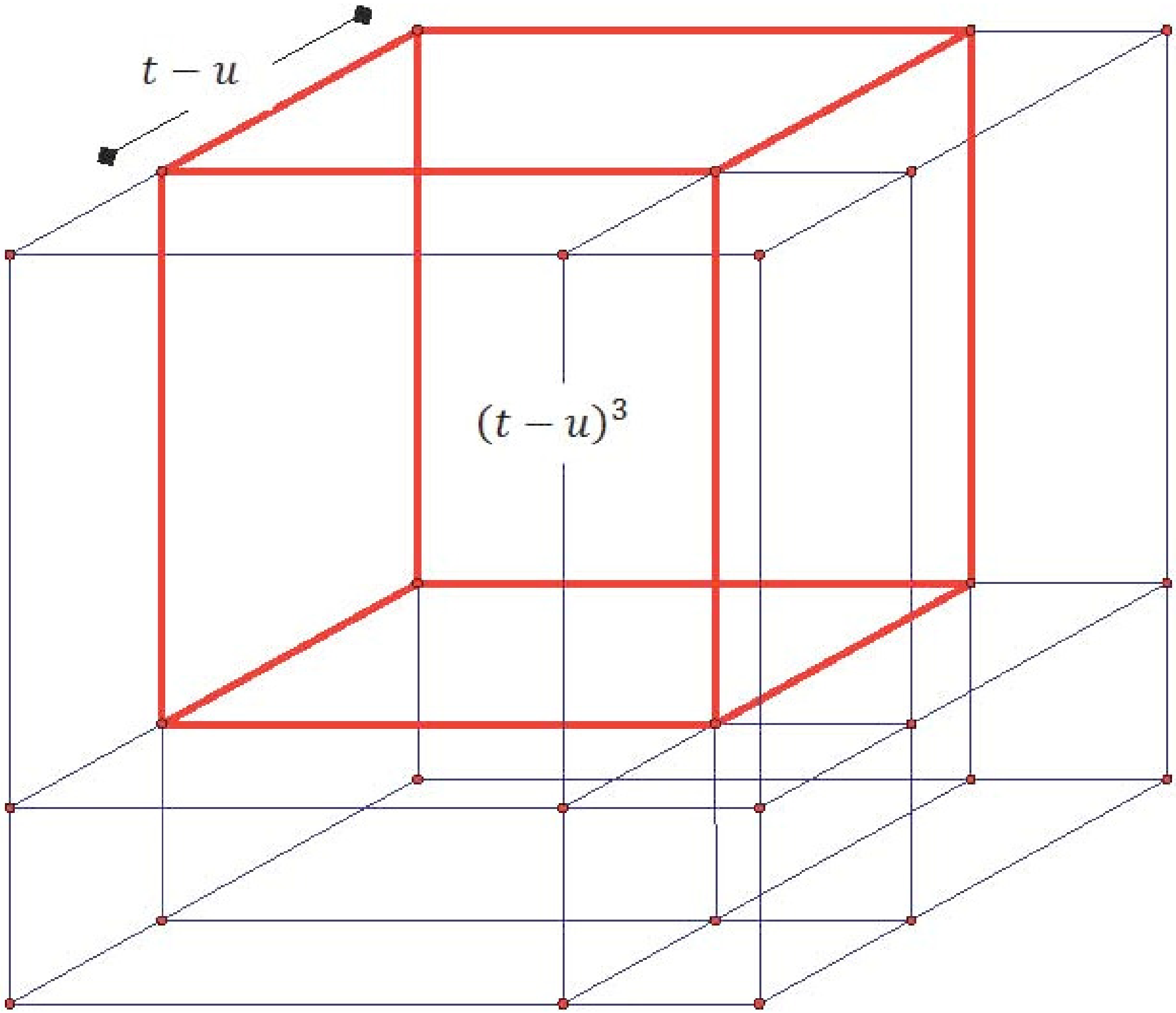
Two Slabs



Base



Big Cube



Tartaglia's Dissection of the Cube

Big cube	$(t - u)^3$
Sm. Cube	u^3
Tower	$u^2(t - u)$
Slab	$t\,u\,(t - u)$
Base	$u\,(t - u)^2$
TOTAL	t^3

So we have….

$$u^3 + u^2(t - u) + 2\,t\,u\,(t - u) + (t - u)^3 + u\,(t - u)^2 = t^3$$

or $(t - u)^3 + 3\,t\,u\,(t - u) = t^3 - u^3$.

Compare: $x^3 + m\,x = n$.

So by thinking of x as $t - u$ and choosing $m = 3\,t\,u$ and $n = t^3 - u^3$,

$$n = t^3 - (m/3t)^3$$

which unfolds into

$$t^6 - nt^3 - m^3/27 = 0.$$

Now by thinking of this equation as a quadratic in t^3, and with a little patience we, along with Tartaglia, will safely arrive at…

Cardano's Equation!

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$