Frechet distance between polygonal curves

Computing the Fréchet distance between two polygonal curves
H Alt, M Godau - International Journal of Computational Geometry & …, 1995 - World Scientific
As a measure for the resemblance of **curves** in arbitrary dimensions we consider the so-called **Fréchet-distance**, which is compatible with parametrizations of the **curves**. For **polygonal** chains P and Q consisting of p and q edges an algorithm of runtime O (pq log (**...**

Figures here, notes follow.

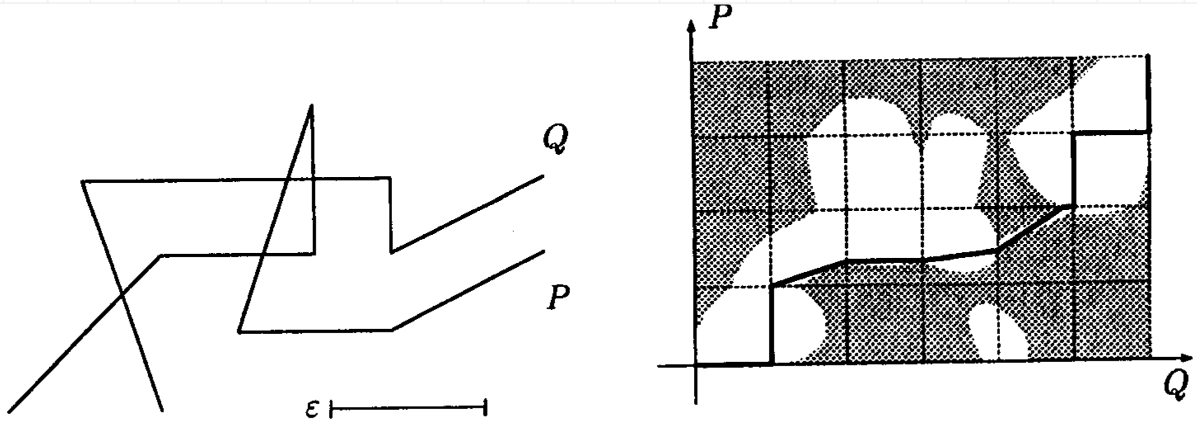

Fig. 3. **Diagram for polygonal chains P, Q and the given ε**

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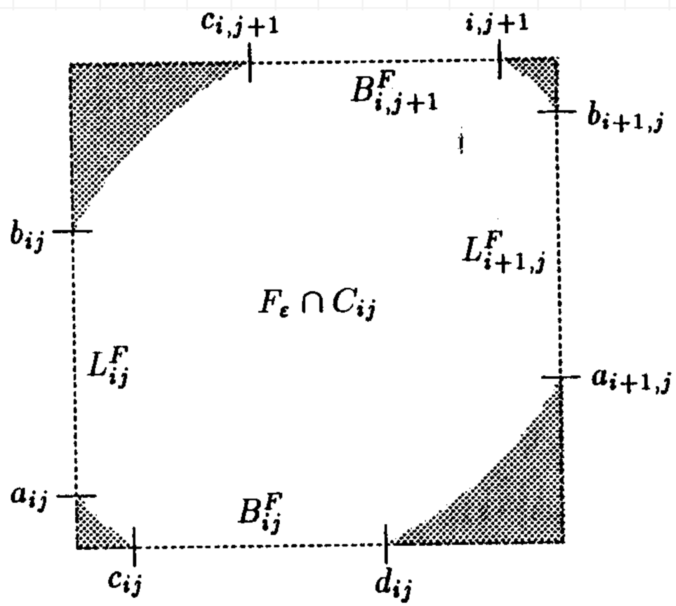

Fig. 4. Intervals of free space on the boundary of a cell.

[Efficient algorithms for geometric optimization](http://scholar.google.ca)
PK Agarwal, M Sharir - ACM Computing Surveys (CSUR), 1998 - dl.acm.org
Abstract We review the recent progress in the design of **efficient algorithms** for various problems in **geometric optimization**. We present several techniques used to attack these problems, such as parametric searching, **geometric** alternatives to parametric searching, **...**
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From: http://scholar.google.ca/scholar?q=Efficient+Algorithms+for+Geometric+Optimization&btnG=&hl=en&as_sdt=0%2C5

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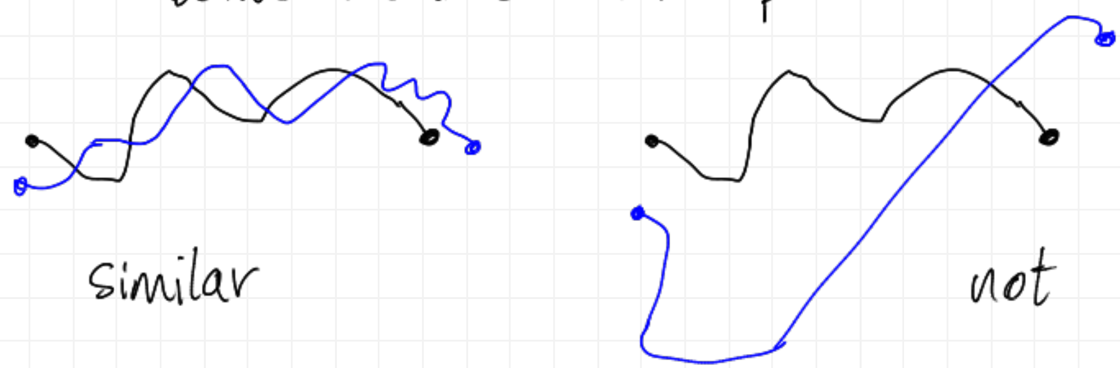
Distances – to measure similarity

e.g. edit distance in strings (can be formulated as distance in a graph, "reconfiguration")

Distance between curves in the plane



similar

not

Hausdorff distance between A and B.

- for each point $a \in A$ take min distance to B
- for each point $b \in B$ take min distance to A

take max of these (need sup, inf in general)

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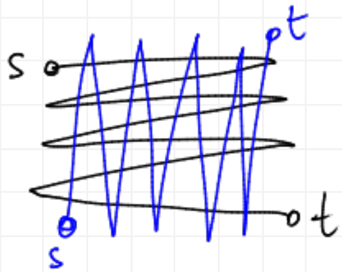
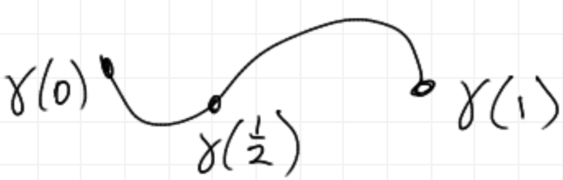
Hausdorff distance does not capture similarity



only takes into account the set of points, not their order on the curve

curve $\gamma : [0,1] \to \mathbb{R}^2$



same curve may have different parameterizations

polygonal curve
composed of n line segments

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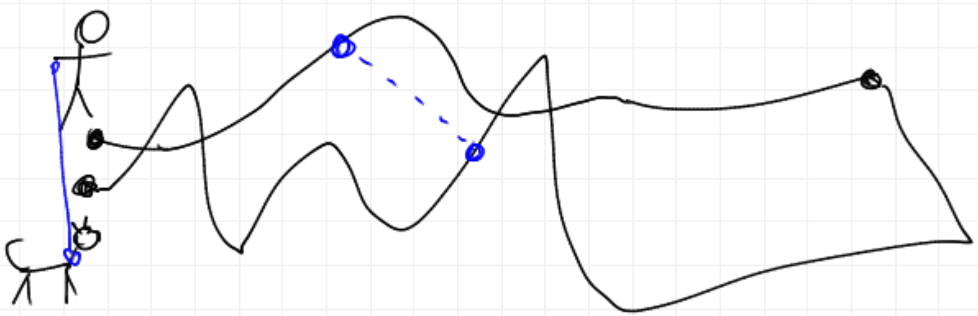


person moves on curve A
dog on curve B
} never backwards

minimum length of leash = Frechet distance

more formally (Frechet 1906)

$$\delta_F(\alpha, \beta) = \inf_{\substack{\text{reparameterizations} \\ \alpha', \beta'}} \underbrace{\max_{t \in [0,1]} \| \alpha'(t) - \beta'(t) \|}_{\text{how long the leash must be}}$$

Frechet distance between polygonal curves

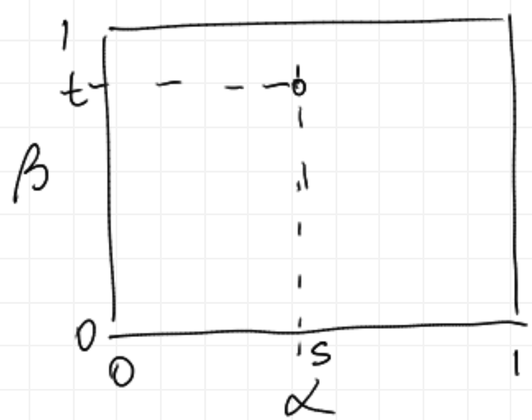
Alt, Godau '95

alg. to compute Frechet distance between 2 polygonal curves with a, b vertices

- decision version $O(ab)$
- computing $O(ab \log ab)$

} in real RAM.

Main Idea



choosing position on α, on β gives a pt. in rectangle

$d(\alpha(s), \beta(t))$

= leash length for those positions.

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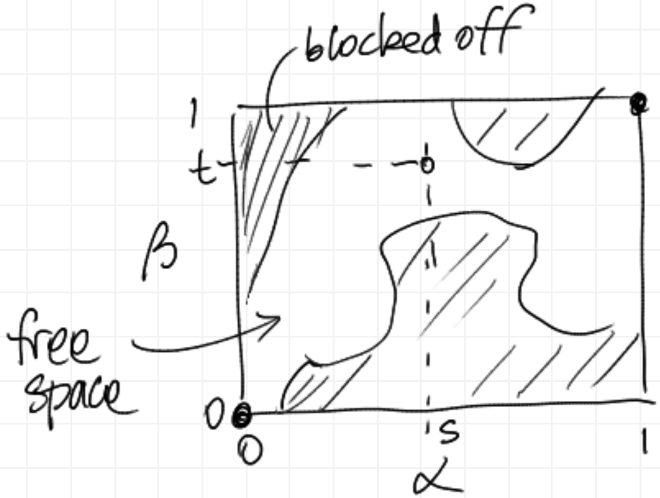

leash $\leq \varepsilon$

For decision problem, block off pts in rectangle where leash is $> \varepsilon$.

ε-diagram

See Fig. 3.

Lemma Frechet distance is $\leq \varepsilon$ iff there is a monotone path from $(0,0)$ to $(1,1)$ in the free space of the ε-diagram.

monotone = x coord never decreases
y " " "

Question: what does the length of the path correspond to?

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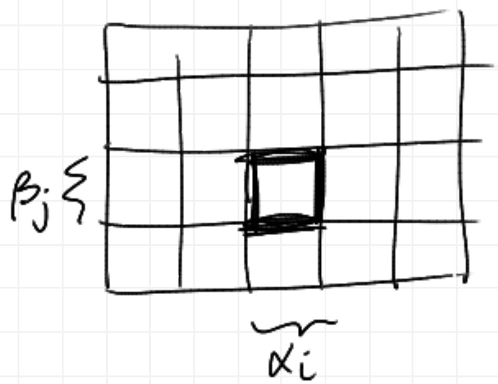
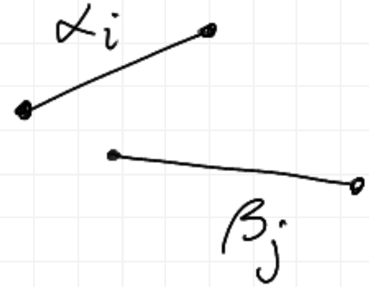
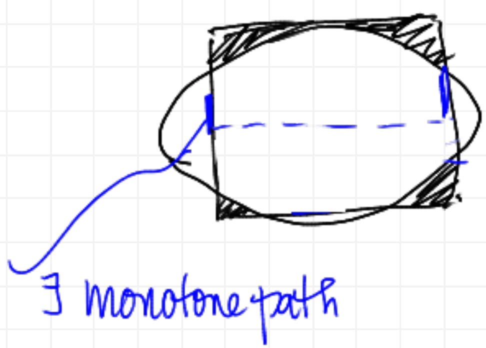
ε-diagram has $a \times b$ rectangles – each corresponds to one segment of α versus one segment of β.





<u>Lemma.</u> Inside one rectangle the free space is an ellipse (Fig.4)

Along each side, the free space is an interval



If we know the subintervals on left & bottom reachable via monotone path from $(0,0)$ then we can compute same for right / top in constant time.

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This solves decision problem in $O(ab)$ time.

To actually compute the Frechet distance:

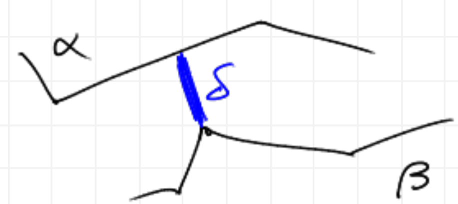
We need $\varepsilon > d(\alpha(0), \beta(0))$ initial & final
$\varepsilon > d(\alpha(1), \beta(1))$ leash lengths.

Starting from this ε_0, imagine increasing ε.

Combinatorial changes at critical values of ε

1. a new passage opens between neighbouring cells
2. a new horizontal/vertical passage opens.

1 corresponds to



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Idea:

- determine all critical values (there are $O(a^2b+b^2a)$ of them)
- sort them
- do binary search for ε (use decision alg.)

all ε values

test this ε and recurse on appropriate side

Time $O\Big(\underbrace{(a^2b+b^2a)\log(ab)}_{\text{sort}} + \underbrace{ab}_{\text{decision}}\ \underbrace{\log(ab)}_{\text{binary search}}\Big)$

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For improved run-time of $O(ab \log(ab))$
use Megiddo's parametric search

(+ Cole's variant using Ajtai-Komlos-Szemeredi sorting network — high constant and impractical).

See survey paper by Agarwal & Sharir.

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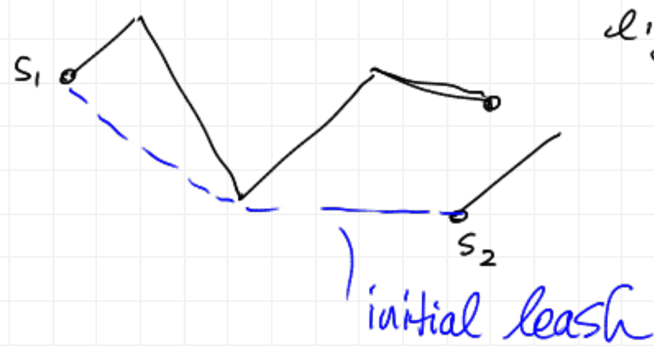
Further work

- can get run time below $O(n^2 \log n)$ – at least for $(1+\varepsilon)$-approx.
- Frechet distance inside a polygon – the leash must remain inside



e.g. the curves themselves might act as boundaries

- Frechet distance between 2 surfaces (up one dimension).