An Overview of Fluid Animation

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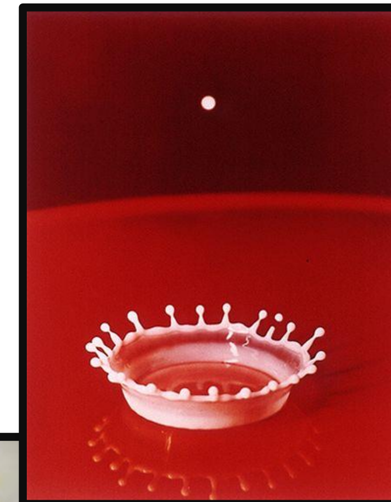
Jan 13, 2016

Reminders

- 1st round presentations start Monday. Graded on…
    1. Knowledge/coverage of technical concepts
    2. Organization
    3. Slide quality
    4. Speaking/presentation skills
- 1st round of paper reviews due Sunday, 5pm.
- Start thinking about project topics.
- Piazza (or email) with any questions.

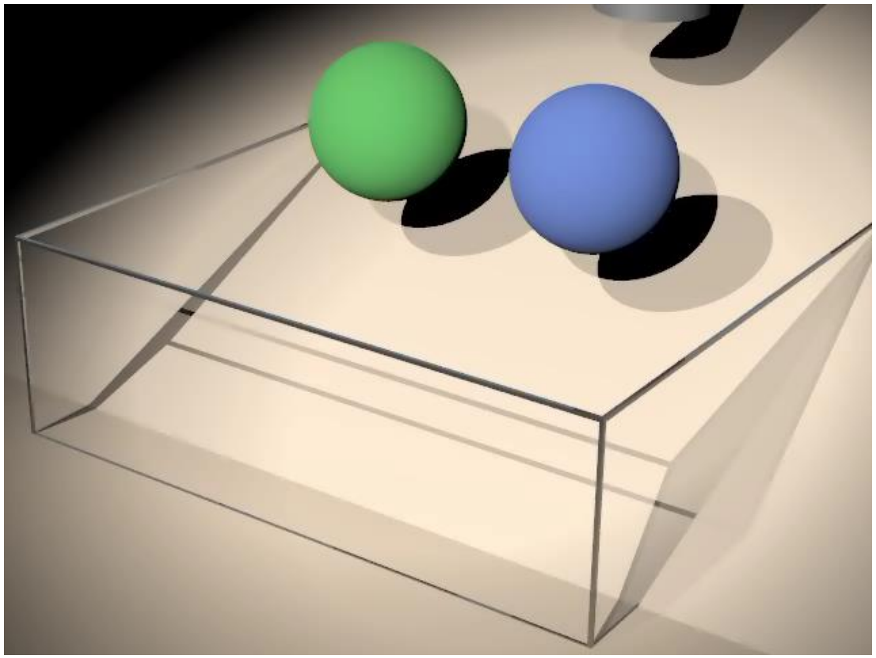
What distinguishes fluids?

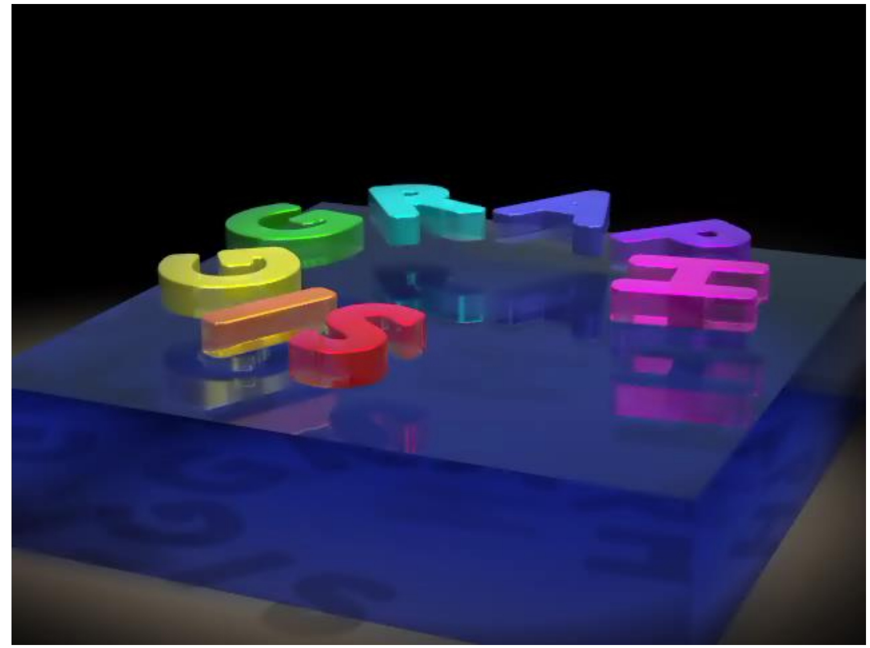


What distinguishes fluids?

- No “preferred” shape.
- Always flows when force is applied.
- Deforms to fit its container.
- Internal forces depend on *velocities*, not displacements/deformation (compare w/ elastic objects)

Examples

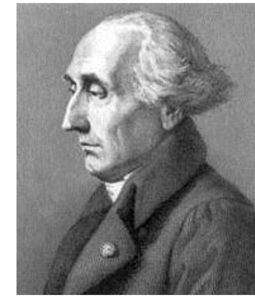




For further detail on today's material, see Robert Bridson's online fluid notes.
http://www.cs.ubc.ca/~rbridson/fluidsimulation/
(There's also a book, which is available in the library.)

Basic Theory

Eulerian vs. Lagrangian

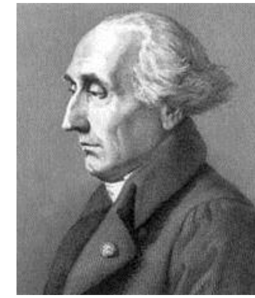


Lagrangian: Point of reference moves *with* the material.

Eulerian: Point of reference is *stationary.*

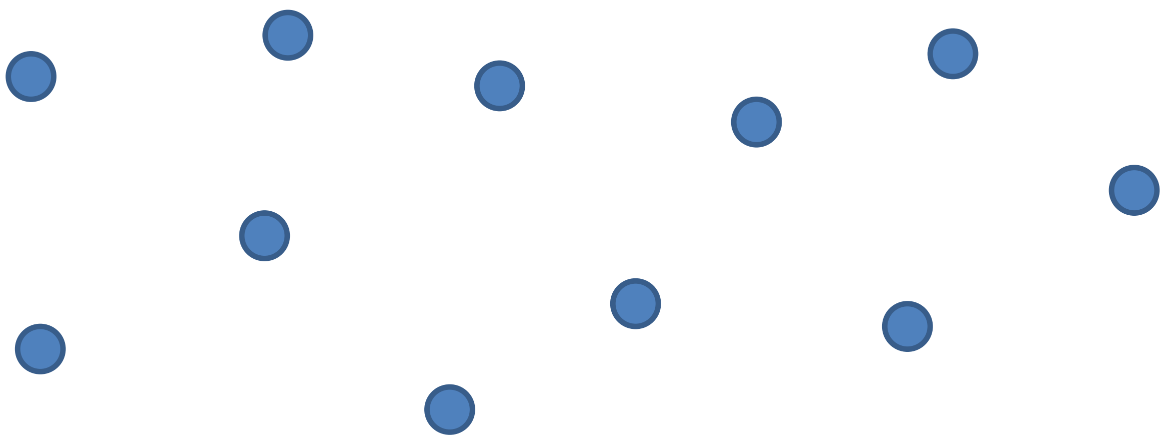
e.g. Weather balloon (Lagrangian) vs. weather station on the ground (Eulerian)

Eulerian vs. Lagrangian

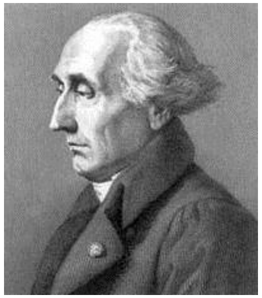


Consider an evolving scalar field (e.g., temperature).

Lagrangian view: Set of ***moving particles***, each with a temperature value.

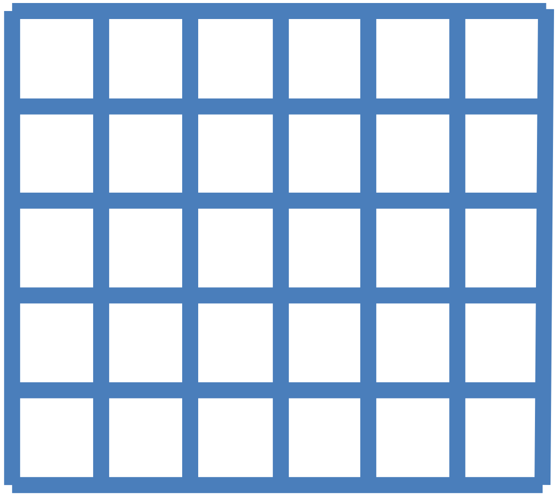


Eulerian vs. Lagrangian



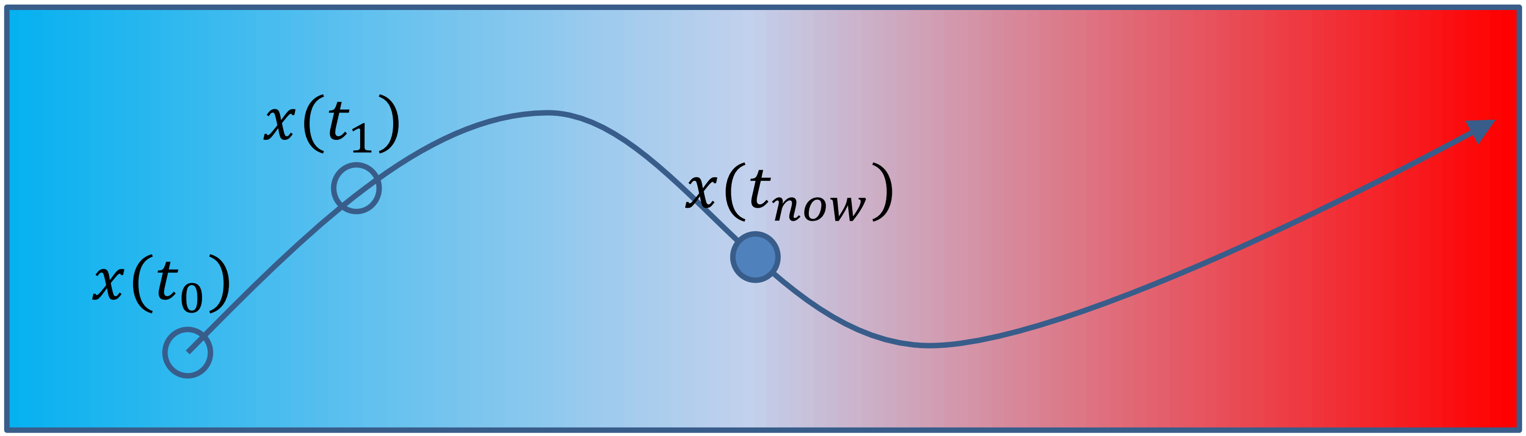
Consider an evolving scalar field (e.g., temperature).

Eulerian view: A fixed grid of temperature values, that temperature ***flows through.***



Relating Eulerian and Lagrangian

Consider the temperature $T(x,t)$ at a point following a given path, $x(t)$.



How can the temperature measured at $x(t)$ change?

1. There is a hot/cold "source" at the current point.
2. Following the path, the point moves to a cooler/warmer location.

Time derivatives

Mathematically:

$$\frac{D}{Dt}T(x(t),t) = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{\partial x}{\partial t}$$

Chain rule!

$$= \frac{\partial T}{\partial t} + \nabla T \cdot \frac{\partial x}{\partial t}$$

Definition of ∇

$$= \frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T$$

Choose $\frac{\partial x}{\partial t} = u$

Material Derivative

This is called the *material derivative*, and denoted $\frac{D}{Dt}$.

(AKA total derivative.)

Change at a point moving along the given path, x(t).

Change due to movement of the point.

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T$$

Change at the current (fixed) point.

Advection

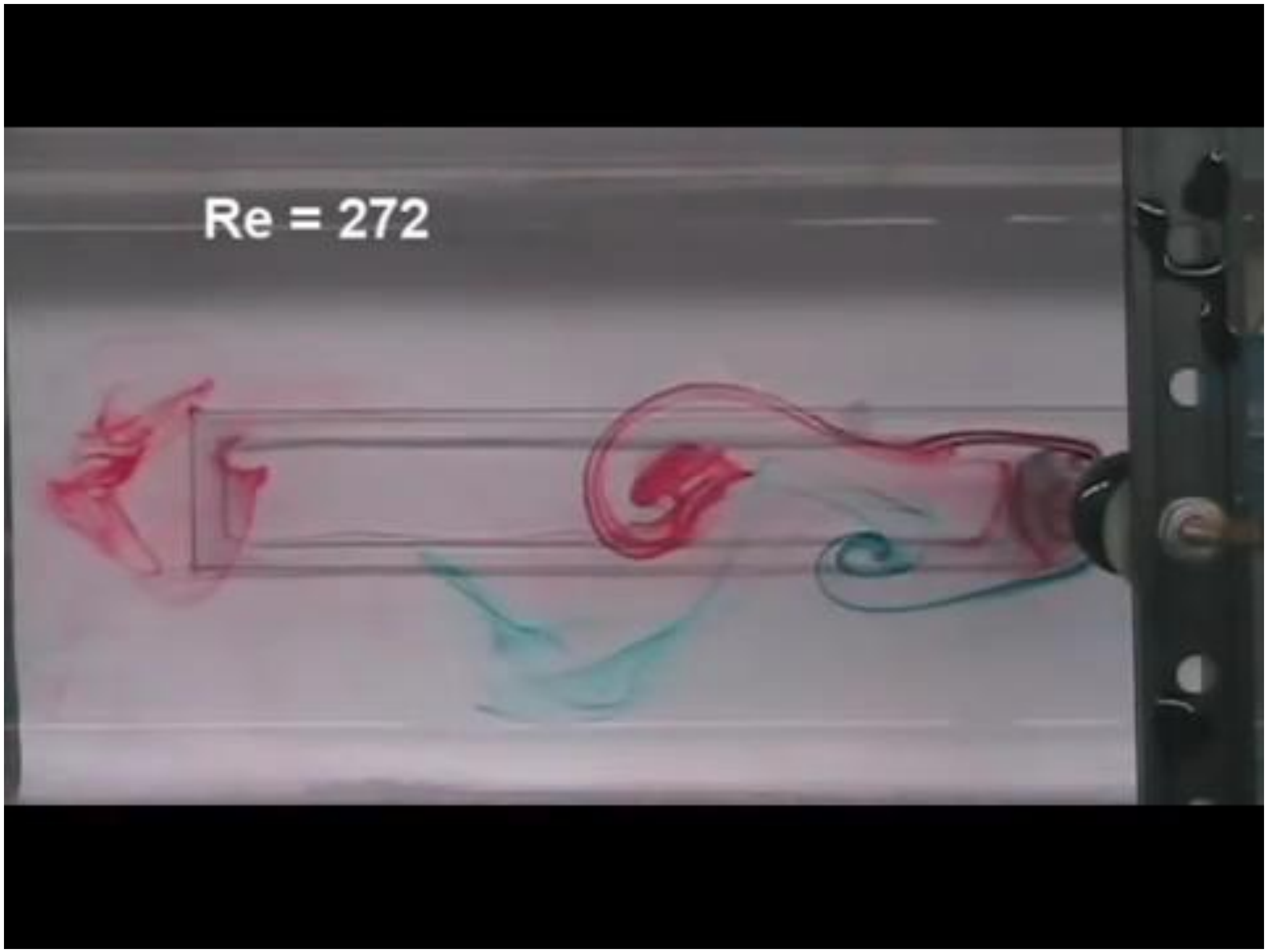
To track a quantity T moving (passively) through a velocity field:

$$\frac{DT}{Dt} = 0 \qquad \text{or equivalently} \qquad \frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = 0$$

This is the *advection equation*.

Think of colored dye or massless particles drifting around in fluid.

Advection

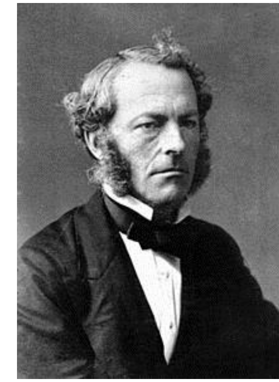


Equations of Motion

For general continuum materials, we essentially had Newton's second law: F = ma.

The *Navier-Stokes equations* are the same equations again, specialized to fluids.

Navier-Stokes



Density × Acceleration = Sum of Forces

$$\rho \frac{D\boldsymbol{u}}{Dt} = \sum_i \boldsymbol{F}_i$$

Expanding the material derivative…

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\rho(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \sum_i \boldsymbol{F}_i$$

What are the forces on a fluid?

Primarily:

- Pressure
- Viscosity
- Simple “external” forces
  – (e.g. gravity, buoyancy, user/artistic forces)

Also:

- Surface tension
- Coriolis
- Possibilities for more exotic fluid types:
  – Elasticity (e.g. silly putty)
  – Shear thickening / thinning (e.g. “oobleck”, ketchup, paints)
  – Electromagnetic forces: magnetohydrodynamics, ferrofluids, etc.
- Various others…

Exotic Fluids - Oobleck

Oobleck Simulation

Oobleck:
Viscoplastic v.s. Shear-Thickening

Simulation parameters (viscoplastic):
ρ: 1000.0 kg/m^3 κ: 109.0 kPa μ: 11.2 kPa σ_Y: 0.1 Pa η: 10.0 m: 1.0 σ_T: 1.0 η_p: 0.3
#points: 519171-534871 grid res.: 157×157×157 dt: 0.5×10^{-5}s subgrid geom. rem.: no

Simulation parameters (shear-thickening):
ρ: 1000.0 kg/m^3 κ: 109.0 kPa μ: 11.2 kPa σ_Y: 0.1 Pa η: 10.0 m: 2.8 σ_T: 1.0 η_p: 0.3
#points: 519171-529365 grid res.: 157×157×157 dt: 0.5×10^{-5}s subgrid geom. rem.: no

[Yue et al. 2015]

Exotic Fluids - Ferrofluid

Fluid equations of motion…

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\rho(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \sum_i \boldsymbol{F}_i$$

Change in velocity at a ***fixed point***

Advection (of *velocity*)

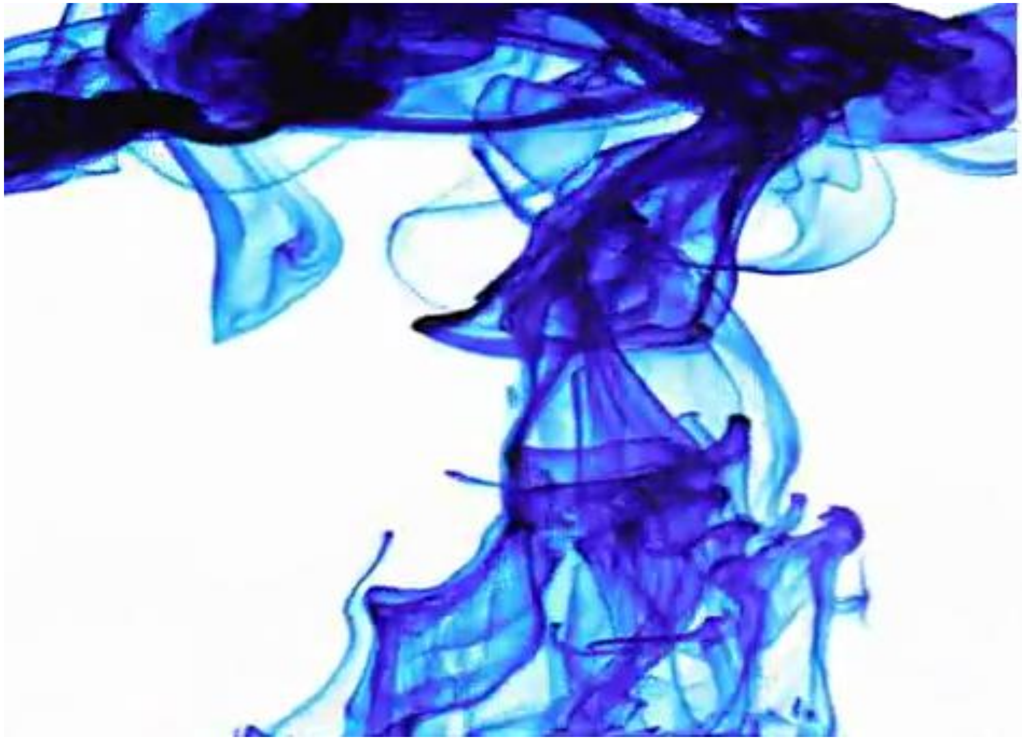
Forces (pressure, viscosity gravity,…)

Operator splitting

Break the full, nonlinear equation into sub-steps:

1. Advection: $\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\rho(\boldsymbol{u} \cdot \nabla \boldsymbol{u})$
2. Pressure: $\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{F}_{pressure}$
3. Viscosity: $\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{F}_{viscosity}$
4. External: $\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{F}_{other}$

1. Advection



Advection

We already considered advection of a passive scalar quantity, T, under velocity $\boldsymbol{u}$.

$$\frac{\partial T}{\partial t} = -\boldsymbol{u} \cdot \nabla T$$

In Navier-Stokes advection term, we have:

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{u} \cdot \nabla \boldsymbol{u}$$

Velocity $\boldsymbol{u}$ is advected (carried along) *by itself*, too!

Advection

That is, (u, v, w) components of velocity $\boldsymbol{u}$ are advected as separate scalars.

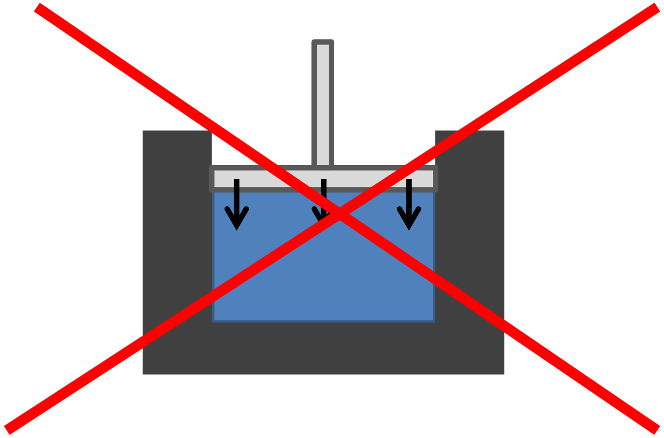
Can often reuse the same numerical method.

2. Pressure

Pressure

What does pressure do?

– Enforces *incompressibility* (fights compression).



Typical fluids (mostly) do not *visibly* compress.

- Exceptions: high velocity, high pressure, ...

Incompressibility

Compressible
velocity field

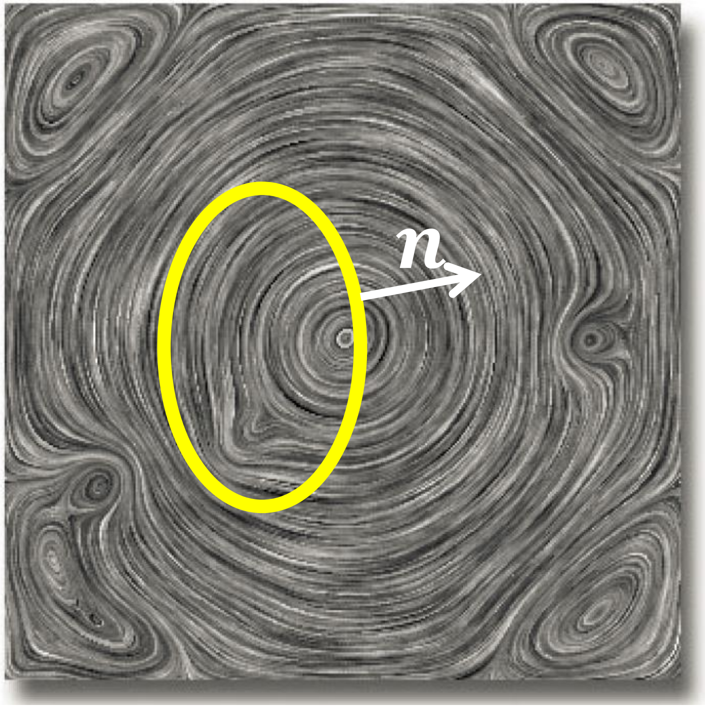
Incompressible
velocity field

Incompressibility

Intuitively, net flow into/out of a given region should be zero (no sinks/sources).

Integrate the net flow across the boundary of a closed region (yellow):

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} = 0$$



Incompressibility

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} = 0$$

By divergence theorem:

$$\iint_{\Omega} \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

But this is true for *any* region, so $\boldsymbol{\nabla} \cdot \boldsymbol{u} = \boldsymbol{0}$ everywhere.

Incompressibility implies $\boldsymbol{u}$ is *divergence-free*.

Pressure

Where does pressure come in?

- Pressure is the force needed to ensure the incompressibility constraint, $\nabla \cdot \boldsymbol{u} = 0$.
- Pressure force has the following form:

$$\boldsymbol{F}_p = -\nabla p$$

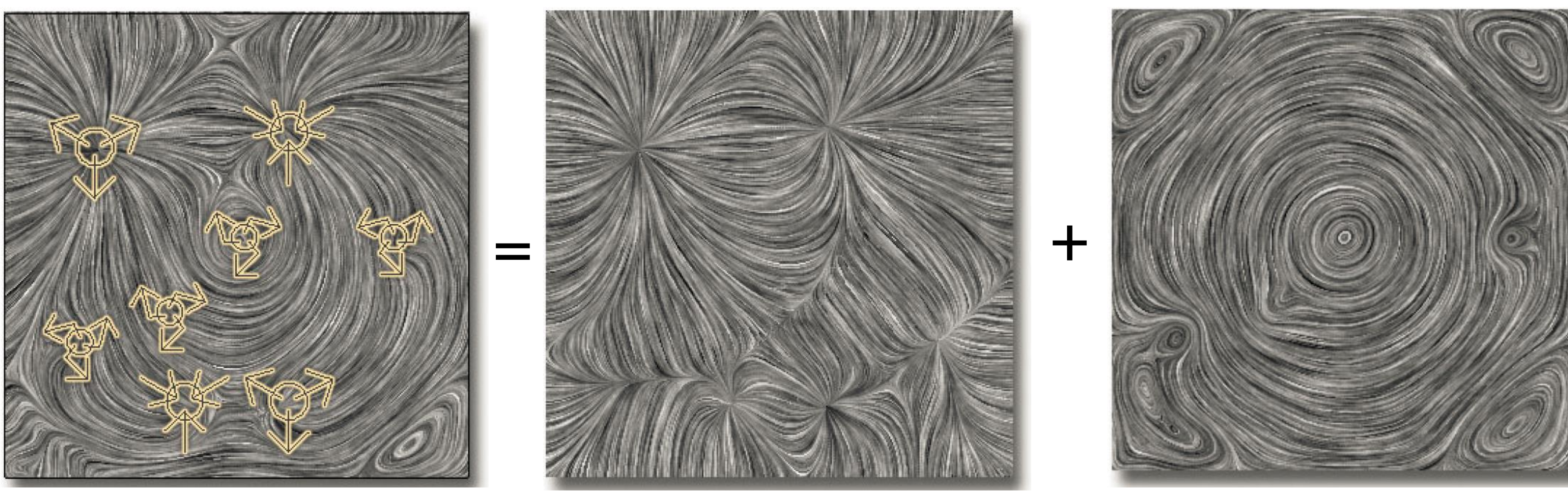
Let’s see why…

Helmholtz Decomposition

Input (Arbitrary)
Velocity Field

Curl-Free
(Irrotational)

Divergence-Free
(Incompressible)



$$u = \nabla p + \nabla \times \varphi$$

$$u_{old} = F_{pressure} + u_{new}$$

Aside: Pressure as Lagrange Multiplier

Interpret as an optimization:

Find the closest $\boldsymbol{u}_{new}$ *to* $\boldsymbol{u}_{old}$ *where* $\nabla \cdot \boldsymbol{u}_{new} = 0$

$$\underset{\boldsymbol{u}_{new}}{\operatorname{argmin}} \frac{\rho}{2} \left\| \boldsymbol{u}_{new} - \boldsymbol{u}_{old} \right\|^2$$
$$\text{subject to} \quad \nabla \cdot \boldsymbol{u}_{new} = 0$$

The *Lagrange multiplier* for the incompressibility constraint is the pressure.

3. Viscosity

High Speed Honey

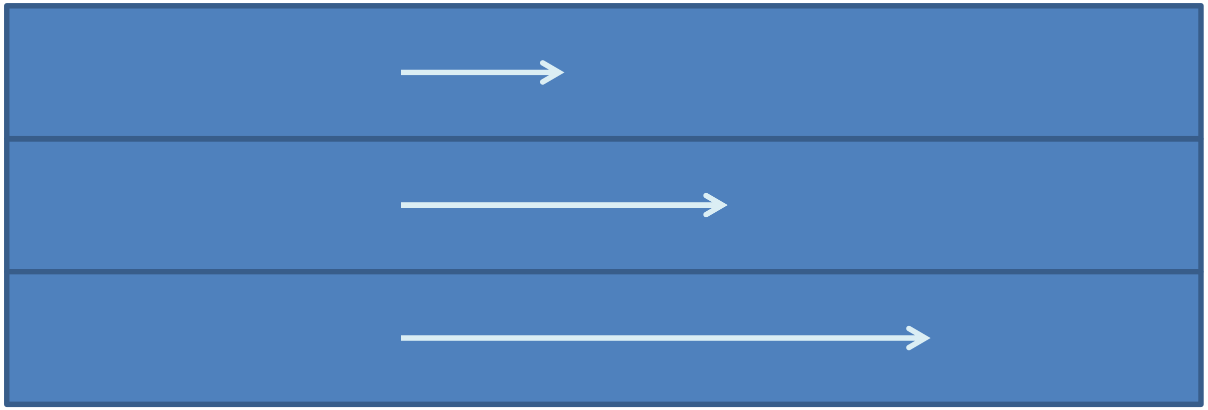
Viscosity

What characterizes a viscous liquid?

- “Thick”, highly *damped* behaviour.
- Strong resistance to flow.

Viscosity

Loss of energy due to internal friction between molecules moving at different velocities.

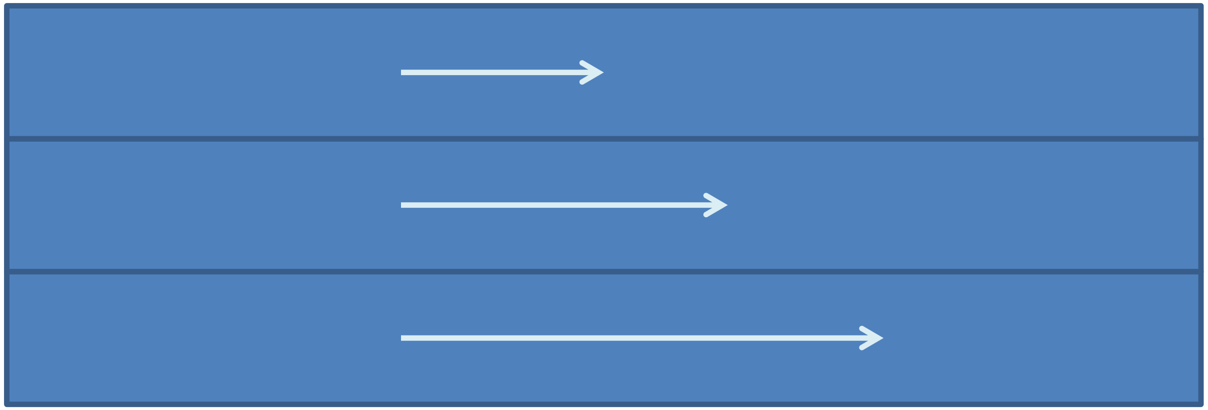


Interactions between molecules causes *shear stress* that…

- opposes *relative* motion.
- causes an exchange of momentum.

Viscosity

Loss of energy due to internal friction between molecules moving at different velocities.



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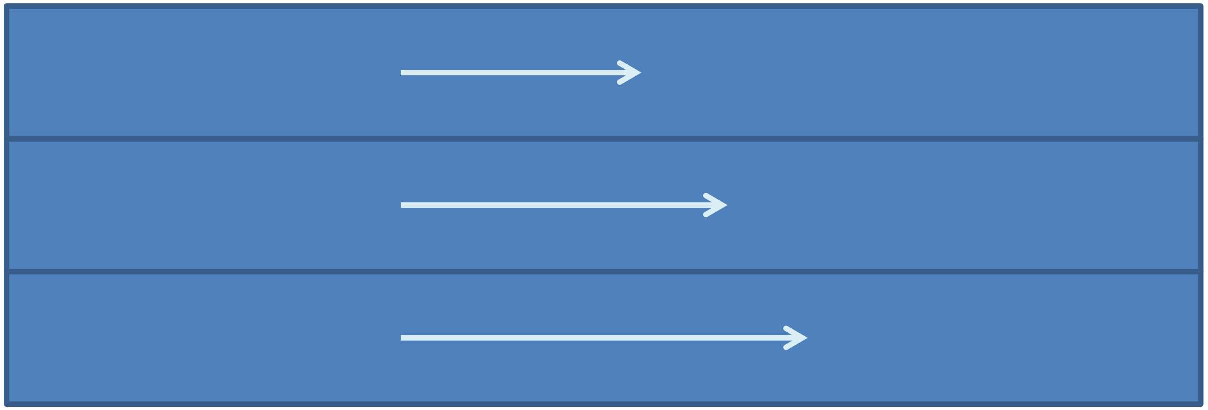

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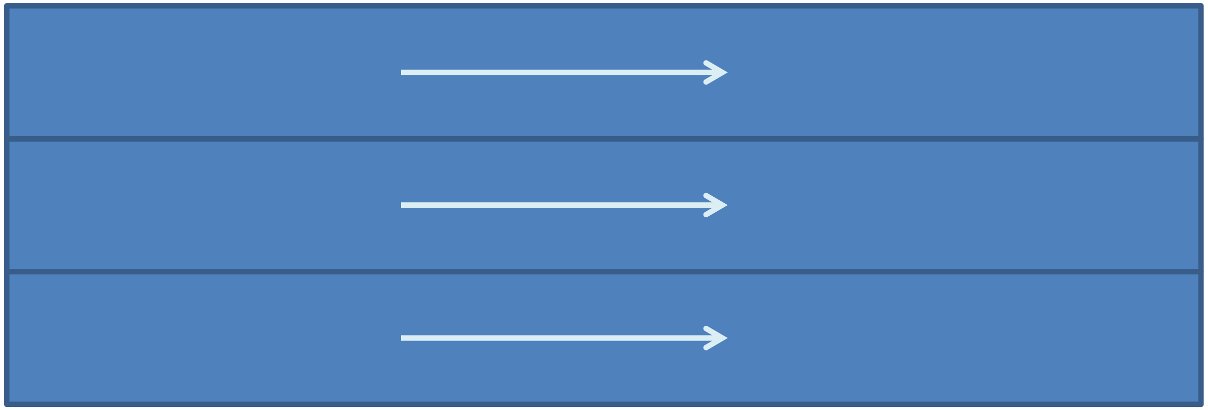

Interactions between molecules causes *shear stress* that…

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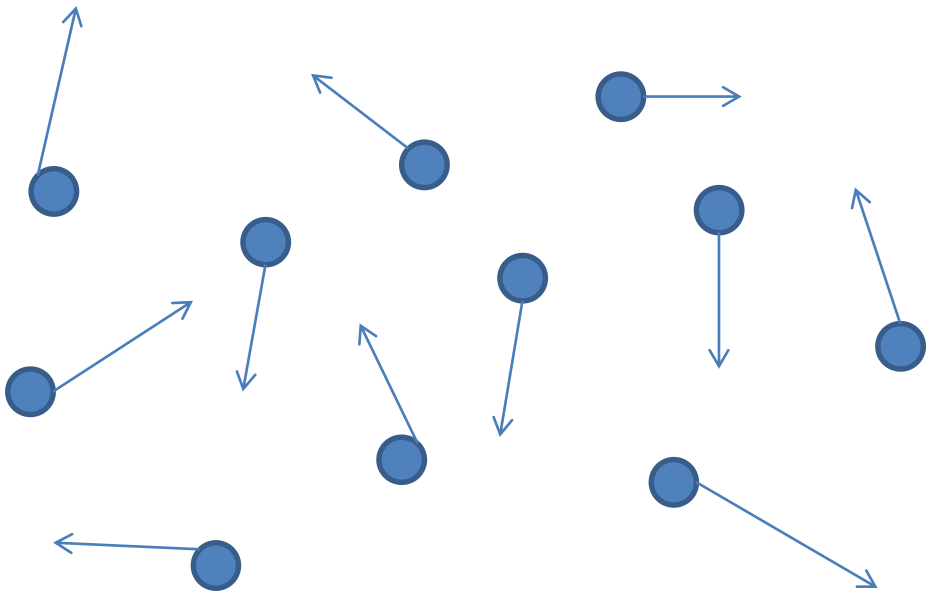
Viscosity

Imagine fluid particles with general velocities.



Each particle interacts with nearby neighbours, exchanging momentum. Velocities gradually tend towards uniformity.

Diffusion

The momentum exchange is related to:

- Velocity gradient, $\nabla \boldsymbol{u}$, in a region.
- Viscosity coefficient, μ.

Net effect is a smoothing or *diffusion* of the velocity over time.

Viscosity

Diffusion is typically modeled using the heat equation:

$$\frac{\partial T}{\partial t} = \alpha \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} T$$

(e.g. modeling dye or heat spreading through a region.)

Diffusion

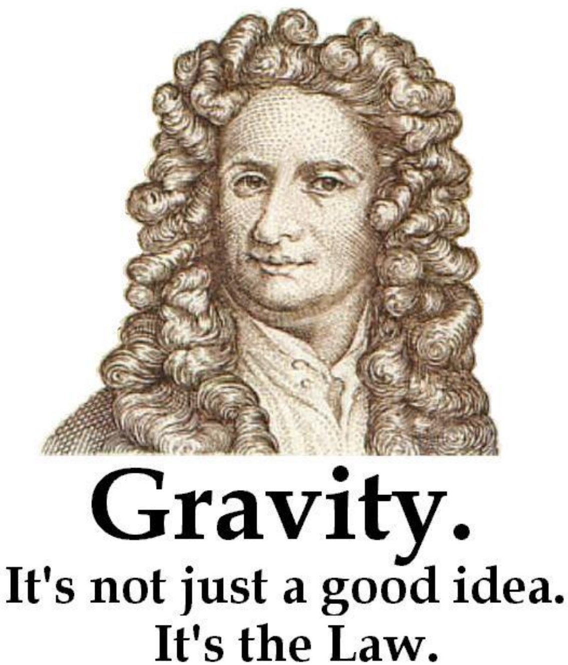
Viscosity

Diffusion applied to velocity gives our viscous force:

$$\boldsymbol{F}_{viscosity} = \rho \frac{\partial \boldsymbol{u}}{\partial t} = \mu \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \boldsymbol{u}$$

Usually, we can diffuse each scalar component of the vector field $\boldsymbol{u} = (u, v, w)$ separately.

4. External Forces



External Forces

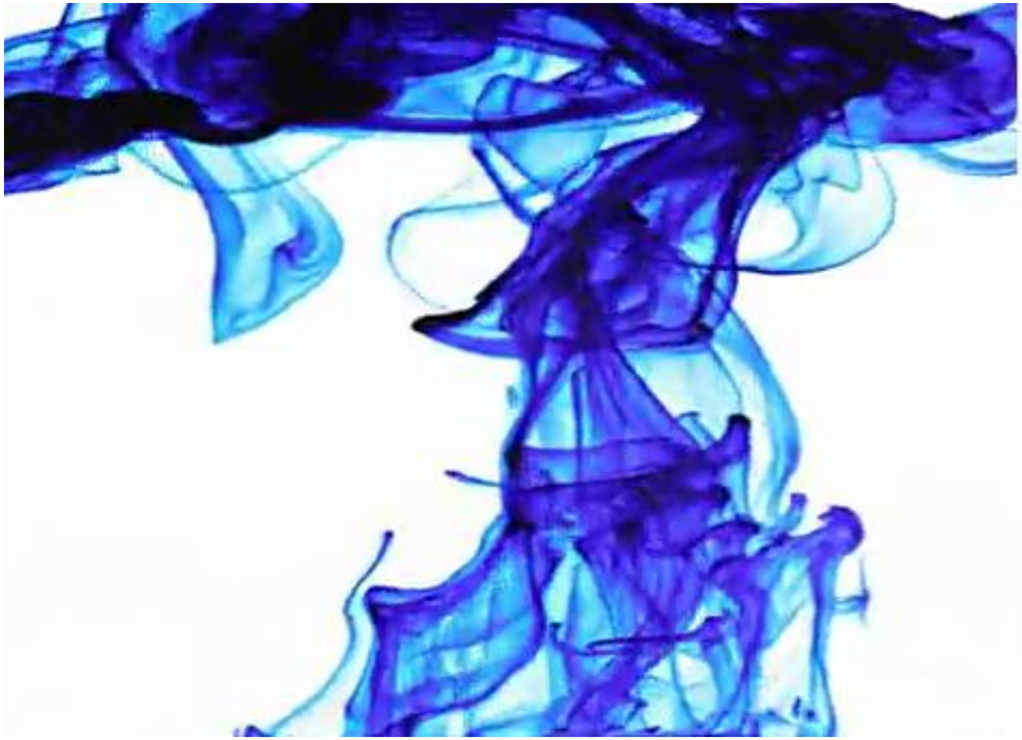
Any other forces you may want.

- Simplest is gravity:
  - $F_g = \rho \boldsymbol{g}$ for $\boldsymbol{g} = (0, -9.81, 0)$
- Buoyancy models are similar,
  - e.g., $F_b = \beta(T_{current} - T_{ref})\boldsymbol{g}$
- Artistic/user controls

e.g. Artistic control – Liquid Monster

Discretizing the Equations for Fluid Animation

1. Advection



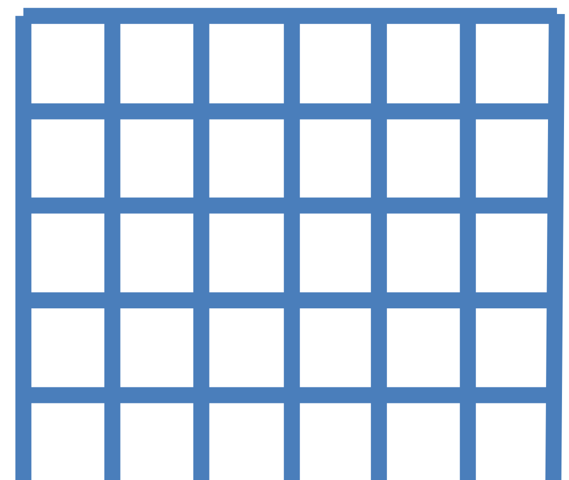
Advection of a Scalar

Consider advecting a quantity, φ

- temperature, color, smoke density, …

according to a (fixed) velocity field $\boldsymbol{u}$.

Allocate a grid (2D array) that stores scalar φ and velocity $\boldsymbol{u}$.



Eulerian

Approximate derivatives with *finite differences*.

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla \varphi = 0$$

FTCS = Forward Time, Centered Space:

$$\frac{{\varphi_i}^{n+1} - {\varphi_i}^{n}}{\Delta t} + u\frac{{\varphi_{i+1}}^{n} - {\varphi_{i-1}}^{n}}{2\Delta x} = 0$$

Unconditionally Unstable!

Lax:

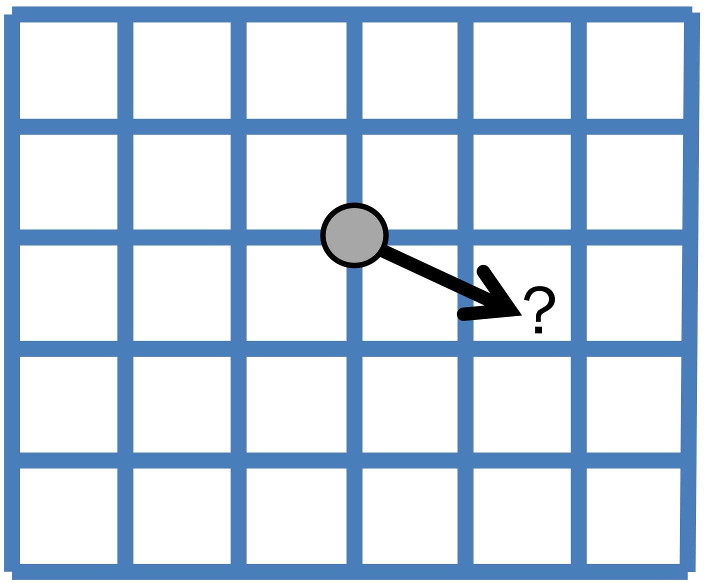
$$\frac{{\varphi_i}^{n+1} - ({\varphi_{i+1}}^{n} + {\varphi_{i-1}}^{n})/2}{\Delta t} + u\frac{{\varphi_{i+1}}^{n} - {\varphi_{i-1}}^{n}}{2\Delta x} = 0$$

Conditionally Stable!

Many possible methods, stability can be a challenge.

Lagrangian

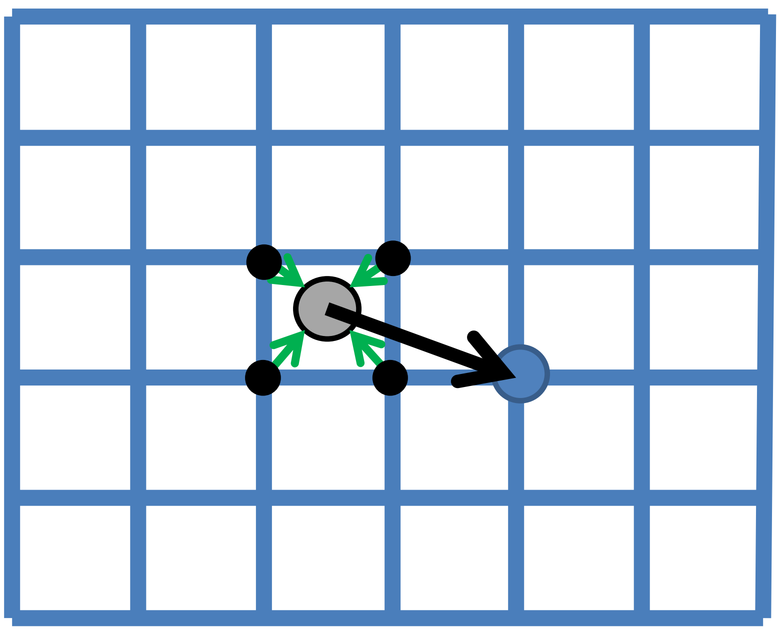
Advect data "forward" from grid points by integrating position according to grid velocity (e.g. forward Euler).



Problem: New data position doesn't necessarily land back on a grid point.

Semi-Lagrangian

- Look ***backwards*** in time from a grid point (blue), to see where its new data is coming ***from*** (black).
- Interpolate data at the previous time position.



Semi-Lagrangian - Details

1. Look up velocity $\boldsymbol{u}_{i,j}$ at grid point.
2. Integrate position for a timestep of $-\Delta t$ (FE).
   - e.g. $x_{back} = x_{i,j} - \Delta t \boldsymbol{u}_{i,j}$
3. (Bilinearly) Interpolate φ at x_{back}, call it φ_{back}.
4. Assign $\varphi_{i,j} = \varphi_{back}$ for the new time.

Unconditionally stable! (Why?)

(Though dissipative – loses energy over time.)

Advection of Velocity

This handles scalars. What about advecting velocity?

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{u} \cdot \nabla \boldsymbol{u}$$

Same method:

- Trace back with current velocity
- Interpolate velocity at that point
- Assign it to the grid point at the *new time*.

Caution: Do not overwrite the velocity field you’re using to trace back! (Make a copy.)

2. Pressure


80
40
psi
120
160
www.ProfessionalEquipment.com
PROFESSIONAL
EQUIPMENT

Recall… Helmholtz Decomposition

Input Velocity field

=

Curl-Free
(irrotational)

+

Divergence-Free
(incompressible)

$$u = \nabla p + \nabla \times \varphi$$

$$u_{old} = F_{pressure} + u_{new}$$

Pressure Projection - Derivation

$$(1)\ \rho \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p \quad \textit{and} \quad (2)\ \nabla \cdot \boldsymbol{u} = 0$$

Discretize (1) in time…

$$\boldsymbol{u}_{new} = \boldsymbol{u}_{old} - \frac{\Delta t}{\rho} \nabla p$$

Then plug into (2)…

$$\nabla \cdot \left(\boldsymbol{u}_{old} - \frac{\Delta t}{\rho} \nabla p \right) = 0$$

Pressure Projection

Implementation:

1) Solve a linear system of equations for *p:*

$$\frac{\Delta t}{\rho} \nabla \cdot \nabla p = \nabla \cdot \boldsymbol{u}_{old}$$

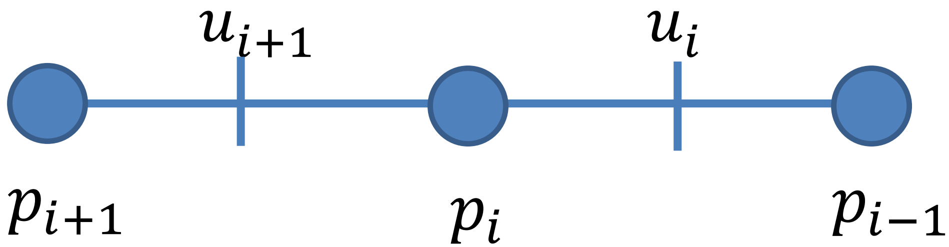
2) Given *p*, plug back in to update velocity:

$$\boldsymbol{u}_{new} = \boldsymbol{u}_{old} - \frac{\Delta t}{\rho} \nabla p$$

Implementation

$$\frac{\Delta t}{\rho} \nabla \cdot \nabla p = \nabla \cdot \boldsymbol{u}_{old}$$

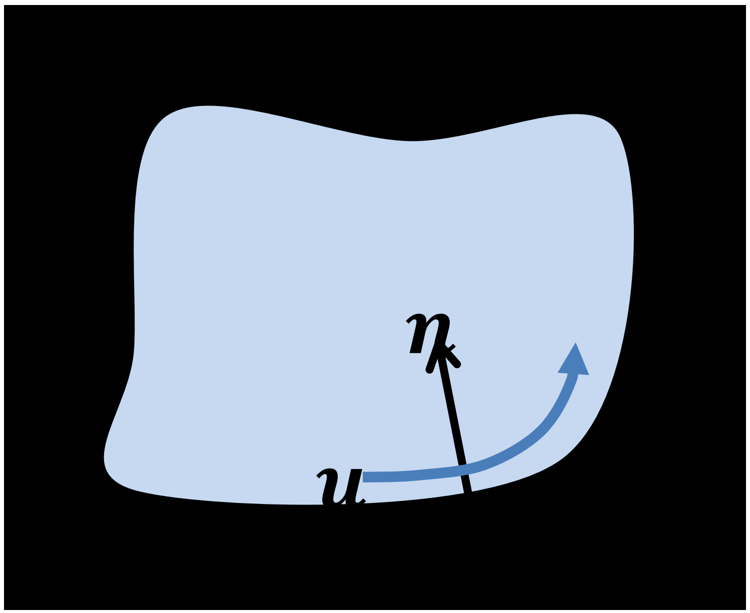
Discretize with finite differences (at staggered positions):



e.g., in 1D:

$$\frac{\Delta t}{\rho}\left(\frac{\frac{p_{i+1}-p_i}{\Delta x} - \frac{p_i - p_{i-1}}{\Delta x}}{\Delta x}\right) = \frac{u_{i+1}{}^{old} - u_i{}^{old}}{\Delta x}$$

Solid Boundary Conditions



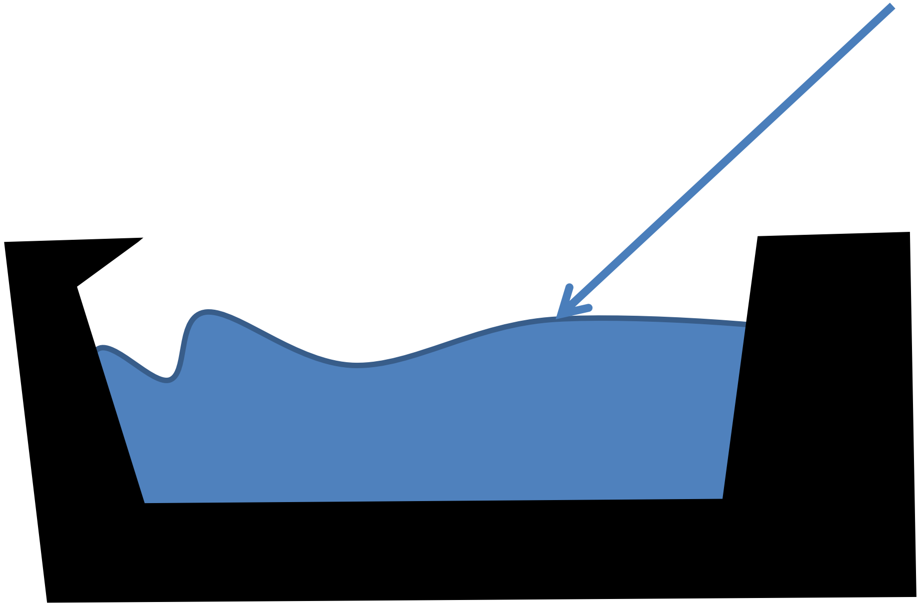
Free Slip:

$$\boldsymbol{u}_{new} \cdot \boldsymbol{n} = 0$$

i.e., Fluid cannot penetrate or flow out of the wall, but may slip along it.

Air ("Free surface") Boundary Conditions

Assume air (liquid exterior) is at some constant atmospheric pressure, $p = p_{atm}$ or $p = 0$.



3. Viscosity

Viscosity

PDE: $\rho \frac{\partial \boldsymbol{u}}{\partial t} = \mu \nabla \cdot \nabla \boldsymbol{u}$

Again, apply finite differences.

Discretized in time:

$$\boldsymbol{u}_{\boldsymbol{new}} = \boldsymbol{u}_{old} + \frac{\Delta t \mu}{\rho} \nabla \cdot \nabla \boldsymbol{u}_{*}$$

$\boldsymbol{u}_{\boldsymbol{old}}$ -> explicit time integration

$\boldsymbol{u}_{\boldsymbol{new}}$ -> implicit time integration

Viscosity – Time Integration

Explicit integration: $\boldsymbol{u}_{\boldsymbol{new}} = \boldsymbol{u}_{old} + \frac{\Delta t \mu}{\rho} \nabla \cdot \nabla \boldsymbol{u}_{old}$

– Compute $\frac{\Delta t \mu}{\rho} \nabla \cdot \nabla \boldsymbol{u}_{old}$ from current velocities.

– Add on to current $\boldsymbol{u}$.

– Quite unstable (stability restriction: $\Delta t \approx O(\Delta x^2)$)

Implicit integration: $\boldsymbol{u}_{\boldsymbol{new}} = \boldsymbol{u}_{old} + \frac{\Delta t \mu}{\rho} \nabla \cdot \nabla \boldsymbol{u}_{new}$

– Stable even for high viscosities, large steps.

– Must solve a system of equations.

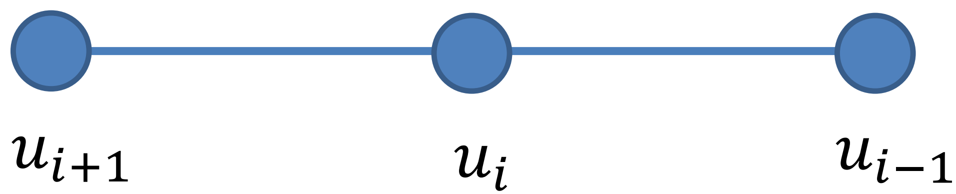
Viscosity – Implicit Integration

Solve for $\boldsymbol{u_{new}}$:

$$\boldsymbol{u_{new}} - \frac{\Delta t \mu}{\rho} \nabla \cdot \nabla \boldsymbol{u}_{new} = \boldsymbol{u}_{old}$$

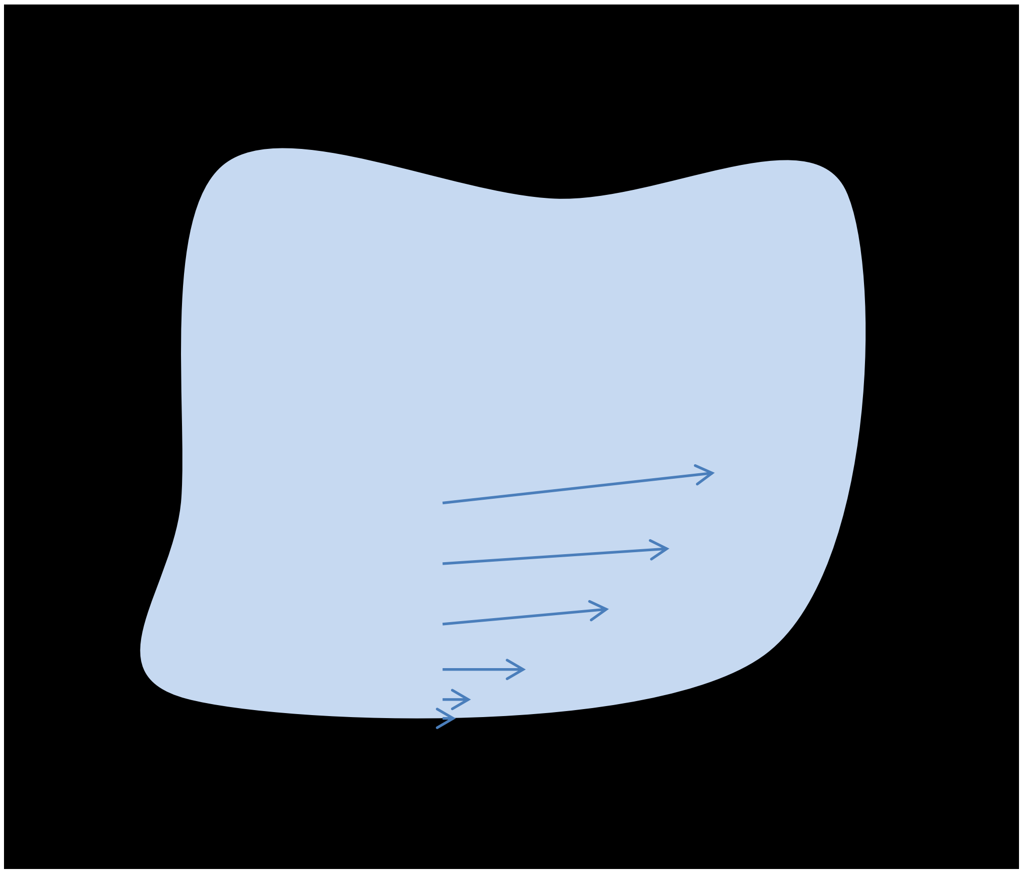
(Apply separately for each velocity component.)

e.g. in 1D:



$$u_i - \frac{\Delta t \mu}{\rho} \left(\frac{\frac{u_{i+1} - u_i}{\Delta x} - \frac{u_i - u_{i-1}}{\Delta x}}{\Delta x} \right) = u_i^{old}$$

Viscosity - Solid Boundary Conditions



No-Slip:

$\boldsymbol{u}_{new} = 0$

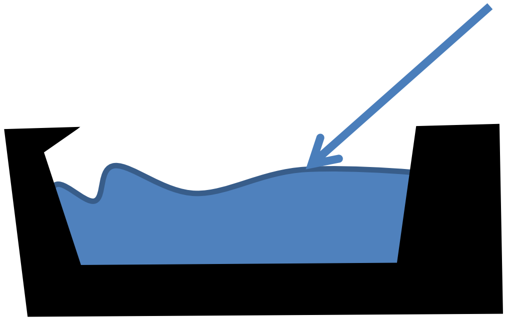
No-slip Condition

Viscosity - Free Surface Conditions

Treat air as negligible, so enforce zero momentum exchange between liquid and "air".

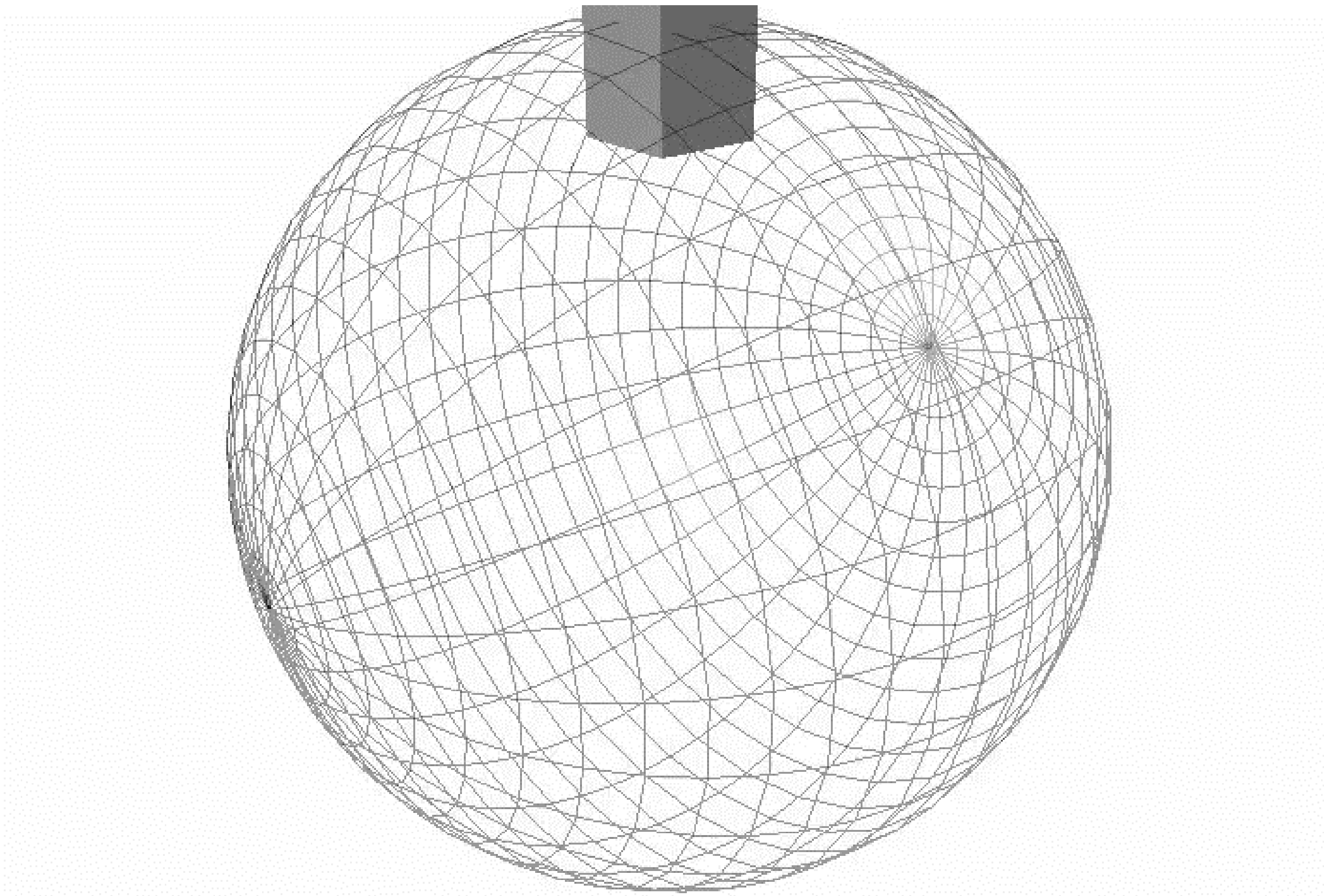
The proper conditions are quite involved:

$$\left(-p\mathbf{I} + \boldsymbol{\mu}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right)\right) \cdot \boldsymbol{n} = \mathbf{0}$$

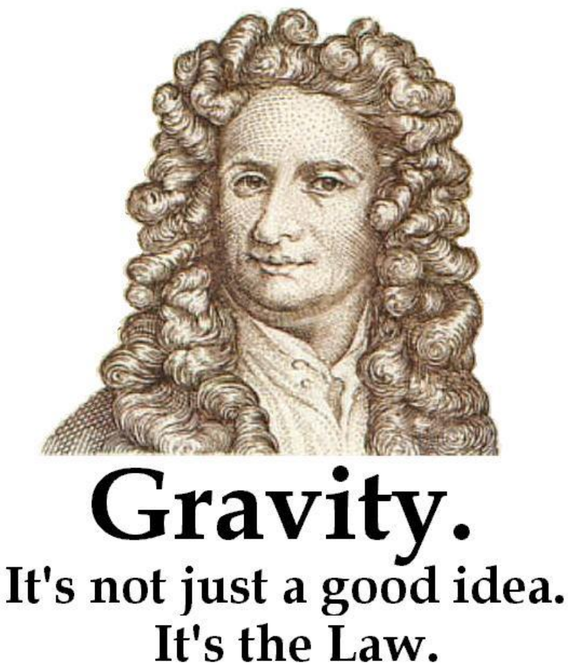


See [Batty & Bridson, 2008] for the standard solution in graphics. (Needed e.g., for honey coiling.)

Viscous coiling simulation



4. External Forces



Gravity

Discretized form is:

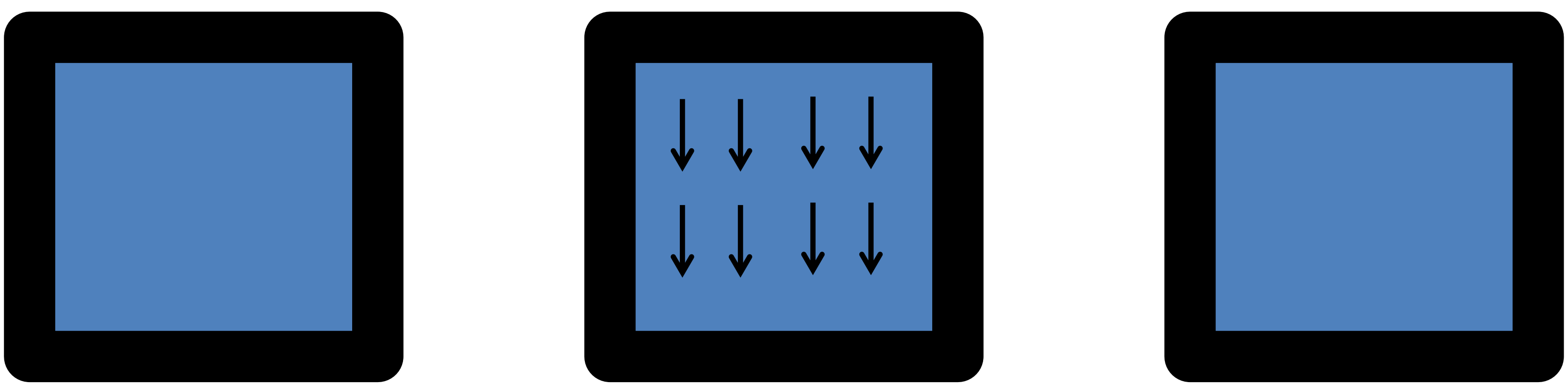
$$\boldsymbol{u}_{\boldsymbol{new}} = \boldsymbol{u}_{old} + \Delta t \boldsymbol{g}$$

Simply increment the vertical velocities at each step!

Gravity

Notice: in a closed fluid-filled container, gravity (alone) won't do anything!

– Incompressibility cancels it out. (Assuming constant density.)



Start

After gravity step

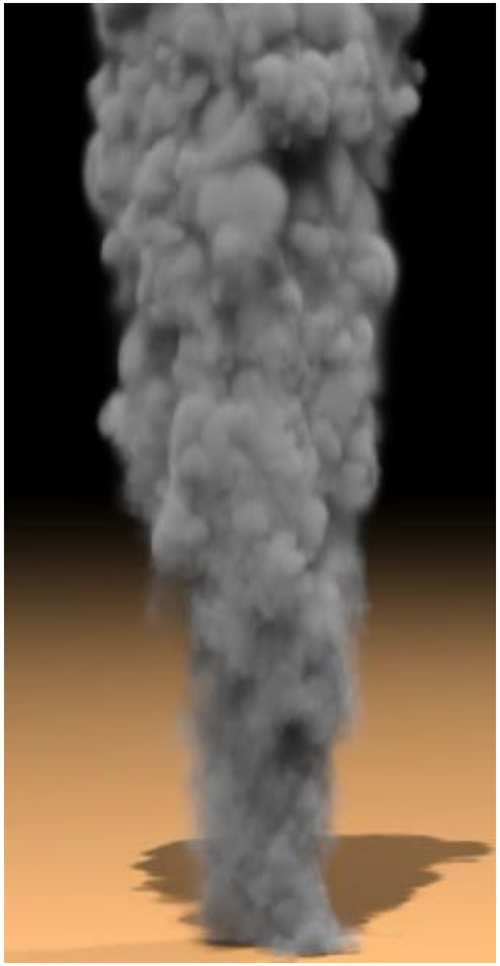
After pressure step

Simple Buoyancy

Track an extra scalar field T, representing local temperature, generated at a heat source.

Apply diffusion to T, and advect it along with the velocity field.

The difference between current and "reference" temperature induces a buoyancy force.



Simple Buoyancy

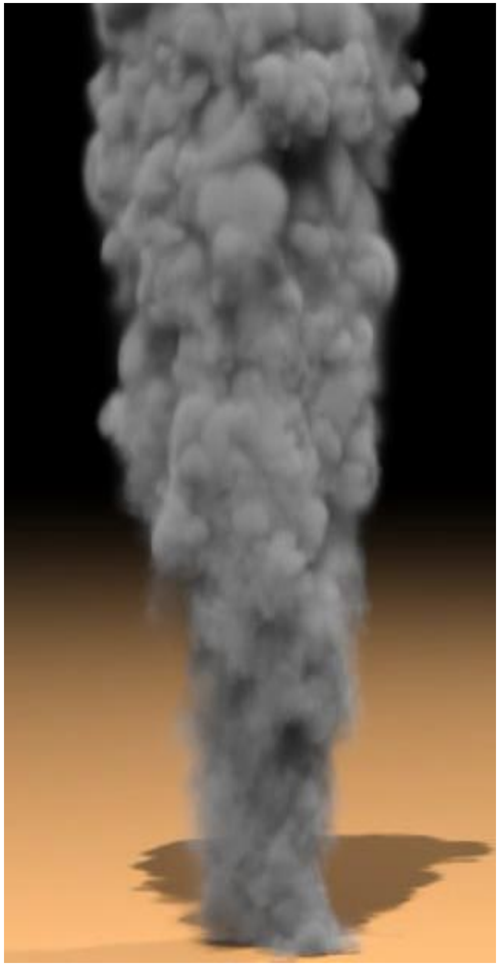
e.g.

$$\boldsymbol{u}_{\boldsymbol{new}} = \boldsymbol{u}_{old} + \Delta t \beta \left(T_{current} - T_{ref}\right) \boldsymbol{g}$$

β dictates the strength of the buoyancy force.

For an enhanced version of this:

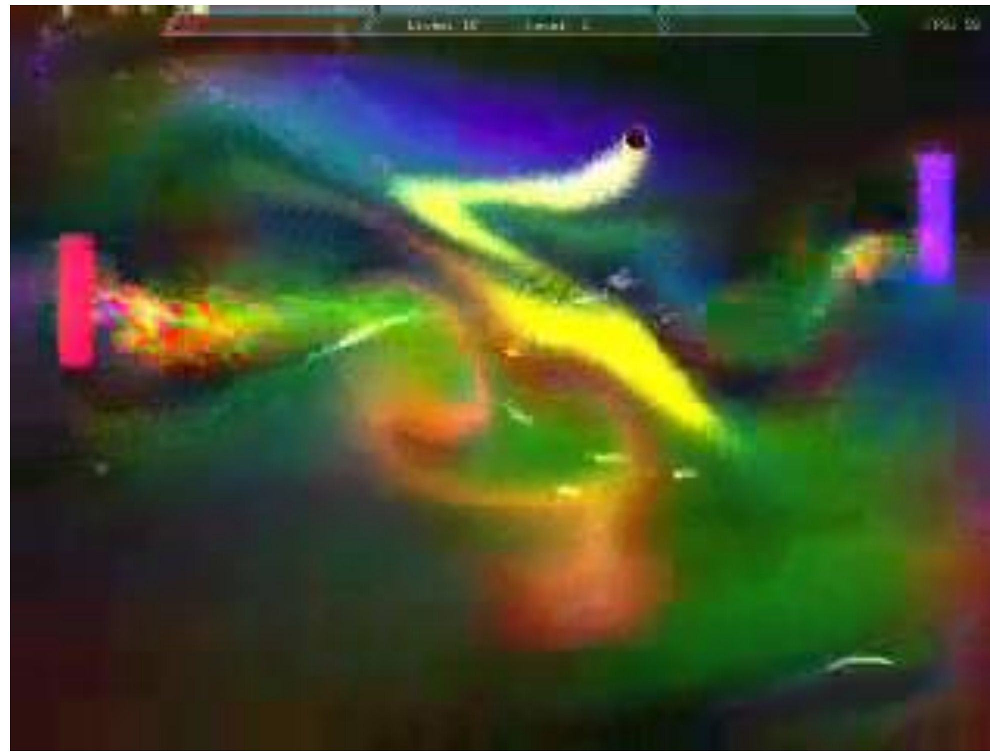
“Visual simulation of smoke”, [Stam et al., 2001].



User Forces

Add whatever additional forces we want:

- Wind forces near a mouse click.
- Paddle forces in *Plasma Pong*.



Plasma Pong game
(eventually taken down due to copyright claim by Atari)

Ordering of Steps

Order is important.

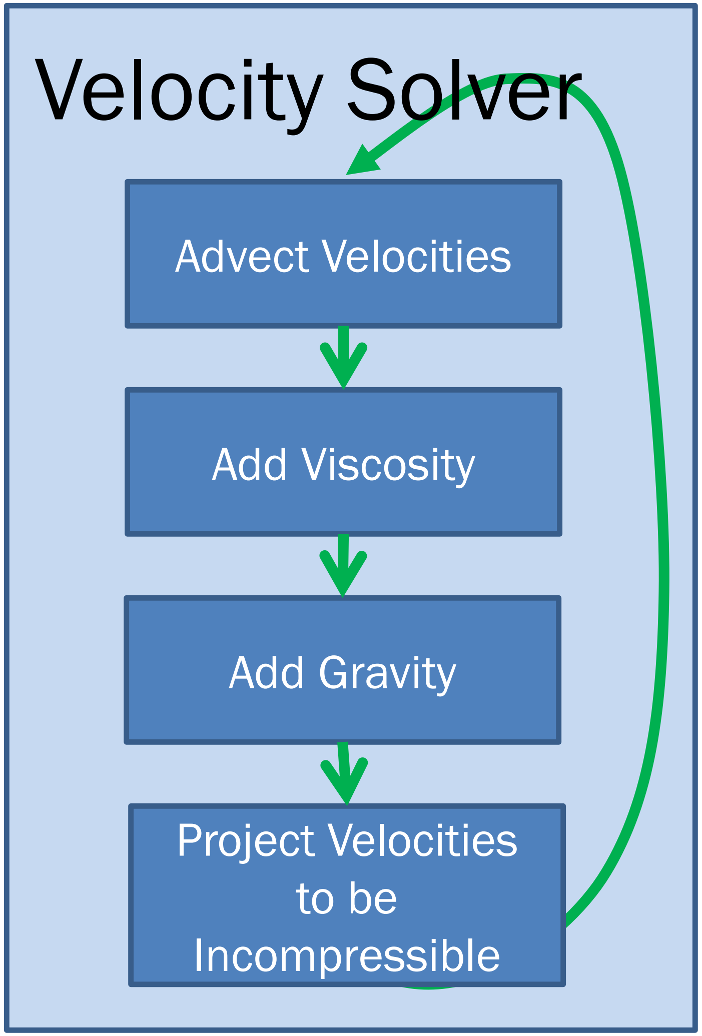
Why?

1) Incompressibility is not satisfied at intermediate steps.
2) Advecting with a divergent field causes volume/material loss or gain!

Ordering of Steps

For example, consider advection in this field:

The Big Picture



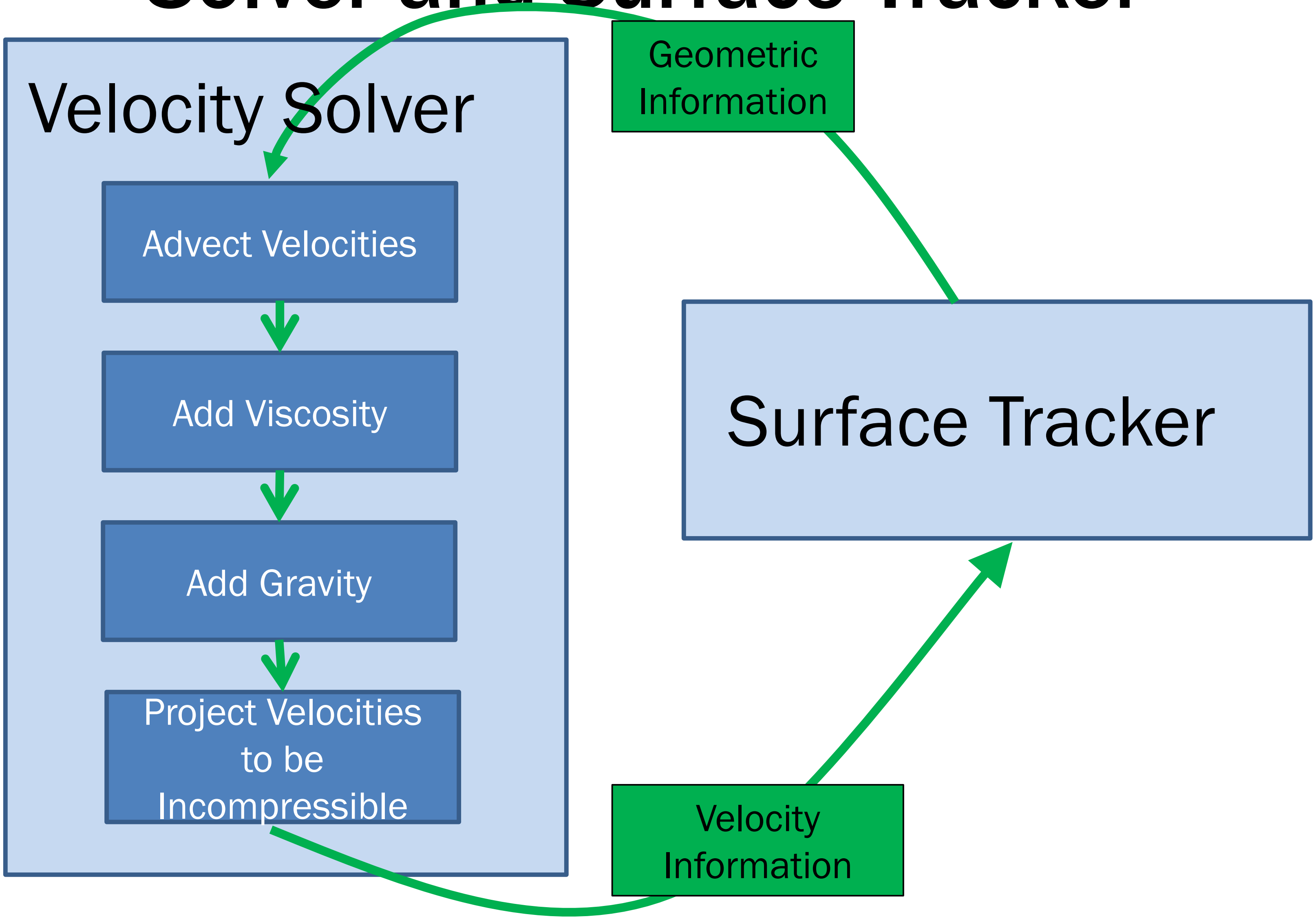
Liquids

Liquids

What's missing?

We still need a *surface representation.*

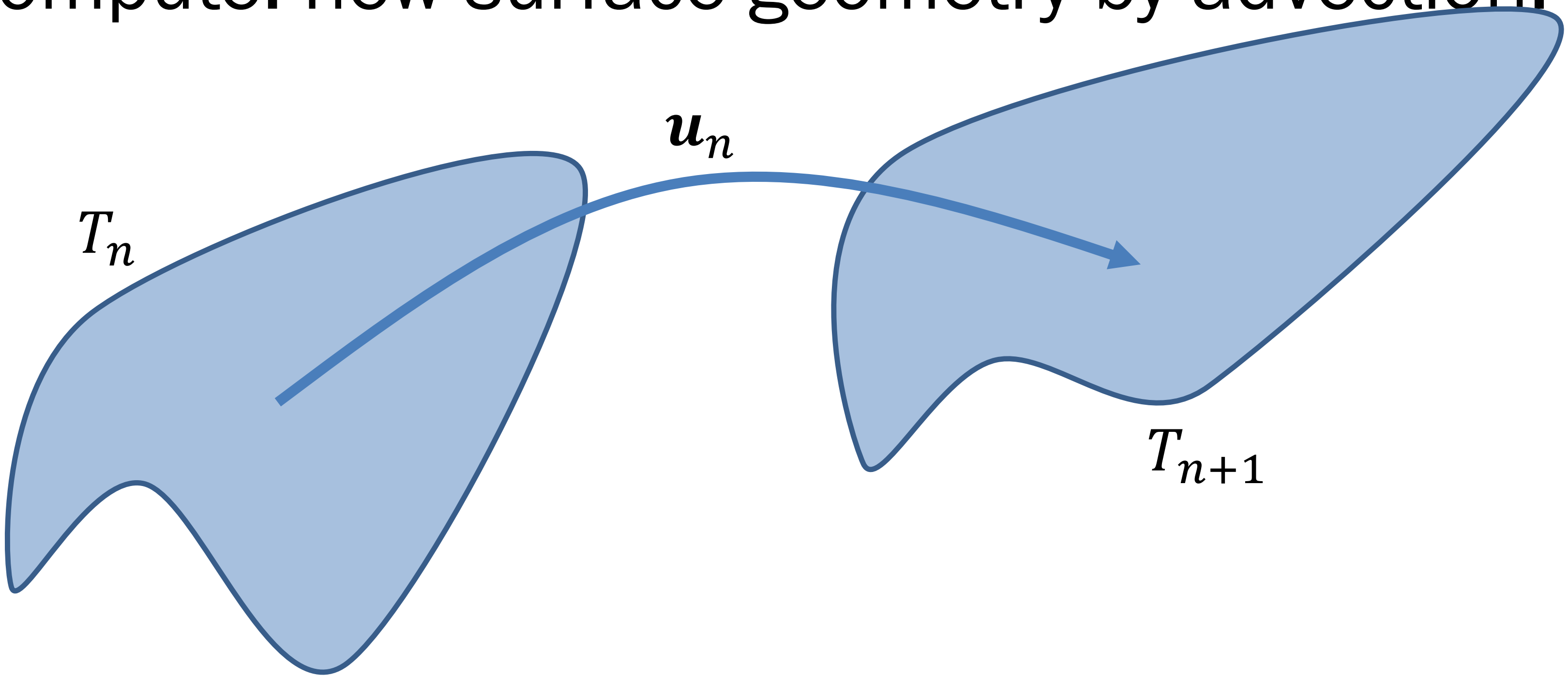
Interaction between Solver and Surface Tracker



Solver-to-Surface Tracker

Given: current surface geometry, velocity field, and timestep.

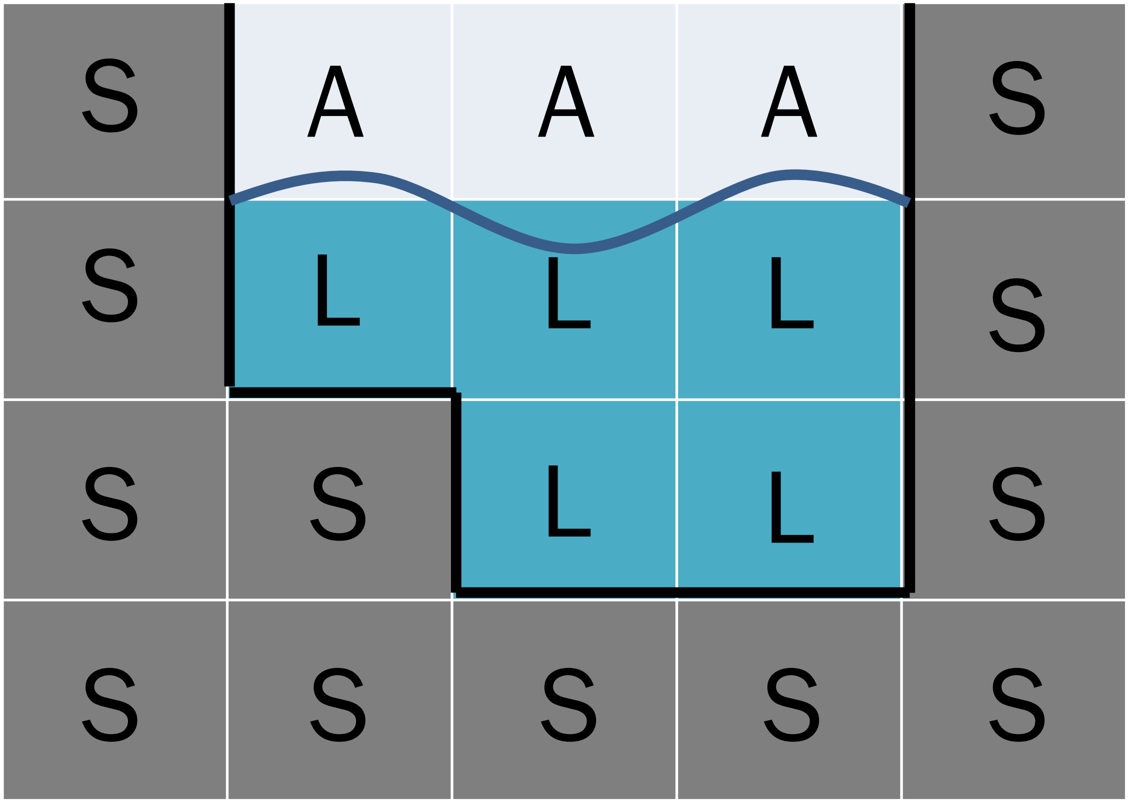
Compute: new surface geometry by advection.



Surface Tracker-to-Solver

Given the surface geometry, identify the type of each cell.

Solver uses this information for boundary conditions.



Surface Tracker

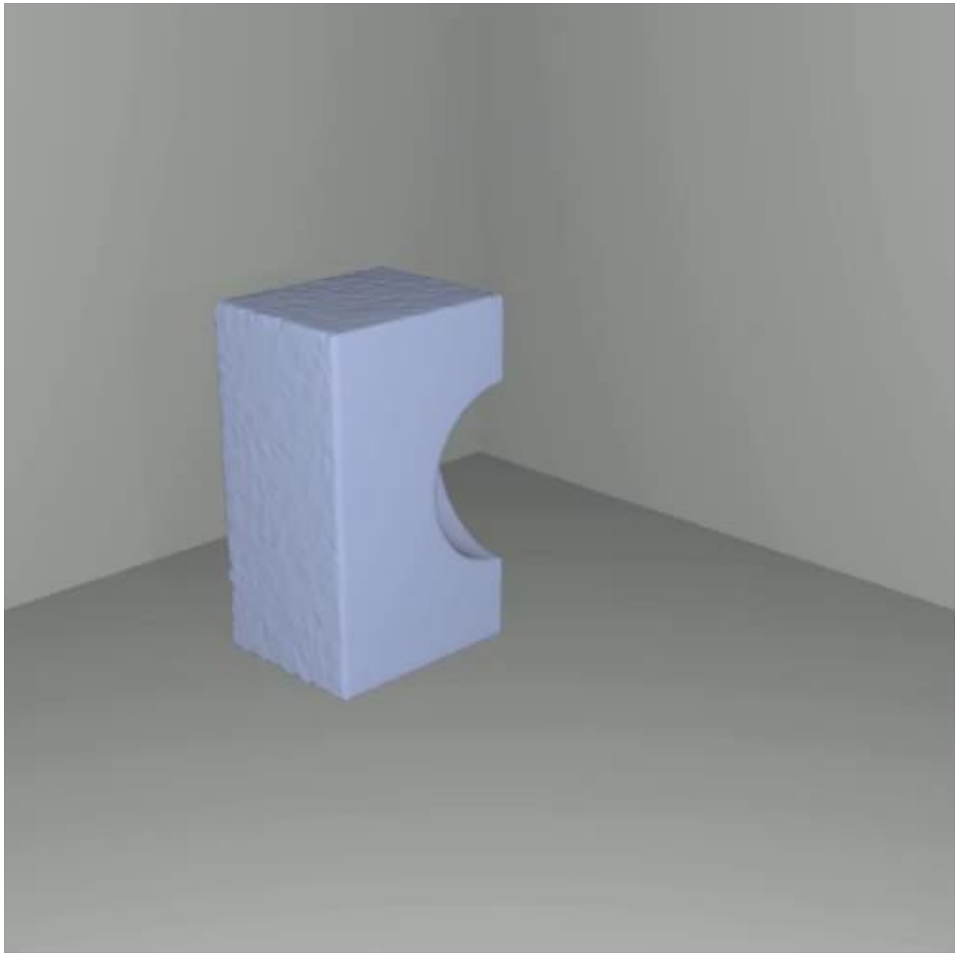
Ideally:
- Efficient
- Accurately follows the velocities
- Handles merging/splitting (“topology changes”)
- Conserves volume
- Retains small features and details
- Gives a smooth surface for rendering
- Provides convenient geometric operations (post-processing?)
- Easy to implement…

Very hard (impossible?) to do all of these at once.

Surface Tracking Options

1. Particles
2. Level sets
3. Volume-of-fluid (VOF)
4. Triangle meshes
5. Hybrids (many of these)

Particles

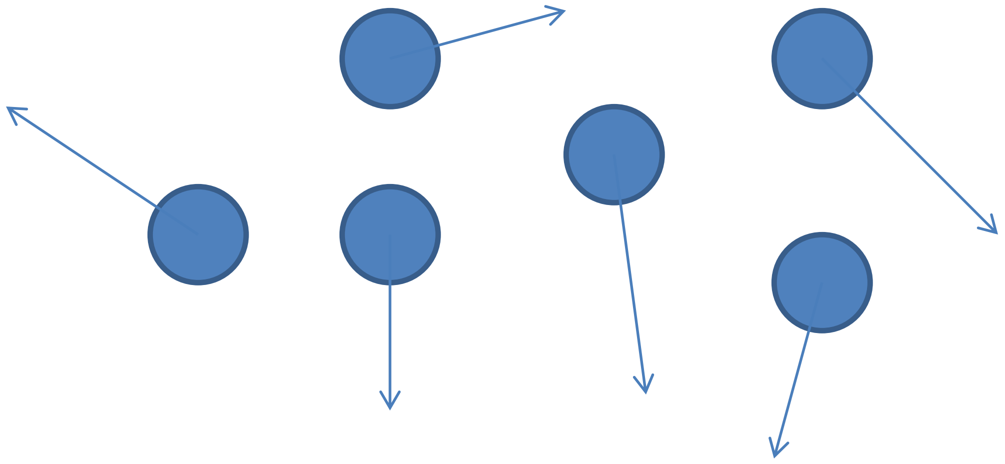


[Zhu & Bridson 2005]

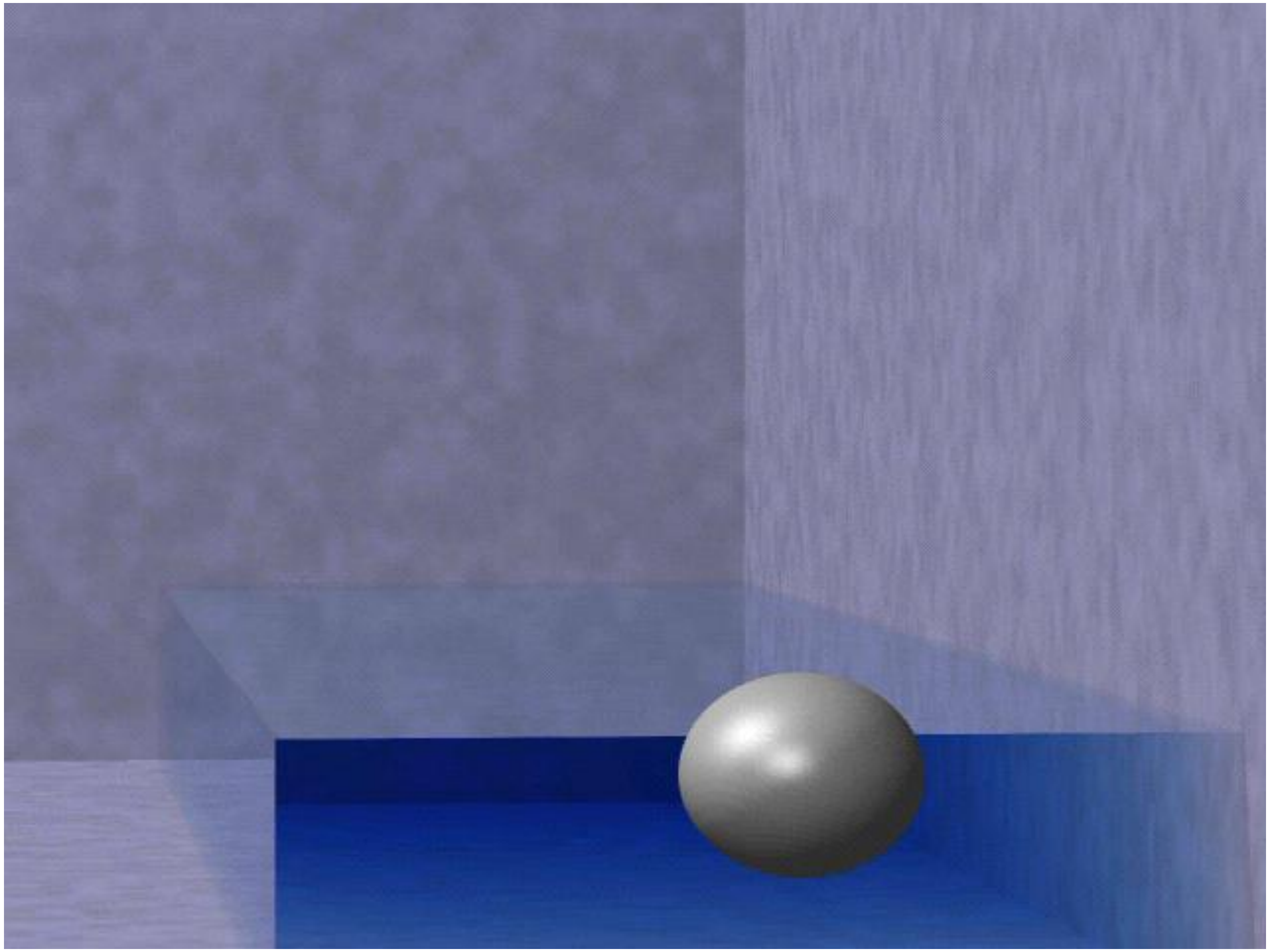
Particles

Perform passive Lagrangian advection on each particle.

For rendering, need to reconstruct a surface.



Level sets

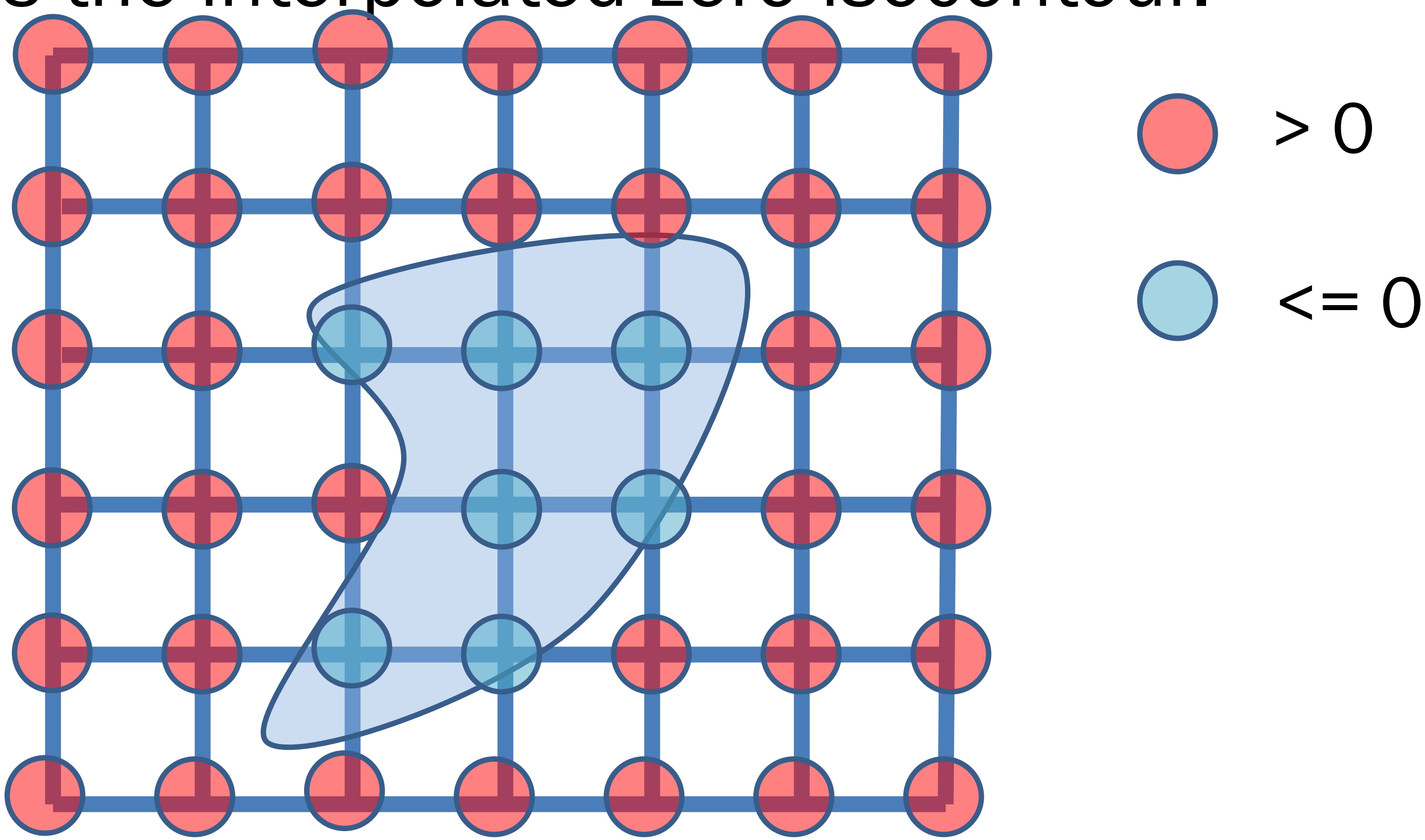


[Losasso et al.
2004]

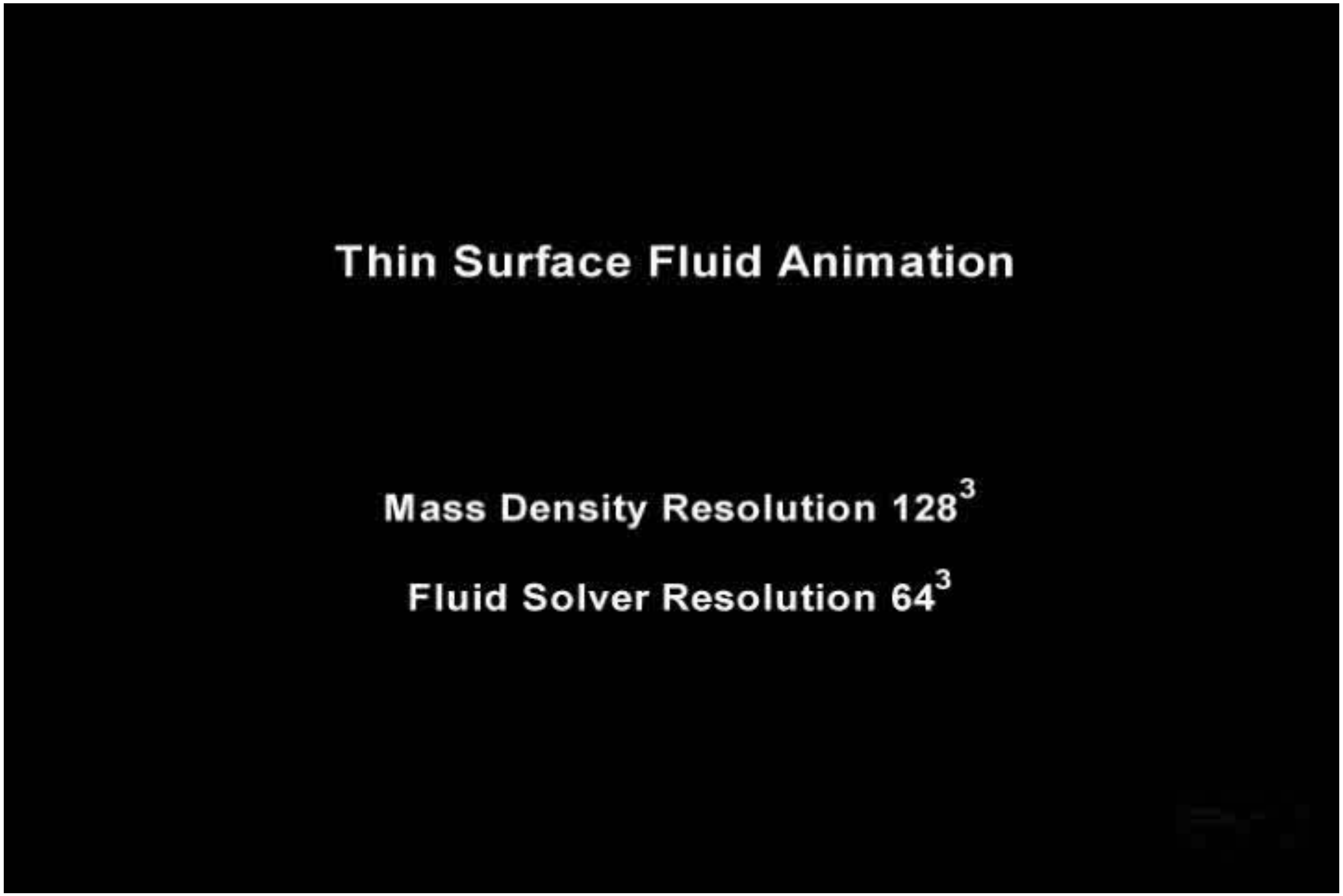
Level sets

Each grid point stores *signed* distance to the surface (inside <= 0, outside > 0).

Surface is the interpolated zero isocontour.



Densities / Volume of fluid

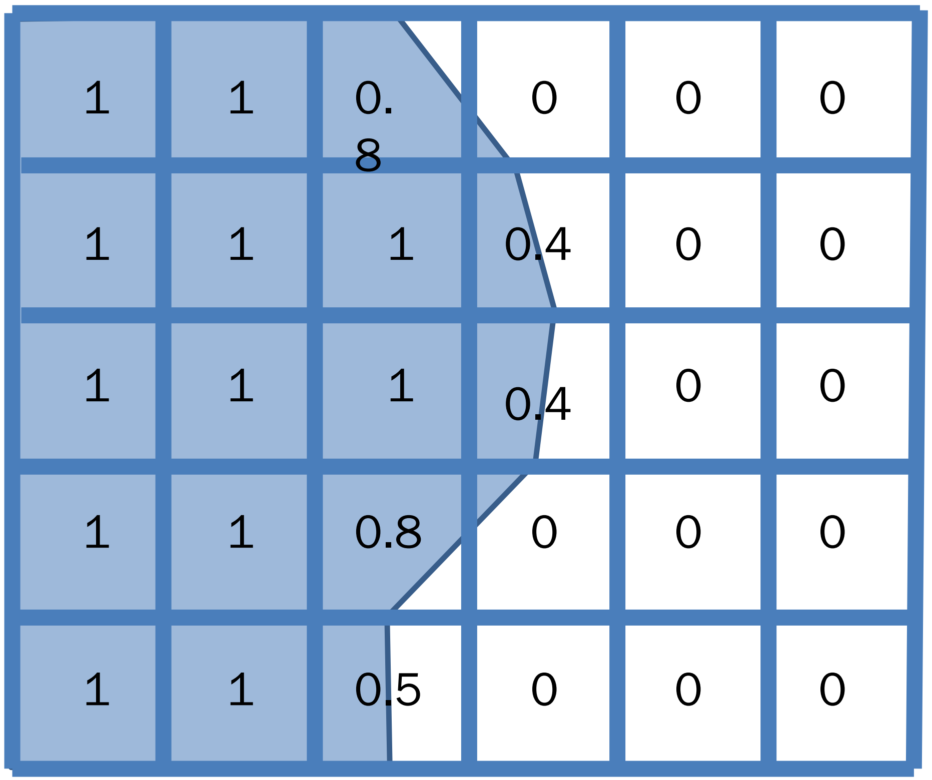


[Mullen et al 2007]

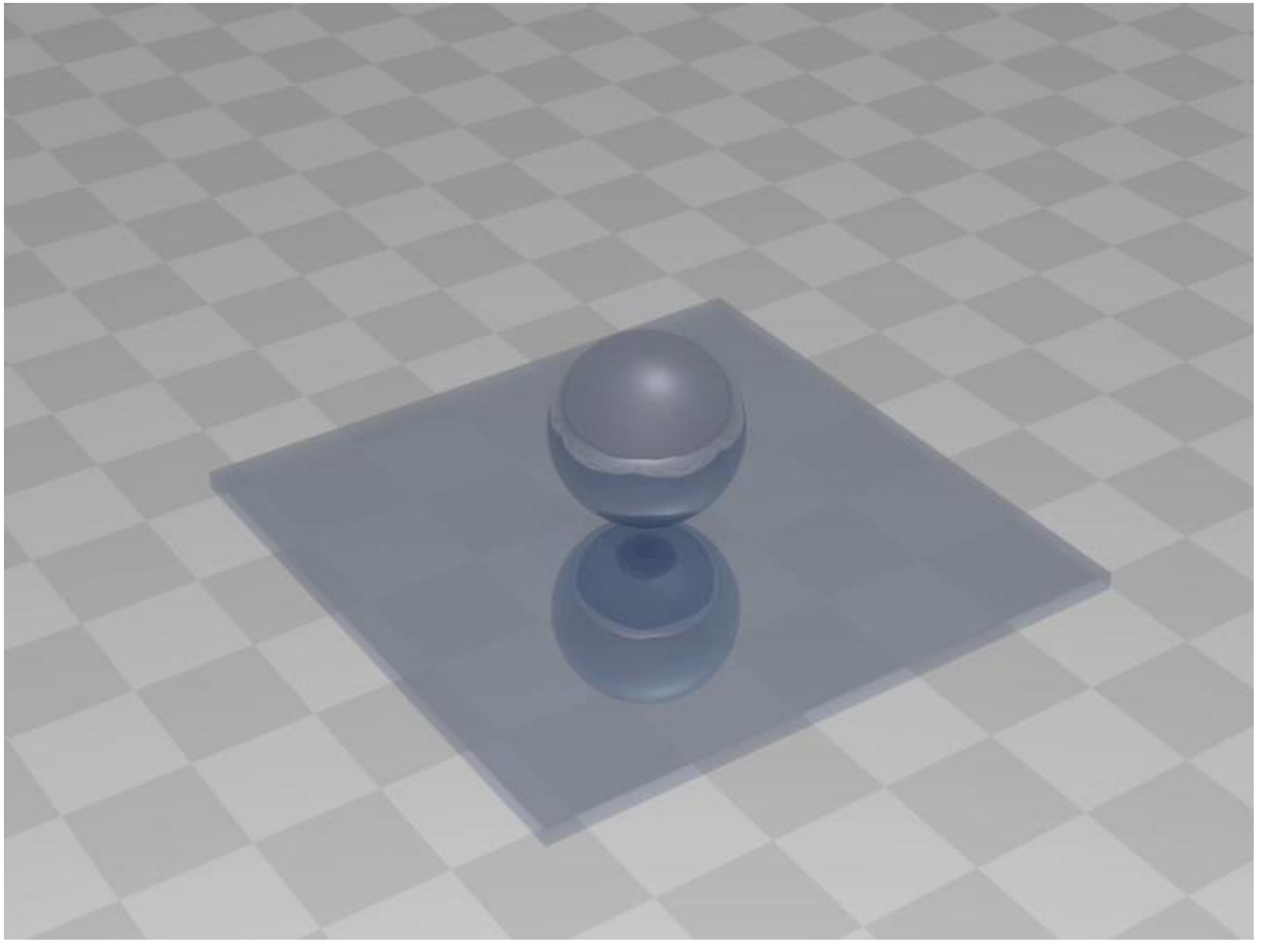
Volume-Of-Fluid

Each cell stores fraction $f \in [0,1]$ indicating how empty/full it is.

Surface is the transition region, $f \approx 0.5$.



Meshes

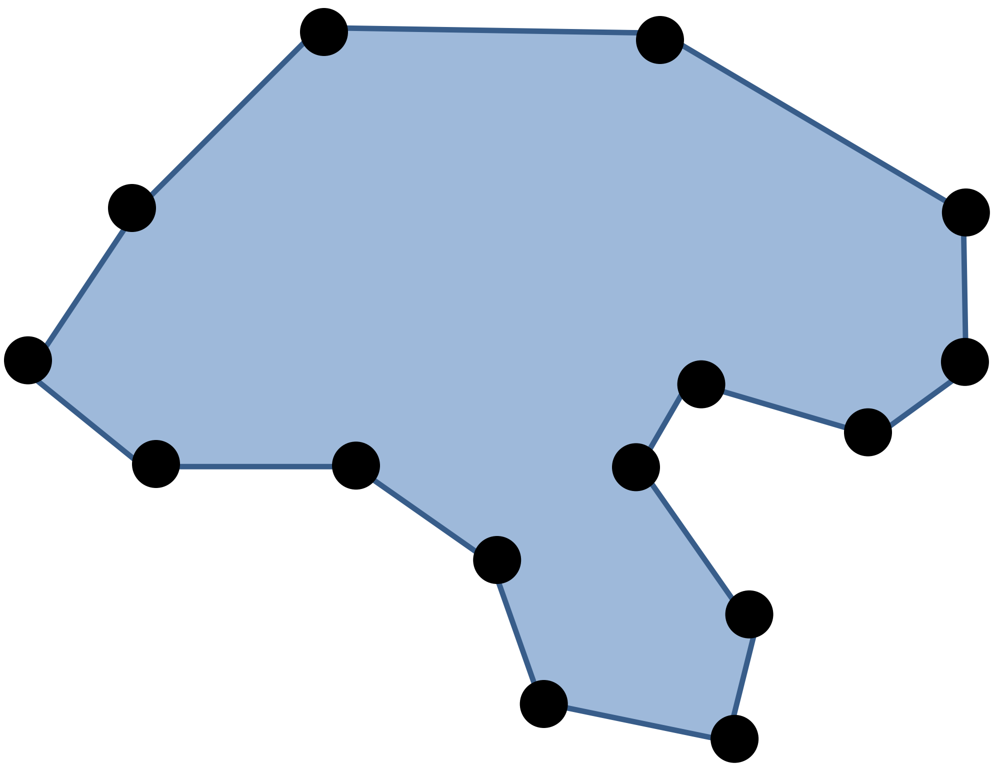


[Brochu et al 2010]

Meshes

Store a triangle mesh.

Advect its vertices, and deal with collisions.

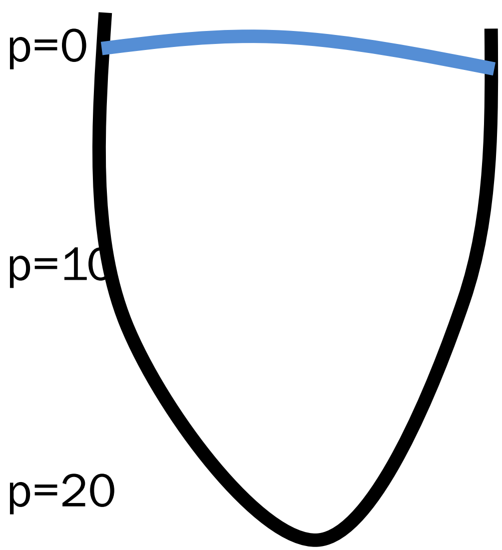


Reminders

- 1st round presentations start Monday. Graded on…
  1. Knowledge/coverage of technical concepts
  2. Organization
  3. Slide quality
  4. Speaking/presentation skills
- 1st round of paper reviews due Sunday, 5pm.
- Start thinking about project topics.
- Piazza (or email) with any questions.

Free Surface Boundary Conditions

Only the pressure *gradient* matters, so simplify by assuming $p = p_{atm} = 0$.



Same (vertical) pressure gradient, ∇p.