CO380

Lecture 7

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Today's Lecture

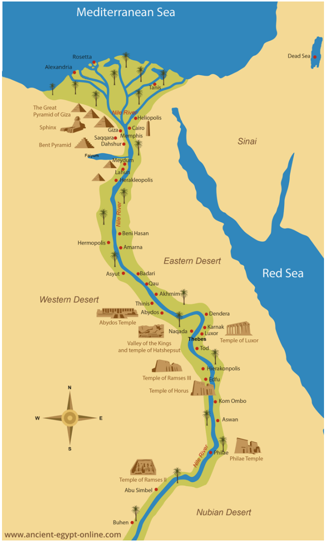
- The problem: The division of loaves
- The place: Antic Egypt
- Egyptian Fractions

The division of loaves

Share 9 loaves of bread amongst 10 men.

Egyptian Geography

- Egyptian civilzation established on the shores of the Nile about 5000 years ago;
- The yearly flood of the Nile brings sediments and nutriments that enrichs the surrounding soils in the valley.
- The Nile becomes a mean of transportation for both material and people and with the flood allowing for crop to grow, Egypt becomes a self-sufficient country (unlike other civilizations)



The Nile and the Practical Problems that Arise

- The Calendar Problem
  - Use of solar (days, seasons, and years) and lunar (months) calendar produces various calendars.
  - Precision is an issue:
    - One of the first calendar would alternate months of 29 and 30 days long (based on the lunar cycle), making for 354 days years. That would result to add a full month after three years.
    - A second calendar would have 12 months of 30 days (based on the sun), and the Egyptians would add 5 days at the end of the year.
- Need for mathematics to record information:
  - Quantities of cereals harvested;
  - Size of cultivated land taxes on those;
  - Work done by workers.

  It is this kind of practical problem that leads the Egyptians to develop a mathematical structure that will allow them to ensure a certain consistency.

The Papyrus

In order to ascertain the accuracy of the information gathered, the Egyptians used parchments, called papyrus, made from plants growing in the neighboring Nile. The majority of Egyptian mathematical knowledge comes from the study of these papyri. Obviously, by their vegetable nature, papyri are fragile documents and have been very poorly preserved. So there are very few resources that are still available today to study Egyptian mathematical techniques. Most of the mathematical information is contained in the following documents:

- Amhes (Rhind) Mathematical Papyrus
- Moscow Mathematical Papyrus

The Amhes Mathematical Papyrus

Figure: Amhes Mathematical Papyrus : Wikipedia

Egyptian Numerical Notation

- Hieroglyphs

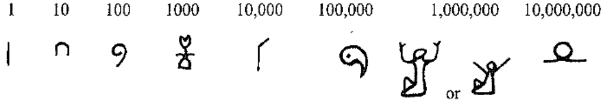


Figure: Hieroglyphs: Burton

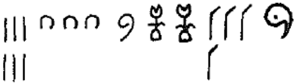


Figure: Representation of a number using hieroglyphs : Burton

Note the absence of a positionnal system.

Egyptian Numerical Notation

- Hieratic Numeration

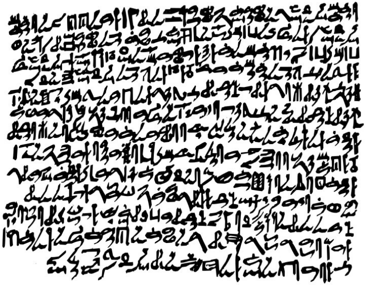


Figure: Hieratic Numeration

This is the notation used on the Amhes Mathematical Papyrus

Egyptian Numerical Notation

- Demotic Numeration

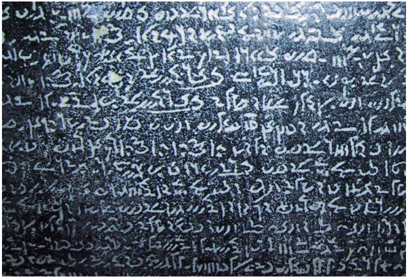


Figure: Demotic Numeration

Egyptian Arithmetic

- Addition

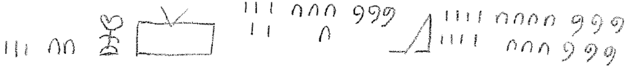


Figure: Addition

- Subtraction

Figure: Subtraction

Note : Subtraction uses the completion process. That is, to subtract 678-345 the Egyptian scribe would think of the problem: what must I add to 345 to get 678?

Egyptian Arithmetic

- Multiplication

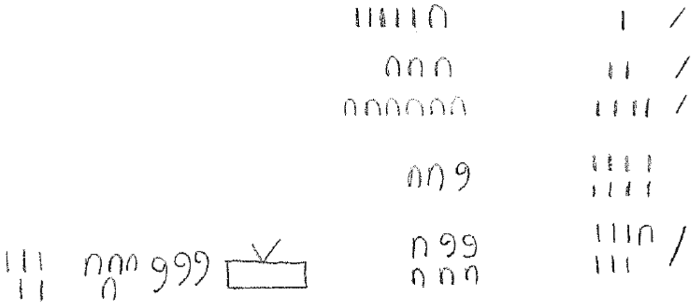


Figure: Multiplication using hieroglyphs

Egyptian Arithmetic

Questions to ask :

- What kind of process are they using?
  - Doubling and adding;
  - Very early use of a binary systrem;
- Would that kind of process always work ?
  - The greedy algorithm:

$$
\begin{aligned}
435 &= 256 + (435 - 256) = 2^8 + 179 \\
435 &= 2^8 + 128 + (179 - 128) = 2^8 + 2^7 + 51 \\
435 &= 2^8 + 2^7 + 32 + (51 - 32) = 2^8 + 2^7 + 2^5 + 19 \\
435 &= 2^8 + 2^7 + 2^5 + 16 + (19 - 16) = 2^8 + 2^7 + 2^5 + 2^4 + 3 \\
435 &= 2^8 + 2^7 + 2^5 + 2^4 + 2 + (3 - 2) = 2^8 + 2^7 + 2^5 + 2^4 + 2^1 + 1 \\
435 &= 2^8 + 2^7 + 2^5 + 16 + 2 + 1 = 2^8 + 2^7 + 2^5 + 2^4 + 2^1 + 2^0
\end{aligned}
$$

We could show that this process is finite and unique.

Egyptian Arithmetic

Show that every positive integer n can be written as a sum of distinct powers of two.

Proof:

Let $P(n)$ be the claim that n can be written as a sum of distinct powers of two. We show, using strong induction, that $P(n)$ is true for all positive integers n.

Base case:

Since $1 = 2^0$, then $P(1)$ is true.

Induction Hypothesis:

Assume for a positive integer k that $P(i)$ is true for all $1 \leq i \leq k$.

Egyptian Arithmetic

Proof (cont'd):

Show that every positive integer n can be written as a sum of distinct powers of two.

Inductive Conclusion:

We consider two cases based on the parity of $k+1$:

- If $k+1$ is even, then $\frac{k+1}{2}$ is an integer, and by the inductive hypothesis, we can express $\frac{k+1}{2}$ by a sum of distinct powers of two. We can then multiply this sum by 2, which simply increases the exponent of each power of two by 1, so this is again a sum of distinct powers of two that is equal to $k+1$.
- If $k+1$ is odd, we have that k is even. By the inductive hypothesis, we can express k as a sum of distinct powers of two. However, since k is even, the sum cannot contain $2^0 = 1$. Thus, we can add 2^0 to this sum, which remains a sum of distinct powers of two, and equals $k+1$.

Egyptian Arithmetic

- Division :
  The Egyptian division is only a variation of the multiplication, a little as we still do today. Then, if we ask to make $559 \div 13$, this is equivalent to asking: what is the integer k such that $k \times 13 = 559$.

$$
\begin{array}{cc}
1 & 13/ \\
2 & 26/ \\
4 & 52 \\
8 & 104/ \\
16 & 208 \\
32 & 416/ \\
\hline
43 & 559
\end{array}
$$

Thus $559 \div 13 = 43$

Rational numbers

The division of natural numbers without rest is a simple operation using the doubling and adding technique. However, when the division produces a remainder, fractions must then be introduced. Egyptian fractions have the peculiarity that they are all unit fractions, that is to say that the numerator is always 1. The exception to this rule is $\frac{2}{3}$ (and $\frac{3}{4}$ in some texts).

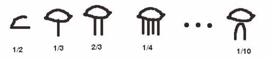


Figure: Symbols for Egyptian Fractions : MacTutor

Subsequently, with the evolution of writing, a simple bar above the number was introduced. For example: $\frac{1}{13} = \overline{13}$.

Horus Eye

- Horus is the son of Osiris, king of the gods.
- Osiris was killed by his brother Set.
- Horus tried to avenge his father, but in doing so he lost an eye that was restored to him by Thoth, the god of wisdom.
- Horus, instead decided to give his eye to his father, who, following his assassination, became the god of hell so that his father could "keep" an eye on the world of the living.
- The lost and restored eye of Horus represents the moon and its other eye, the sun.

Horus Eye and Fractions

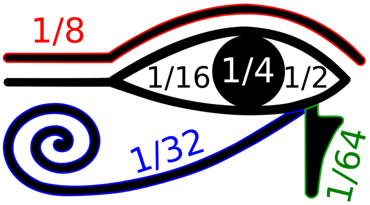


Figure: Horus Eye

- Unit fractions
- Idea of geometric series
- Binary, again...
- The Egyptian measuring system for volume, the *hekat*, uses division that are using unit fractions with denominators being powers of 2 from 2 to 64.

Representing a rational number as a sum of unit fractions:

Represent $\frac{7}{11}$ as a sum of unit fractions. To do so, we will use the fact that if a, b, c are all positive integers such that $a < b < c$, then $\frac{1}{a} > \frac{1}{b} > \frac{1}{c}$. We will use, once again, the greedy algorithm. First we need to find the largest unit fraction that can fit in $\frac{7}{11}$. Since $1 < \frac{11}{7} < 2$, then $1 > \frac{7}{11} > \frac{1}{2}$. So the first unit fraction is $\frac{1}{2}$. We then subtract :

$$\frac{7}{11} - \frac{1}{2} = \frac{14 - 11}{22} = \frac{3}{22}$$

Thus :

$$\frac{7}{11} = \frac{1}{2} + \frac{3}{22}$$

We then repeat the process with $\frac{3}{22}$. $7 < \frac{22}{3} < 8$, then $\frac{1}{7} > \frac{3}{22} > \frac{1}{8}$ and $\frac{3}{22} - \frac{1}{8} = \frac{1}{88}$.
We conclude that :

$$\frac{7}{11} = \frac{1}{2} + \frac{1}{8} + \frac{1}{88}$$

Representing a rational number as a sum of unit fractions

- That method is credited to Fibonacci.
- We (I?) don't know exactly how the Egyptians were actually getting their unit fractions.
- It has been shown that every rational number smaller than 1 can be expressed as a sum of unit fractions.
- That sum is not unique. There are actually an infinite number of ways of writing a fraction as a sum of unit fractions (consider the identity $\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$).

Representing a rational number as a sum of unit fractions

Not a formal proof of Fibonacci's process:

Let $\frac{a}{b} < 1$, where a, b are positive integers.

1. $\frac{b}{a} = q + r$ where q is an integer such that $q < \frac{b}{a} < q + 1$. Thus $\frac{1}{q} > \frac{a}{b} > \frac{1}{q+1}$.
2. The first unit fraction is $\frac{1}{q+1}$.
3. The difference of $\frac{a}{b} - \frac{1}{q+1}$ is $\frac{a(q+1)-b}{b(q+1)}$.
4. Before the subtraction, the numerator was a and after it is $a(q+1) - b$. Now, note that the difference of both numerator is $a - a(q+1) - b = b - aq$. Using the fact that $q < \frac{b}{a}$, or $aq < b$, then $b - aq > 0$. That tells us that the numerator is still positive but decrease after each iteration. So this process can't go indefinitely and so the numerator will reach eventually reach 1.

Dividing with remainder

Ex 1: $27 \div 8$

1	$8/$
2	$16/$
$\overline{2}$	4
$\overline{4}$	$2/$
$\overline{8}$	$1/$
$1 + 2 + \overline{4} + \overline{8}$	27

$$\text{So } 27 \div 8 = 3 + \frac{1}{4} + \frac{1}{8}$$

Dividing with remainder

Ex 2: $27 \div 36$

$$
\begin{array}{cc}
1 & 36 \\
\bar{\bar{3}} & 24/ \\
\bar{3} & 12 \\
\bar{6} & 6 \\
\overline{12} & 3/ \\
\hline
\bar{\bar{3}} + \overline{12} & 27
\end{array}
$$

$$
\text{So } 27 \div 36 = \frac{2}{3} + \frac{1}{12}
$$

Dividing with remainder

Ex 3: $13 \div 18$

$$
\begin{array}{cc}
1 & 18 \\
\bar{\bar{3}} & 12/ \\
\overline{18} & 1/ \\
\hline
 & \\
\bar{\bar{3}} + \overline{18} & 13
\end{array}
$$

$$\text{So } 13 \div 18 = \frac{2}{3} + \frac{1}{18}$$

The $\frac{2}{N}$ table

The different papyrus have tables which have facilitated the work of the scribes. One of these tables is in the Rhind papyrus. Since the Egyptians used only unit fractions, with the exception of $\frac{2}{3}$, and their arithmetic was based mainly on doubling, fractions $\frac{2}{N}$ were frequently encountered. This table certainly has an advantage for Egyptian scribes in order to increase the efficiency of their calculations. It lists the summation of unit fractions for fractions for which the denominator varies from $N = 3$ to $N = 101$.

The $\frac{2}{N}$ table

2/3 = 1/2 + 1/6	2/5 = 1/3 + 1/15	2/7 = 1/4 + 1/28
2/9 = 1/6 + 1/18	2/11 = 1/6 + 1/66	2/13 = 1/8 + 1/52 + 1/104
2/15 = 1/10 + 1/30	2/17 = 1/12 + 1/51 + 1/68	2/19 = 1/12 + 1/76 + 1/114
2/21= 1/14 + 1/42	2/23 = 1/12 + 1/276	2/25 = 1/15 + 1/75
2/27 = 1/18 + 1/54	2/29 = 1/24 + 1/58 + 1/174 + 1/232	2/31 = 1/20 + 1/124 + 1/155
2/33 = 1/22 + 1/66	2/35 = 1/30 + 1/42	2/37 = 1/24 + 1/111 + 1/296
2/39 = 1/26 + 1/78	2/41 = 1/24 + 1/246 + 1/328	2/43 = 1/42 + 1/86 + 1/129 + 1/301
2/45 = 1/30 + 1/90	2/47 = 1/30 + 1/141 + 1/470	2/49 = 1/28 + 1/196
2/51 = 1/34 + 1/102	2/53 = 1/30 + 1/318 + 1/795	2/55 = 1/30 + 1/330
2/57 = 1/38 + 1/114	2/59 = 1/36 + 1/236 + 1/531	2/61 = 1/40 + 1/244 + 1/488 + 1/610
2/63 = 1/42 + 1/126	2/65 = 1/39 + 1/195	2/67 = 1/40 + 1/335 + 1/536
2/69 = 1/46 + 1/138	2/71 = 1/40 + 1/568 + 1/710	2/73 = 1/60 + 1/219 + 1/292 + 1/365
2/75 = 1/50 + 1/150	2/77 = 1/44 + 1/308	2/79 = 1/60 + 1/237 + 1/316 + 1/790
2/81 = 1/54 + 1/162	2/83 = 1/60 + 1/332 + 1/415 + 1/498	2/85 = 1/51 + 1/255
2/87 = 1/58 + 1/174	2/89 = 1/60 + 1/356 + 1/534 + 1/890	2/91 = 1/70 + 1/130
2/93 = 1/62 + 1/186	2/95 = 1/60 + 1/380 + 1/570	2/97 = 1/56 + 1/679 + 1/776
2/99 = 1/66 + 1/198	2/101 = 1/101 + 1/202 + 1/303 + 1/606	

Figure: The $\frac{2}{N}$ table : Wikipedia

Representing a rational number as a sum of unit fractions

A few notes:

- The study of the various Egyptian documents makes it possible to understand how the scribes succeeded in performing different calculations.
- It is nevertheless true that certain aspects of their arithmetic remain unexplained. For example, several historians have tried to determine how the $2 \div N$ table was designed, and despite several assumptions, there does not seem to be any single method that led to the creation of this table.
- There are certain elements of constancy such as the one found in fractions of the form $\frac{2}{3k}$. Indeed, each of the decompositions is of the form: $\frac{2}{3k} = \frac{1}{2k} + \frac{1}{6k}$.
- It should be noted that not all divisions can be performed using any of these methods. Unfortunately, the lack of documents does not allow us to know how the Egyptians divided by 10.
- The heaviness of the calculations leading to the resolution of simple problems shows the inefficiency of the Egyptian fractional system in the context of mathematics as studied today. However, we must not forget that there may be other techniques used by the scribes that we can not witness today, since the majoirté of the documents did not survive the test of time.

Solution to the Distribution of Loaves Problem: AMP Problem 6

In Egyptian mathematics, always remembering that only unit fractions or $\frac{2}{3}$ are used, we find that $\frac{9}{10} = \frac{2}{3} + \frac{1}{5} + \frac{1}{30}$ portion of bread for each of the 10 people. So, the problem is to create 10 portions of $\frac{1}{30}$, which will take $\frac{1}{3}$ of bread, ten portions of $\frac{1}{5}$ Of bread, which will take two loaves. There are then six complete loaves and $\frac{2}{3}$ of another loaf. The last division is to divide the six complete loaves into $\frac{2}{3}$ and $\frac{1}{3}$ of bread. Finally, the distribution is as follows: Seven of the people will receive $\frac{2}{3} + \frac{1}{5} + \frac{1}{30}$ of bread and the other three will receive $\frac{1}{3}$ of bread in addition to a piece of $\frac{1}{5}$ of a bread and another of $\frac{1}{30}$ of a bread. Here is a representation of the solution:

Solution to the Distribution of Loaves Problem

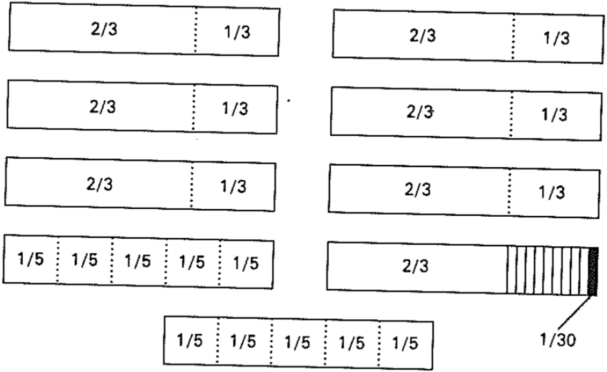


Figure: Distribution of Loaves Problem

Problem 33 of the AMP

The sum of a certain quantity together with its two thirds, its half, and its one-seventh becomes 37. What is the quantity?

Problem 33 of the AMP: The Solution

$$\begin{array}{cc}
1 & 1+\overline{\overline{3}}+\overline{2}+\overline{7} \\
2 & 4+\overline{3}+\overline{4}+\overline{28} \\
4 & 8+\overline{\overline{3}}+\overline{2}+\overline{14} \\
8 & 18+\overline{3}+\overline{7} \\
16 & 36+\overline{\overline{3}}+\overline{4}+\overline{28}
\end{array}$$

Applying $\overline{\overline{3}}, \overline{4}, \overline{28}$ to 42 we have :

$$\begin{array}{cc}
1 & 42 \\
/\overline{\overline{3}} & 28 \\
\overline{2} & 21 \\
/\overline{4} & 10+\overline{2} \\
/\overline{28} & 1+\overline{2}
\end{array}$$

Problem 33 of the AMP: The Solution

The total is 40; there remains 2, or $\overline{21}$ of 42. As $1+\overline{\overline{3}}+\overline{2}+\overline{7}$ applied to 42 gives 97, we shall have as a continuation of our first muliplication

$$\begin{array}{cll} \overline{97} & \overline{42} & \text{or 1 as part of 42} \\ /\overline{56}+\overline{679}+\overline{776} & \overline{21} & \text{or 2 as part of 42} \end{array}$$

This $\overline{21}$ with the product already obtained will make teh total 37.

Thus the required quantity is $16+\overline{56}+\overline{679}+\overline{776}$

Interesting Problems

a. Show that

$$\frac{2}{n} = \frac{1}{3}\left(\frac{1}{n}\right) + \frac{5}{3}\left(\frac{1}{n}\right)$$

whenever $5|n$.

b. If $7 \mid n$, conjecture a formula similar to part a. for fractions of the form $\frac{2}{n}$ and prove that your conjecture is correct.

Interesting Problems

c. Show that for all positive integers $n > 2$, $\frac{2}{n}$ can be written as a sum of two non-zero unit fractions with different denominators.

d. It can be shown in a similar way that fractions of the form $\frac{3}{n}$ can be expressed as a sum of three unit fractions with different denominators. However, not all fractions of the form $\frac{3}{n}$ can be expressed as a sum of two unit fractions with different denominators, $\frac{3}{7}$ is an example of that fact. Show that $\frac{3}{7}$ can't be expressed as a sum of two unit fractions with different denominators.