Elasticity & Discretization

Jan 11, 2016

Logistics

- I’ve created and posted the schedule based on sign-ups.
- 1st review due Sunday at 5pm; pick one of the 4 rigid body papers from the schedule. Submit your review to the LEARN DropBox.
- Other?

Elasticity

Elasticity

An elastic object is one that, when deformed, seeks to return to its original reference or rest configuration.

Previously: we saw discrete mass/spring models.

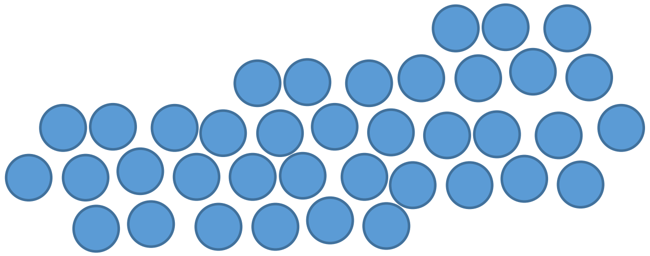
Today: more principled ***continuum mechanics*** approach.

Generalizes 1D elasticity (springs) to 3D objects.

Will loosely follow Sifakis' SIGGRAPH course: http://run.usc.edu/femdefo/sifakis-courseNotes-TheoryAndDiscretization.pdf

Continuum Mechanics

View the material under consideration as a *continuous* mass, rather than a set of *discrete* particles/atoms.



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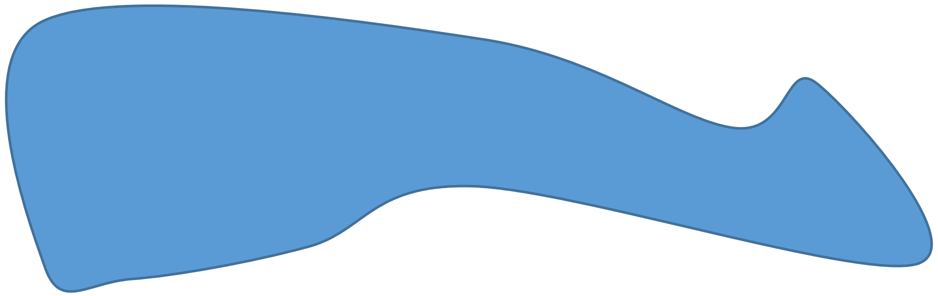


Useful for both solids and fluids.

Not always applicable: e.g., at small scales, during some kinds of fracture, for objects that are composed of large discrete elements, etc.

Elasticity - Springs

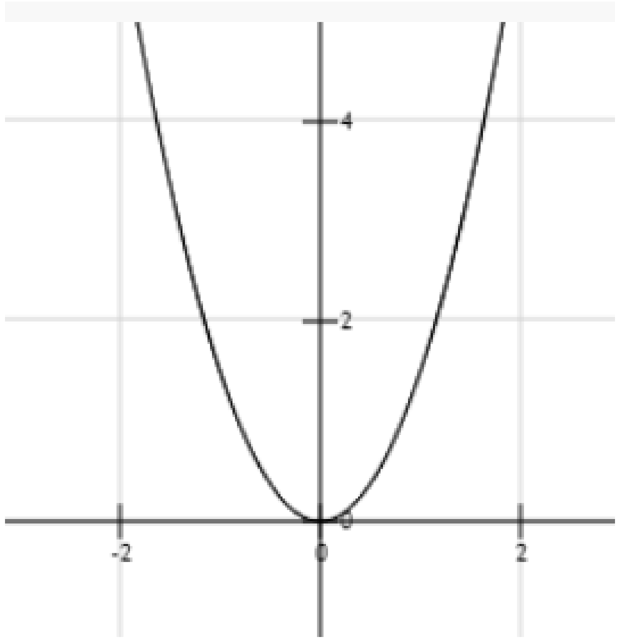
Recall: The linear spring force is dictated by displacement, $\Delta x = L - L_0$, away from rest length (Hooke's law):

$$F = -k\Delta x.$$

This force is related to the spring's potential energy:

$$\mathrm{U} = \frac{1}{2}k(\Delta x)^2.$$

The force acts to drive potential energy towards zero, by reducing the displacement, Δx.



Conservative Forces

The spring force is an example of a **conservative force** – it depends only on the *current state* (i.e., it is “path-independent”).

In this case, the force F is given by the gradient of a potential energy U:

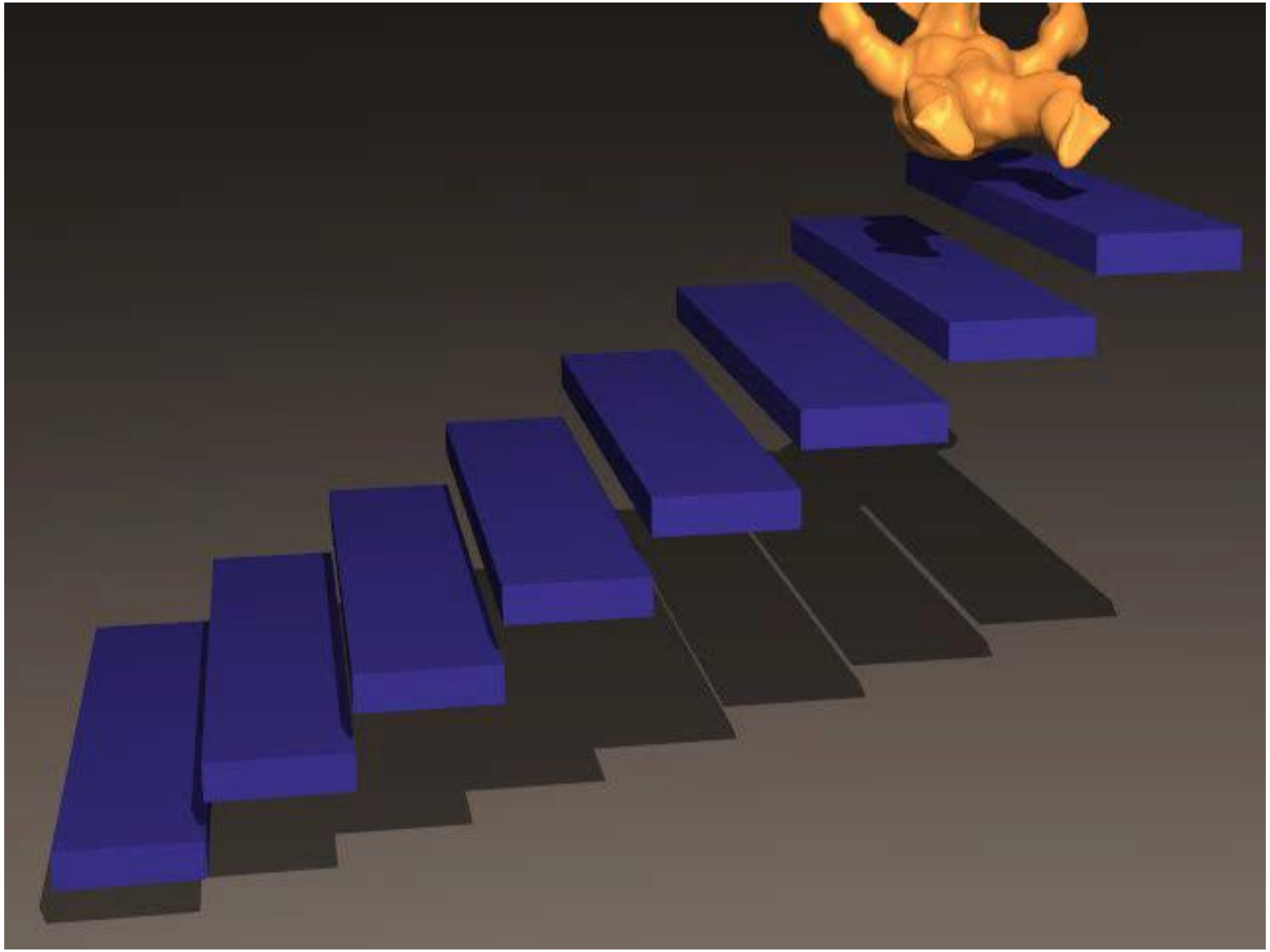
$$F = -\nabla U.$$

For our continuum elastic material, we seek a potential energy that is zero when our 3D object is undeformed.

Elasticity – 3D

How can we generalize the spring to (three-dimensional) *volumes* of material?

First, we need a way to describe 3D deformations.



Deformation Map

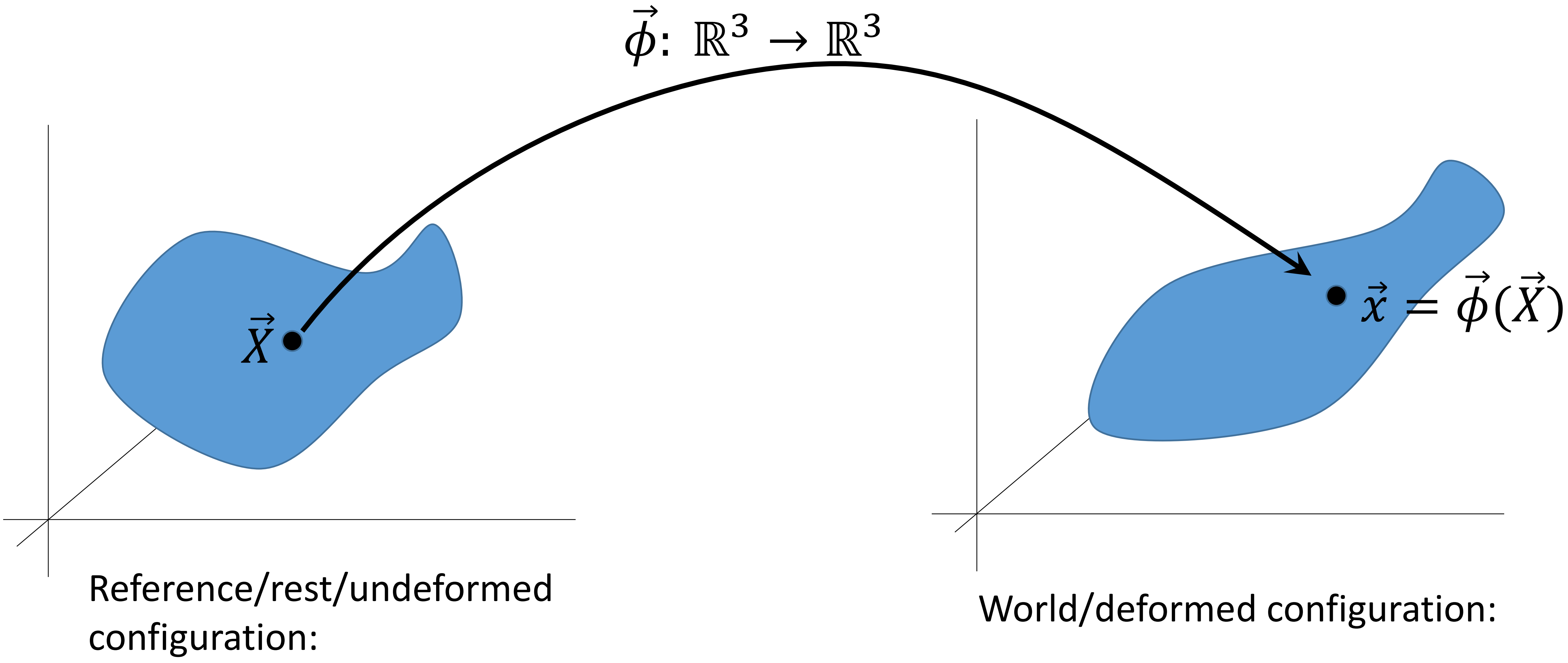
A function $\vec{\phi}$ that maps points from the reference configuration ($\vec{X}$) to current position in world space ($\vec{x}$).

$$\vec{\phi}\colon \mathbb{R}^3 \to \mathbb{R}^3$$

Purpose is similar to the state/transform of a rigid body.

However, each (infinitesimal) point in the body can now have a *different* transformation.

Deformation Map, $\vec{\phi}$



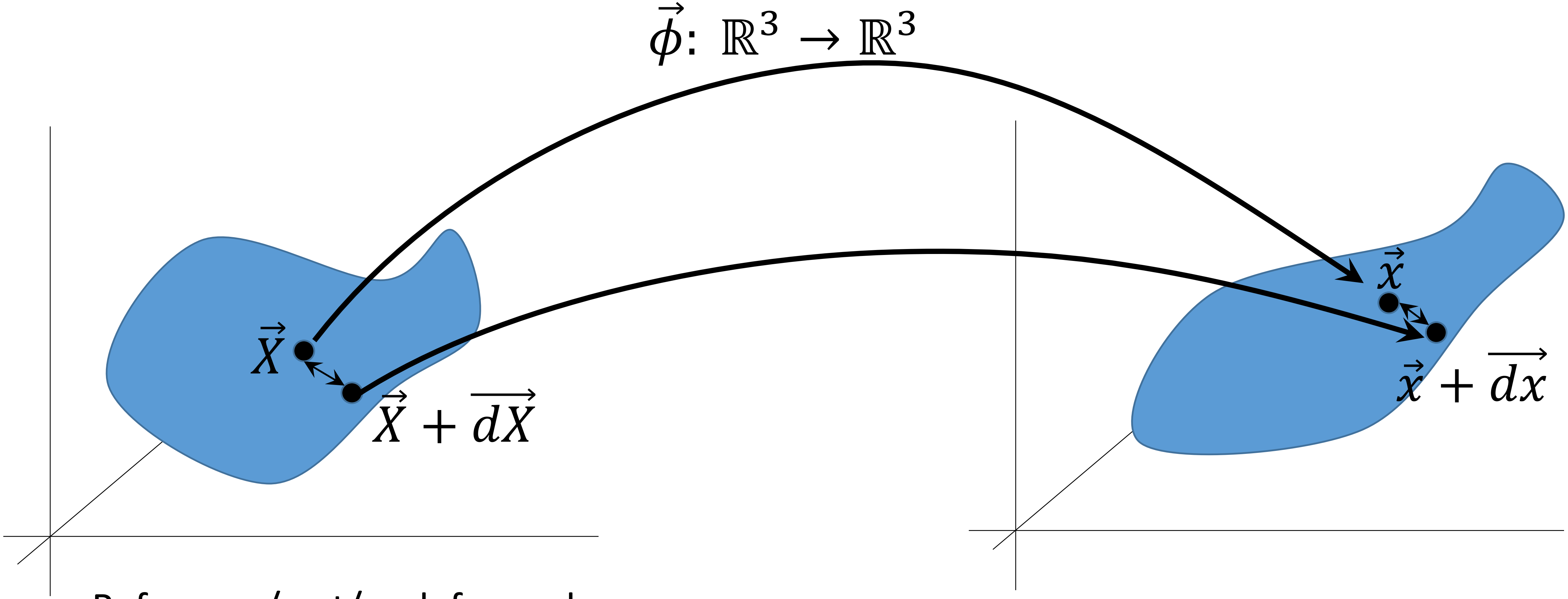
Deformations

The deformation map says where points in the material have moved to.

However, to determine forces due to deformation, we need to know how nearby points have moved ***relative to one another***.

The tool we need is the *deformation gradient*.

Deformation Gradient, $F = \frac{\partial \vec{\phi}}{\partial \vec{X}}$



Reference/rest/undeformed configuration:

World/deformed configuration:

Deformation Gradient

For some offset position from $\vec{X}$, say $\vec{X} + \overrightarrow{dX}$, what is the corresponding world position?

Offset in rest space

$$\vec{x} + \overrightarrow{dx} = \vec{\phi}(\vec{X} + \overrightarrow{dX}) \approx \vec{\phi}(\vec{X}) + \frac{\partial \vec{\phi}}{\partial \vec{X}} \overrightarrow{dX} = \vec{x} + \boldsymbol{F}\overrightarrow{dX}$$

Taylor expand…

Deformation gradient

Deformation gradient, $\boldsymbol{F} = \frac{\partial \vec{\phi}}{\partial \vec{X}}$, describes how particle positions have changed relative to one another.

Deformation Gradient

It is given by the 3×3 matrix (tensor):

$$\boldsymbol{F} = \frac{\partial \vec{\phi}}{\partial \vec{X}} = \begin{pmatrix} \frac{\partial \phi_1}{\partial X_1} & \frac{\partial \phi_1}{\partial X_2} & \frac{\partial \phi_1}{\partial X_3} \\ \frac{\partial \phi_2}{\partial X_1} & \frac{\partial \phi_2}{\partial X_2} & \frac{\partial \phi_2}{\partial X_3} \\ \frac{\partial \phi_3}{\partial X_1} & \frac{\partial \phi_3}{\partial X_2} & \frac{\partial \phi_3}{\partial X_3} \end{pmatrix}$$

Deformation Gradient: $\boldsymbol{F} = \frac{\partial \overrightarrow{\phi}}{\partial \vec{X}}$

Examples of simple deformations:

Translation: $\vec{x} = \vec{\phi}(\vec{X}) = \vec{t} + \vec{X}$, implies $\boldsymbol{F} = I$.

- World pos $\vec{x}$ is rest pos $\vec{X}$ plus a translation, $\vec{t}$. Zero *relative* motion between points.

Uniform Scaling: $x = \phi(X) = sX$ implies $\boldsymbol{F} = sI$.

- World pos $\vec{x}$ is rest pos $\vec{X}$ times a constant, s.

Rotation: $x = \phi(X) = RX$ implies $\boldsymbol{F} = R$.

- World pos $\vec{x}$ is rest pos $\vec{X}$ rotated by matrix R.

One Possible Potential Energy

What if we use $\boldsymbol{F}$ directly to construct a potential energy?

$$\mathrm{U}(\boldsymbol{F}) = \frac{k}{2}\|\boldsymbol{F} - I\|_F^2$$

Frobenius norm, $\|\boldsymbol{A}\|_F = \sqrt{\sum_i \sum_j a_{i,j}^2}$

Resulting forces ($-\nabla\mathrm{U}$) will drive $\boldsymbol{F}$ towards I, i.e., a deformation that is (just) a translation.

What's wrong with this?

Strain Measures

Want a deformation measure that *ignores* rotation (and translation), but captures other deformations.

Can we extract this from $\boldsymbol{F}$?

Recall: Rotation matrices are *orthogonal*, $R^T R = I$.

A useful measure is the Green/Lagrange strain tensor, $\mathrm{E} = \frac{1}{2}(\boldsymbol{F}^T \boldsymbol{F} - I)$.

Like $\boldsymbol{F}$, but ignores translation *and* rotation, while retaining shear/stretch/compression info.

Strain Measures

But, Green strain is nonlinear (quadratic), so more costly.

For *small* deformations, can use small/infinitesimal/Cauchy strain:

$$\epsilon = \frac{1}{2}(\boldsymbol{F}^T + \boldsymbol{F}) - I$$

(A linearization of Green strain.)

Many other strain tensors exist (and these two have many names)…

Big Picture – So Far

- Deformation map ϕ describes map from rest to world state.
- Deformation gradient $\boldsymbol{F} = \frac{\partial \vec{\phi}}{\partial \vec{X}}$ describes deformation (minus translation).
- Strains ϵ or E describe deformation (minus rotation).
- Then what…

Remaining questions:

- What are the equations of motion ("$F = ma$") for our continuous blob of material?
- How to get from strains to forces?

Equations of Motion

Consider $F = ma$ for a small, continuous "blob" of material.

$$\int_{\Omega} F_{body} dX + \int_{\partial\Omega} T dS = \int_{\Omega} \rho \ddot{x} dX$$

F_{body}: "body" forces that act throughout the material (e.g. gravity, magnetism, etc.). i.e., force per unit *volume* (i.e., force density).

T: tractions, i.e., force per unit *area* acting on a surface.

ρ: density.

Ω is the material region being considered, with surface/boundary $\partial\Omega$.

Traction

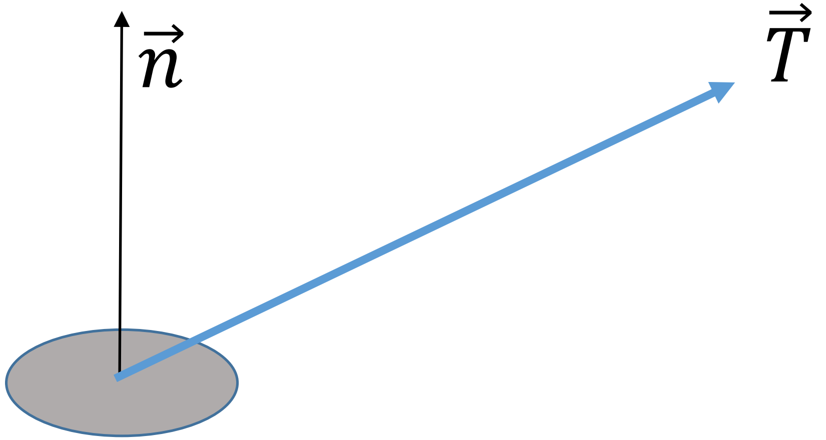
Traction T is a force (vector) per unit area on a small piece of surface.

$$\vec{T}(\vec{X},\vec{n}) = \lim_{A\to 0} \frac{\vec{F}}{A}$$

Cauchy's postulate:

Traction is a function of position $\vec{X}$ and normal $\vec{n}$.

i.e., doesn't depend on curvature, area or other properties.



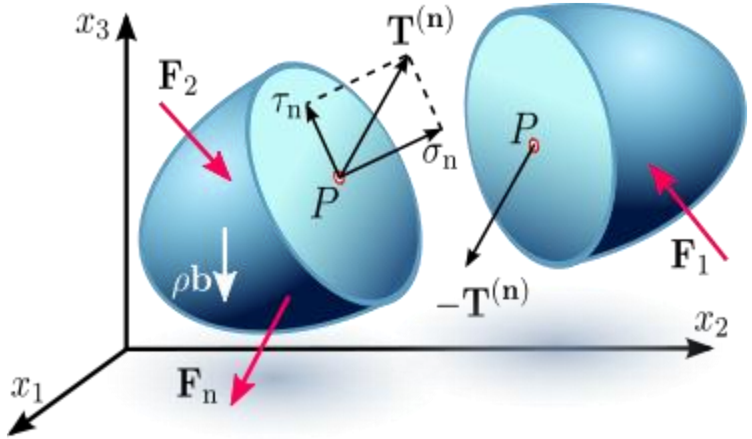
Consists of normal/pressure component along $\vec{n}$, and tangential/shear components perpendicular to it.

Traction

Consider the *internal* traction on any slice through a volume of material.

This describes the forces acting on this plane between the two “sides”.

Note: $T(x,n) \ = \ -T(x,-n)$ (by Newton’s 3rd).



Traction

We can characterize internal forces by considering tractions on 3 perpendicular slices (i.e., normals along x, y, z, directions).

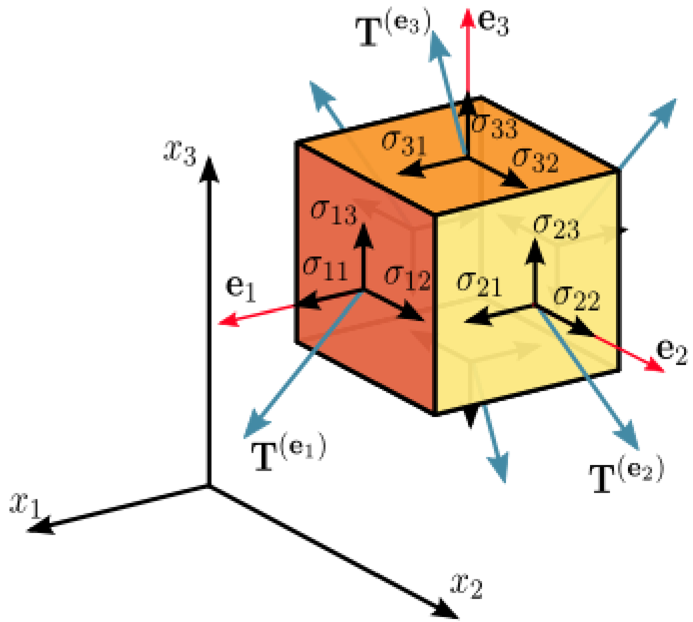
3 components per traction along 3 axes gives us 9 components.

This gives us the Cauchy stress tensor, σ.

Traction on any plane can be recovered via

$$T = \sigma n$$

where n is the normal of the plane.



Stress

The 3×3 stress tensor describes all the forces acting within a material (at a given point).

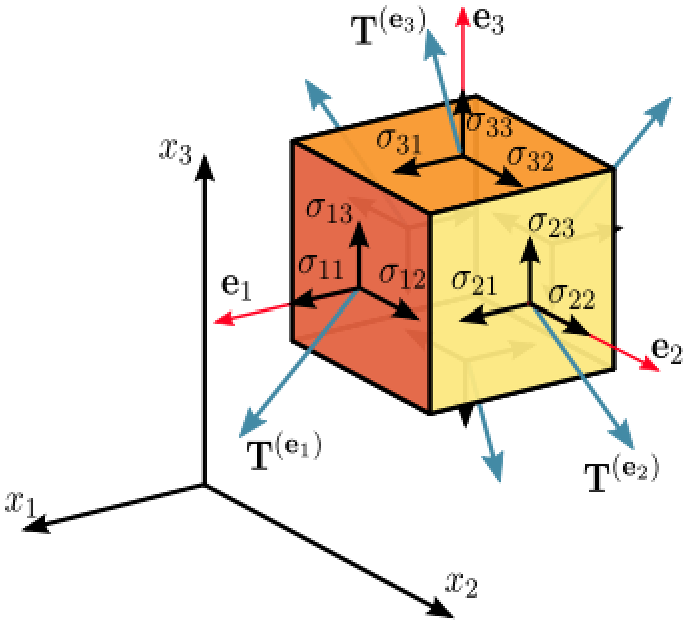
$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

σ can be shown to be symmetric, i.e., $\sigma_{yx} = \sigma_{xy}$, etc. (from conservation of angular momentum.)

Stress – Physical meaning?

Diagonal components correspond to compression/extension (normal) forces.

Off-diagonal components correspond to shear forces.



Equations of motion

$$\int_{\Omega} F_{body} dX + \int_{\partial\Omega} T dS = \int_{\Omega} \rho \ddot{x} dX$$

- Plug in $T = \sigma n$…

$$\int_{\Omega} F_{body} dX + \int_{\partial\Omega} \sigma n dS = \int_{\Omega} \rho \ddot{x} dX$$

- Integrate by parts (divergence theorem) to eliminate surface integral:

$$\int_{\Omega} F_{body} dX + \int_{\Omega} \nabla \cdot \sigma dX = \int_{\Omega} \rho \ddot{x} dX$$

In limit of small Ω, $F_{body} + \nabla \cdot \sigma = \rho \ddot{x}$, for every infinitesimal point.

Big Picture – So Far

- Deformation map ϕ describes map from rest to world state
- Deformation gradient F= describes deformations (minus translation)
- Strains ϵ or E describe deformation (minus rotation)
- Stress σ describes forces in material
- PDE $F_{body} + \nabla \cdot \sigma = \rho \ddot{x}$ describes how to apply stress to get motion
- (Later, will discretize the PDE to get discrete equations to solve.)

Last missing step!

Constitutive models

Strain E/ ϵ describes deformations of a body.

Stress σ describes (resulting) forces within a body.

Constitutive models dictate the stress-strain *relationship* in a material.

i.e., Given some deformation, what stresses (forces) does it induce?

e.g., Explains why rubber responds differently than concrete.

Linear Elasticity - simplest *isotropic* model

(i.e., behaves the same in all directions.)

Hooke's law in 3D, for small strain, ϵ.

Potential Energy:

$$U(F) = \mu\epsilon{:}\epsilon + \frac{\lambda}{2}\mathrm{tr}^2(\epsilon)$$

Stress:

$$\sigma = 2\mu\epsilon + \lambda\mathrm{tr}(\epsilon)I$$

μ, λ are the *Lamé parameters,* loosely analogous to 1D spring stiffness k.

"tr" is the trace operator (sum of diagonals of tensor/matrix)

":" is a tensor double dot product, where $A{:}B = \mathrm{tr}(A^T B)$

Linear Elasticity

Derives from the simplest possible linear relationship between σ and ϵ.

- Flatten the 3x3 tensors ϵ and σ into vectors.
- Isotropy and symmetry of ϵ/σ reduce 81 coeffs down to 2 independent parameters (e.g., μ and λ).

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yx} \\ \sigma_{yy} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{zy} \\ \sigma_{zz} \end{bmatrix} = [9x9\ coefficient\ matrix] \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{xy} \\ \epsilon_{xz} \\ \epsilon_{yx} \\ \epsilon_{yy} \\ \epsilon_{yz} \\ \epsilon_{zx} \\ \epsilon_{zy} \\ \epsilon_{zz} \end{bmatrix}$$

Other Elastic moduli

A more common/intuitive (but interconvertible) parameter pair is *Poisson's ratio,* ν, and *Young's modulus, E, or Y.* (Careful overloading E).

$$\mu = \frac{Y}{2(1+\nu)}$$

and…

$$\lambda = \frac{Y\nu}{(1+\nu)(1-2\nu)}$$

Elastic Moduli – Young’s modulus

Young’s modulus:

- Ratio of stress-to-strain along an axis.
- Should be consistent with linear spring.

Elastic Moduli – Poisson’s ratio

- Poisson’s ratio is negative ratio of transverse to axial strain
  - If stretched in one direction, how much does it compress in the others?
  - Expresses tendency to preserve volume.
  - Lies in range $[-1, 0.5]$.

  - 0.5 = incompressible (e.g., rubber)
  - 0 = no compression (e.g., cork)
  - < 0 is possible, though weird...
  - Called *auxetic* materials.

The “linear” in linear elasticity

- Describes the stress-strain relationship.
- But, strain itself could still be either linear (small strain, ϵ) or nonlinear (Green strain, E) in the deformation.

Use E instead of ϵ with the same equations gives:

$$U(F) = \mu E : E + \frac{\lambda}{2}\mathrm{tr}^2(E)$$

Better for larger deformations/rotations. (AKA St. Venant Kirchhoff model.)

Other common models

- Corotational linear elasticity:
  - Try to pre-factor out the rotational part of strain in a different way; treat the remainder with linear elasticity.
  - We’ll see this idea in the “Interactive Virtual Materials” paper.

- Neo-Hookean elasticity:
  - St.V-K breaks down under large compression (stops resisting).
  - Neo-Hookean is a nonlinear model that corrects this.

A Taste of Common Discretization Methods

Discretization

Need to turn our continuous model describing infinitesimal points…

$$F_{body} + \nabla \cdot \sigma = \rho \ddot{x}$$

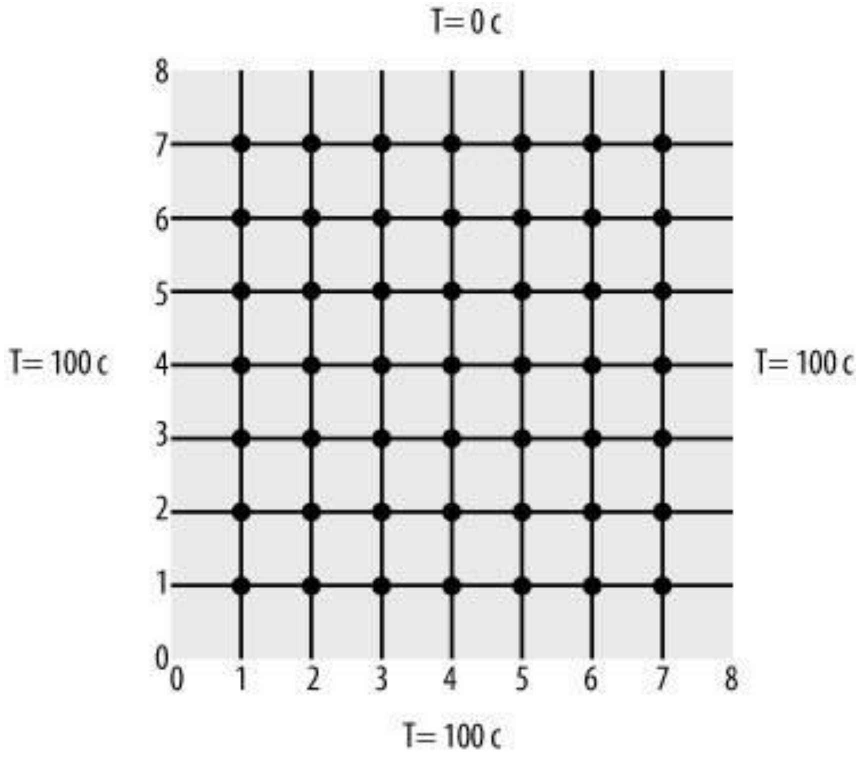
…into a discrete model that approximates it (and can be computed!)

Standard choices: Finite difference (FDM), finite volume (FVM), and finite element methods (FEM).

I’ll give a brief flavour, but… there’s a vast literature & theory. (See e.g. Numerical PDE course, CS778.)

Finite differences

Dice the material/domain into a grid of points holding the relevant data.



Replace all (continuous, spatial) derivatives with (discrete) finite difference approximations.

e.g.,

$$\frac{dy}{dx} \approx \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

Time Discretization

- Notice: Time integration schemes (FE, RK2, BE, etc.) are discretizations of time derivatives, along the 1D time axis.

- E.g., Forward Euler uses a 1st order (one-sided) finite difference:

$$\frac{dx}{dt} \approx \frac{x_{i+1} - x_i}{\Delta t}$$

- We distinguish time discretization and spatial discretization, and focus on the latter now.

e.g. 1D Heat equation

Continuous equation:

$$\frac{\partial \phi}{\partial t} - \alpha \frac{\partial^2 \phi}{\partial x^2} = 0$$

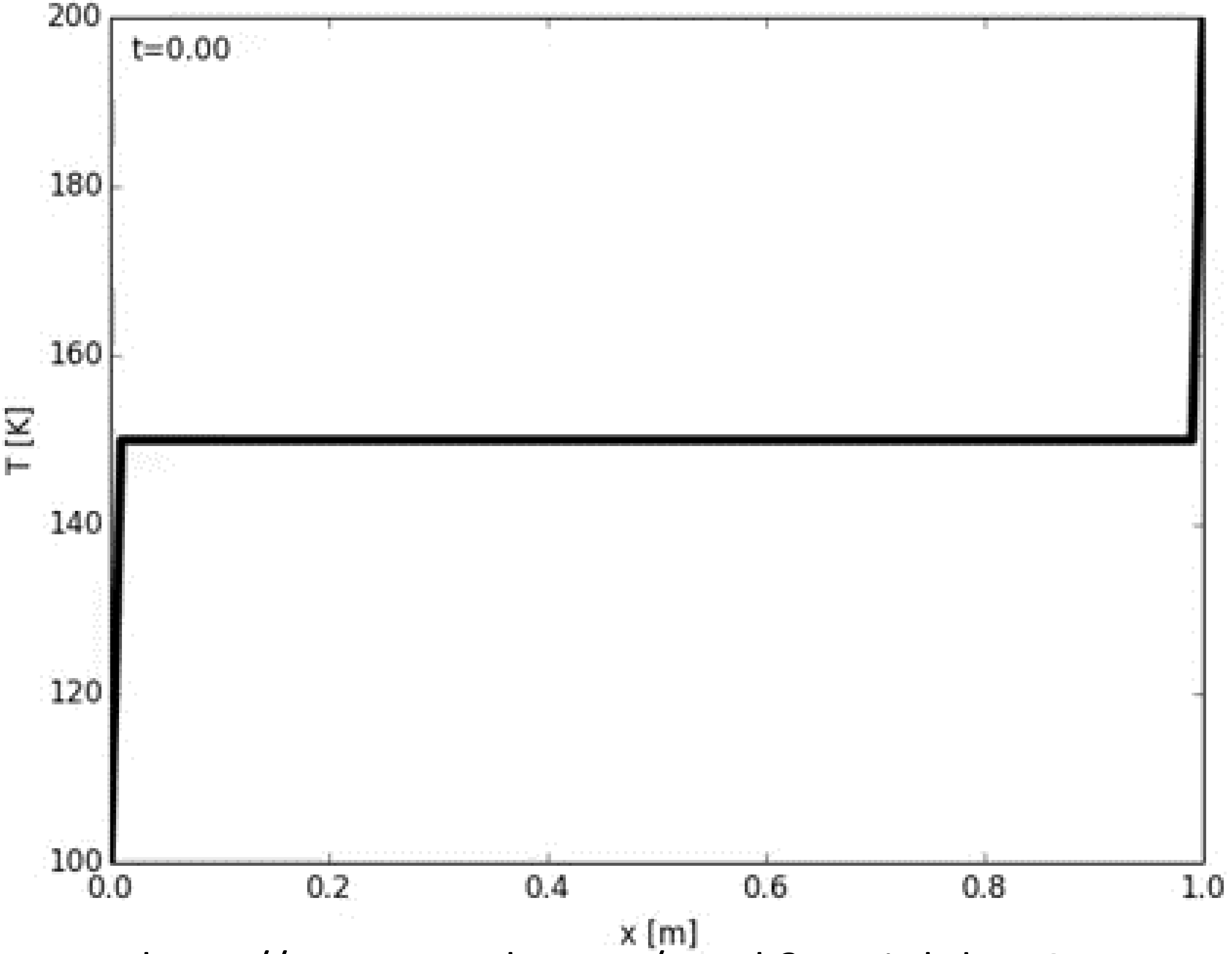
Discretize time derivative (w/ forward Euler):

$$\phi_{t+\Delta t} = \phi_t + \Delta t \alpha \frac{\partial^2 \phi_t}{\partial x^2}$$

Discretize spatial derivatives with F.D.:

$$\phi_{t+\Delta t} = \phi_t + \Delta t \alpha \frac{\left(\frac{\phi_t^{i+1} - \phi_t^{i}}{\Delta x} - \frac{\phi_t^{i} - \phi_t^{i-1}}{\Delta x} \right)}{\Delta x}$$

e.g. 1D Heat(diffusion) equation

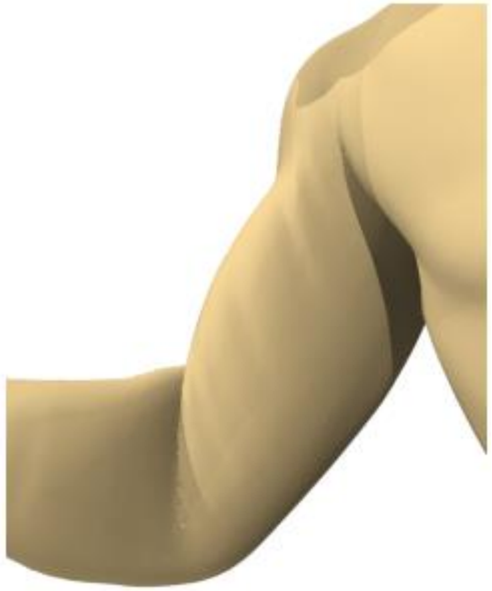


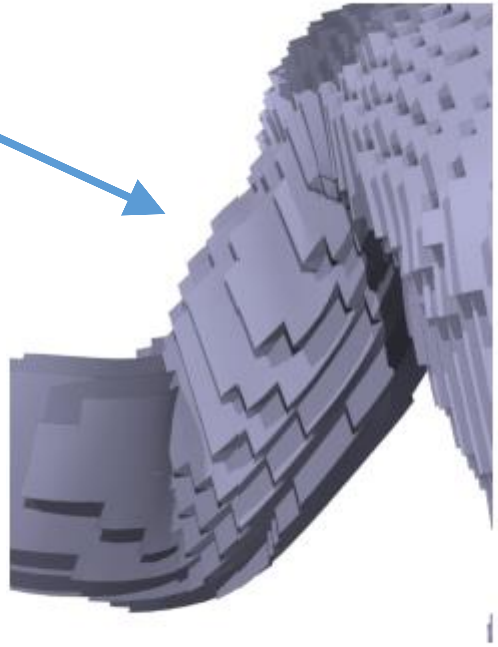
https://www.youtube.com/watch?v=mjDhdyxnOwo

Finite differences

Very common for fluids… less so for solids.

Few graphics papers use FDM for solids: e.g., “An efficient multigrid method for the simulation of high-resolution elastic solids”





Advantages: often simpler to implement, nice grid structure offers various optimizations, cache coherent memory accesses…

Disadvantages: trickier for irregular shapes and boundaries not aligned with axes

Finite volume

- Divide the physical domain up into a set of non-overlapping “control volumes.”
- Could be irregular, tetrahedra, hexahedra, general polyhedral, etc.

- Approximate the *integrated/average* value of quantities within the cell (rather than point values like F.D.).
- Consider the exchange of data between adjacent cells.

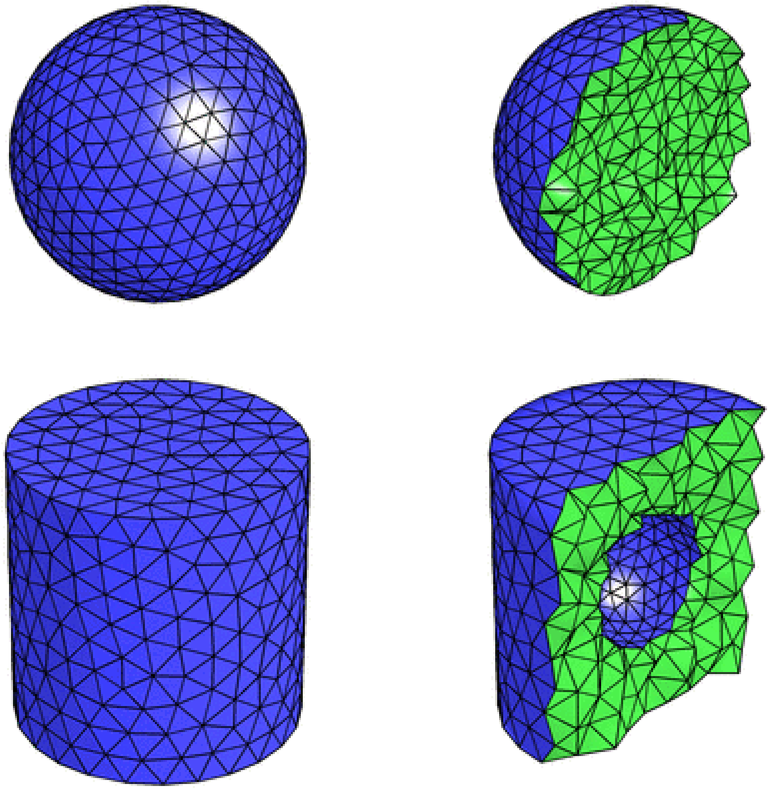


Figure from the *DistMesh* gallery:
http://persson.berkeley.edu/distmesh/gallery_images.html

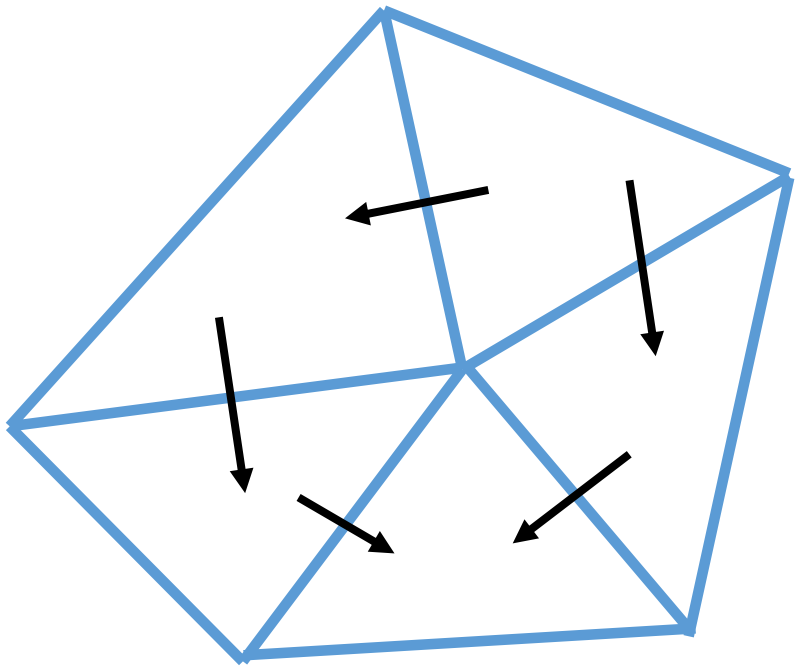
Finite volume – Conservation laws

Useful for conserved quantities; ensures the exact “flow” leaving one cell enters the next.

- E.g. mass, heat, momentum, etc.

Applies to equations in “conservation form”:

$$\frac{d\phi}{dt} + \nabla \cdot f(\phi) = 0$$



Particularly common for fluids.

Reminder: Divergence operator, $\nabla \cdot$

For vector field $\vec{u}(\vec{x})$, divergence is a signed scalar measuring net flow out of a point.

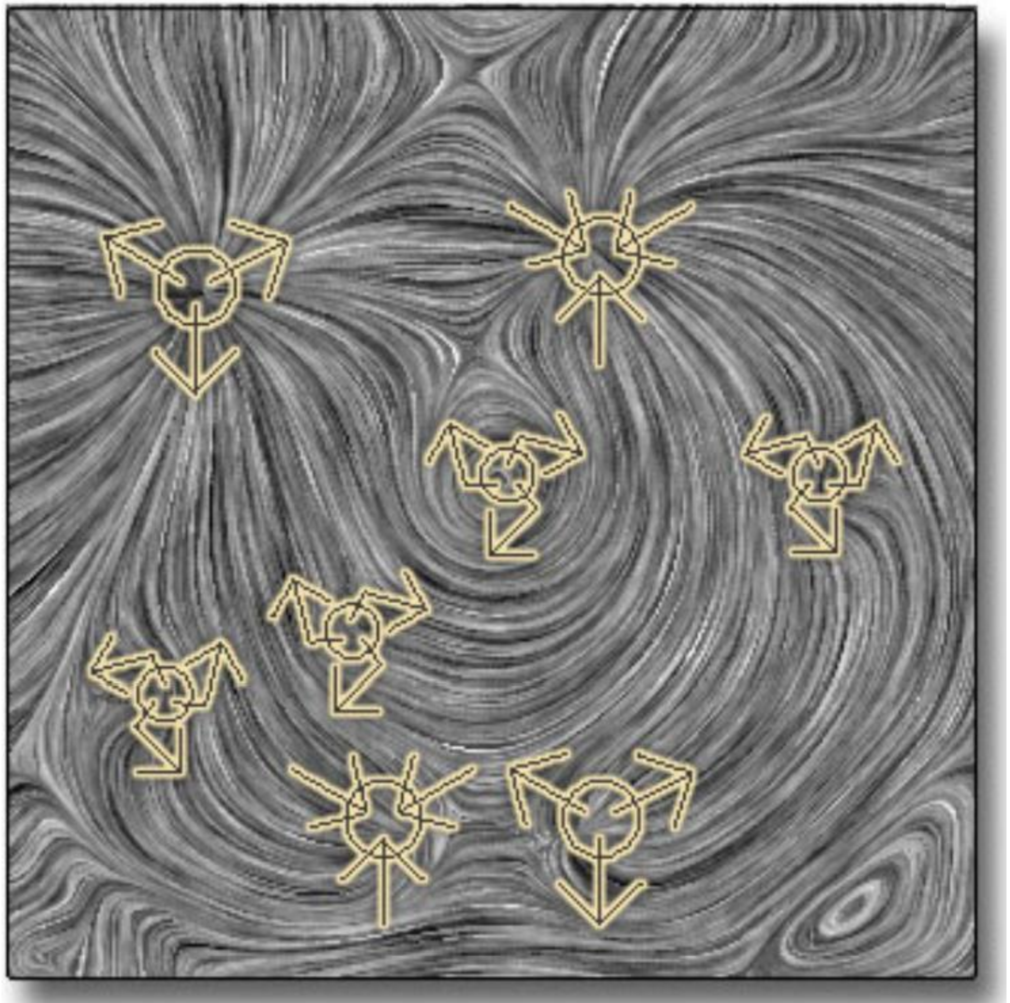
i.e. to what degree that point is a source (v.s. a sink) for the vector field.

$$\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Flowing out: positive

Flowing in: negative

Neither: zero ("divergence-free")



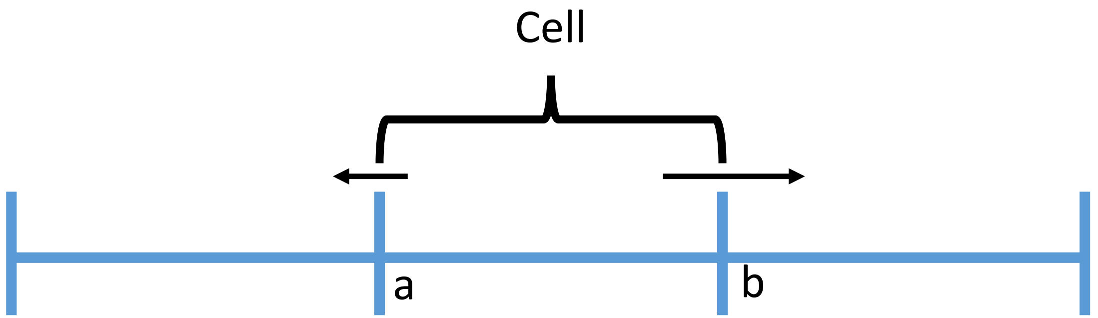
[Tong et al. 2003]

e.g. 1D Heat equation

Instead of differential form, use an *integral* form of the governing equations over a cell.

$$\left(\frac{d}{dt}\int \phi\right) - \int \frac{\partial}{\partial x}\left(\alpha \frac{\partial \phi}{\partial x}\right) = 0$$

The amount of a quantity (ϕ) in a cell changes due to the amount flowing across its boundaries (sides).

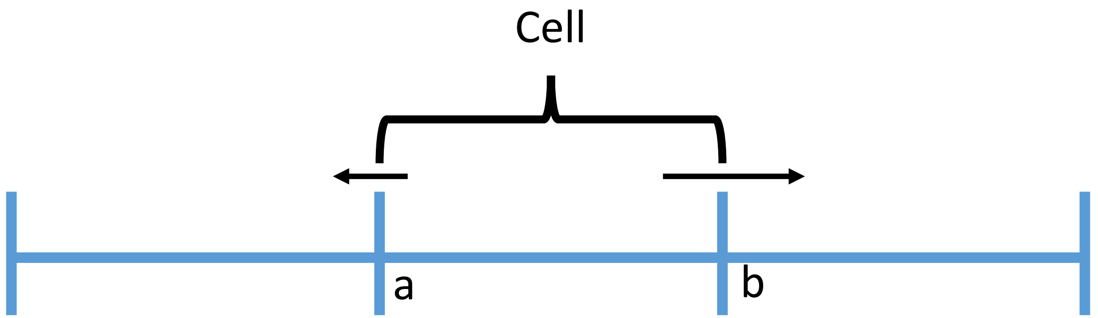


e.g. 1D Heat equation

$$\left(\frac{d}{dt}\int_a^b \phi\, dx\right) - \int_a^b \frac{\partial}{\partial x}\left(\alpha \frac{\partial \phi}{\partial x}\right) dx = 0 \quad \underset{\text{By FTOC}}{\rightarrow} \quad \left(\frac{d}{dt}\int_a^b \phi\, dx\right) - \left[\alpha \frac{\partial \phi}{\partial x}\right]_a^b = 0$$

We estimate the difference in “flux” $\alpha \frac{\partial \phi}{\partial x}$ at the right and left side of the cell.

This tells us how much the total ϕ in the cell, $\int_a^b \phi\, dx$, changes per unit time.



Finite volume – Equations of motion

Return to the *integral* form of our equations of motion…

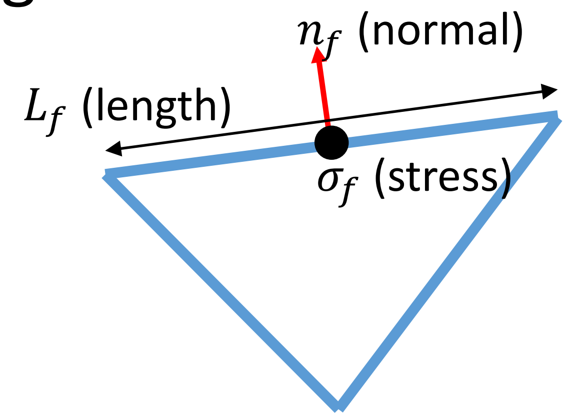
$$\int_\Omega F_{body} dX + \int_\Omega \nabla \cdot \sigma dX = \int_\Omega \rho \ddot{x} dX$$

Convert divergence term into *surface* integrals by divergence th'm.

e.g.

$$\int_\Omega \nabla \cdot \sigma dX = \int_{\partial\Omega} \sigma \cdot n \, dS$$

$$\approx \sum_{faces\ f} (\sigma_f \cdot n_f) L_f$$

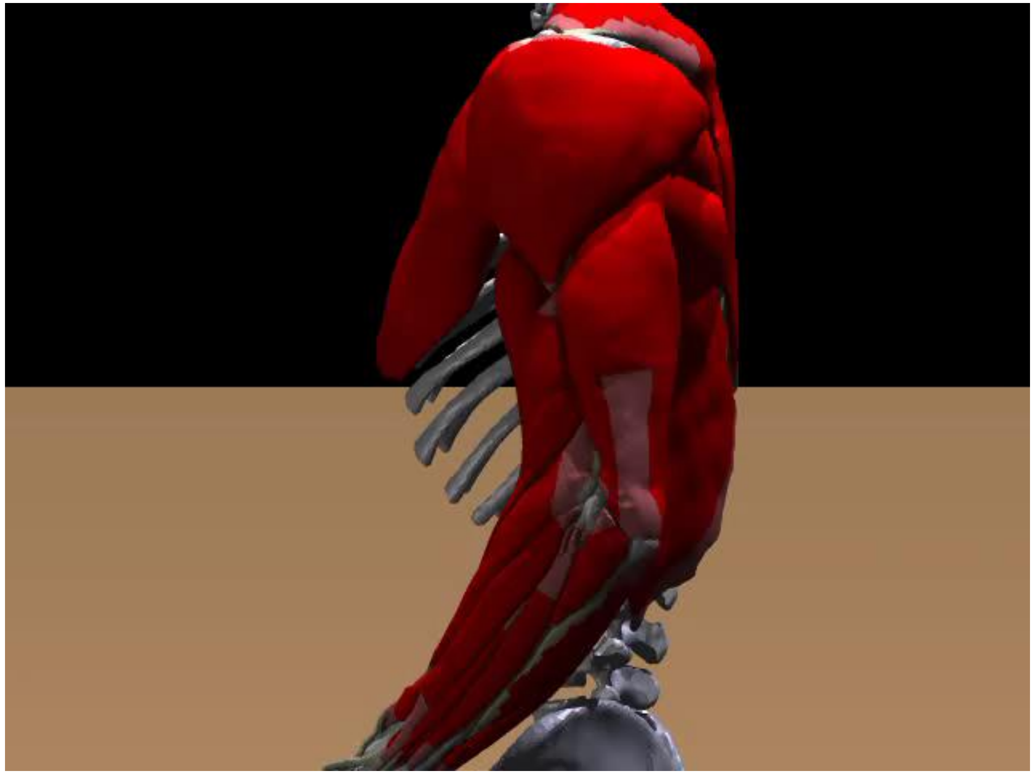


Integrate remaining terms to get volume-averaged quantities per cell.

Finite volume – Elasticity

We'll see details of FV applied to elastic objects in the paper:

"Finite Volume Methods for the Simulation of Skeletal Muscle"



Finite element methods

Core idea:

We can’t really solve the *infinite dimensional*, continuous problem describing all points in the material!

Instead find a solution that we *can* represent, in some ***finite*** dimensional subspace.

Concretely:

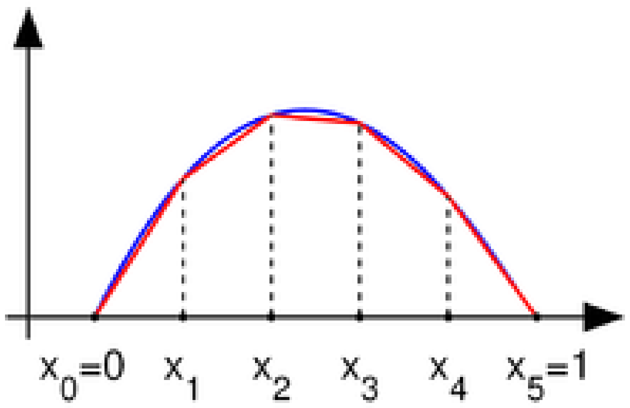
1. choose an approximate representation of continuous functions on a discrete mesh.
2. find the “best” solution possible among all functions that representation can describe.

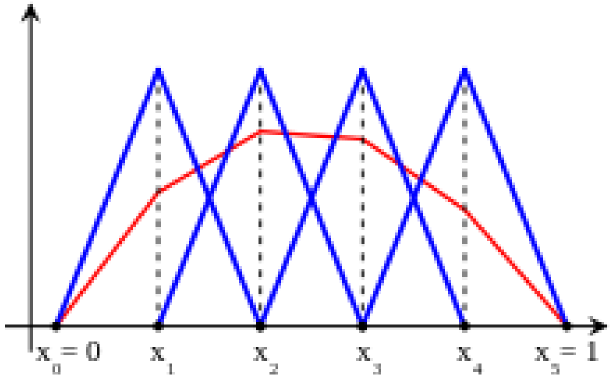
Finite elements – basis functions

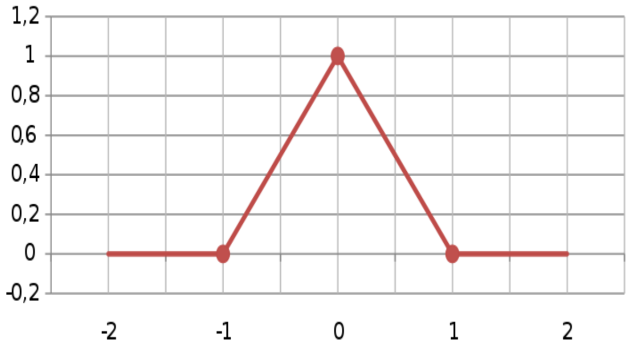
In 1D, consider the space of functions representable by (piecewise) linear interpolation on a set of grid points.

Just a linear combination of scaled and translated “hat” functions at each gridpoint, called a *basis function*.

Many others bases are possible (e.g. higher order polynomials).







Finite elements

Then, any function u *in this space* can be described by:

$$u(x) = \sum_{k=1}^{n} u_k v_k(x)$$

where u_i are the coefficients, and $v_k(x)$ are the basis functions, ("hats" in our case.)

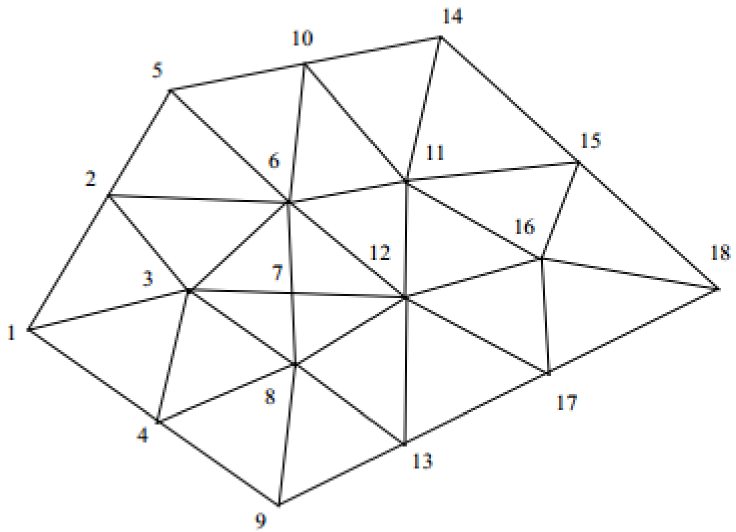
To find a solution to a problem, we want to find the discrete coefficients, u_k.

The approximated continuous solution is recovered by interpolation.

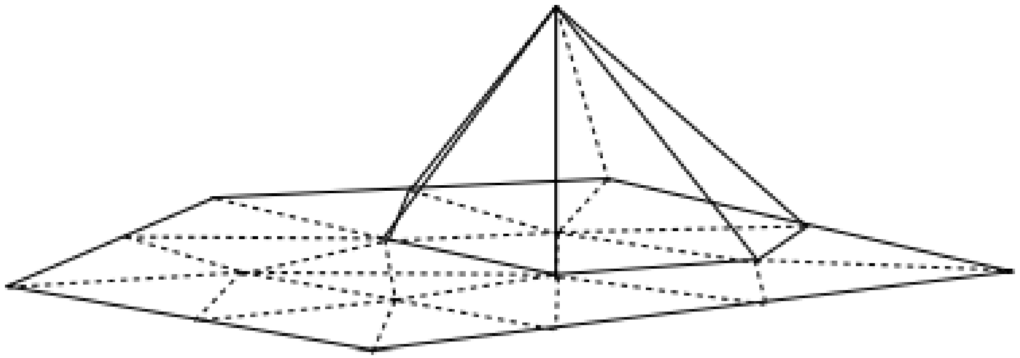
Higher dimensional functions

This generalizes to higher dimensions, where our scalar function u depends on multiple variables (e.g. x, y, z)

e.g., two dimensions:



2D mesh with numbered nodes.



A single linear “hat” basis function in 2D.

Finite elements

Take a 1D model problem: $\frac{d^2u}{dx^2} = f$ on [0,1], with $u(0) = 0, u(1) =$0.

For given f, find u.

For a proper solution, it will also be true that

$$\int \frac{d^2u}{dx^2} v dx = \int f v dx$$

for all “test functions” v (that are smooth and satisfy the BC).

We require this, rather than pointwise satisfaction of original equation.

Finite elements

Integrate LHS by parts (with zero boundaries) to get:

$$\int \frac{du}{dx}\frac{dv}{dx}\,dx = \int f v\,dx$$

This is called the *weak form* of the PDE.

Now, we will replace u, f, and v with our space of discrete, piecewise linear functions.

Finite elements

Specifically, we insert:

- $u(x) = \sum_{k=1}^{n} u_k v_k(x)$
- $f(x) = \sum_{k=1}^{n} f_k v_k(x)$
- $v(x) = v_j(x)$ for j = 1 to n (i.e. a set of functions spanning the space)

From our (linear/hat) basis functions, we can work out derivatives of $u(x)$ and $v(x)$.
We can also *exactly* find the following inner products:

$$\langle v_j, v_k \rangle = \int v_j v_k \, dx$$

$$\phi(v_j v_k) = \int \frac{dv_j}{dx} \frac{dv_k}{dx} \, dx$$

Finite elements

After plugging into $\int \frac{du}{dx}\frac{dv}{dx} = \int f v$ and rearranging, this yields a set of *n discrete equations* of the form:

$$\sum_{k=1}^{n} u_k \, \phi(v_j, v_k) = \sum_{k=1}^{n} f_k \, \langle v_j, v_k \rangle$$

Unknown coefficients

Known inner products

Known input data

Known inner products

Final system

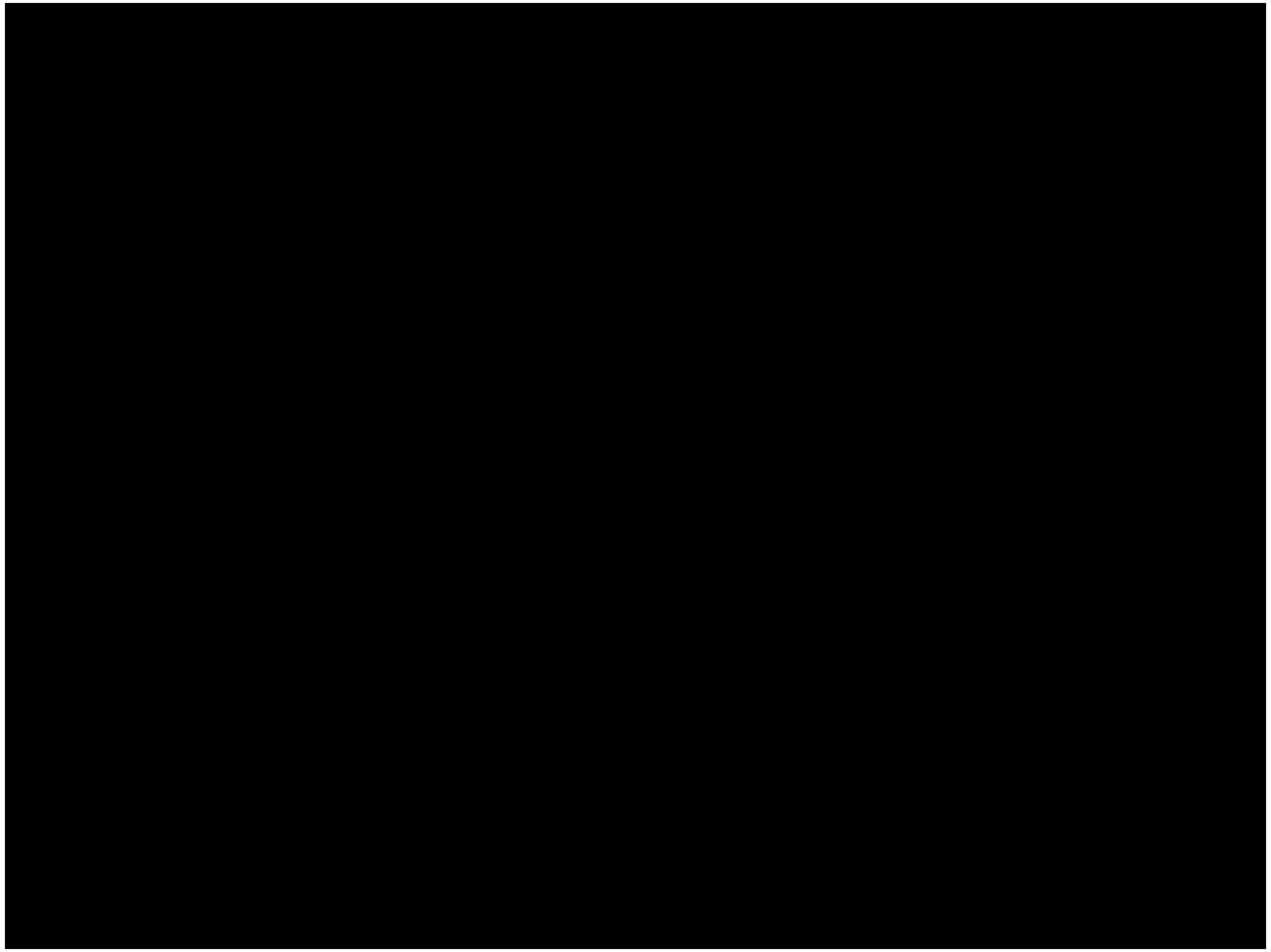
Letting $\mathbf{u}$ be the vector of unknown coefficients, and $\mathbf{b}$ the RHS vector, this becomes a matrix equation:

$$\mathrm{L}\mathbf{u} = \mathbf{b}$$

where the entries of L are just the $\phi(v_j, v_k)$'s we defined.

See paper "Graphical Modeling and Animation of Brittle Fracture" for details of an early application of FEM to elasticity in graphics.

Example – FEM with fracture

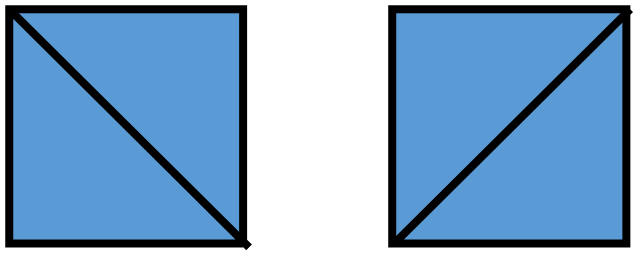


“Graphical modeling and animation of brittle fracture”, O’Brien et al. 1999

FD/FV/FE elasticity v.s. mass-spring

- In all cases (w/ implicit time integration) we get a possibly nonlinear system of equations to solve for data stored on a discrete mesh/grid.

- However, for FD/FV/FE:
  - we can use physically meaningful/measurable parameters.
  - as the mesh resolution is increased, we approach true/real analytical solutions.
  - behaviour becomes independent of the mesh structure (under refinement!)
    - E.g. eliminates bias due to triangle edge directions. With springs, the mesh structures below behave differently regardless of how fine the mesh is.



Summary

- The equations of elasticity give us a more consistent and principled approach to evolving continuous deformable bodies.
- The most common approaches to **discretizing** the resulting PDE are the finite difference, volume, and element methods.