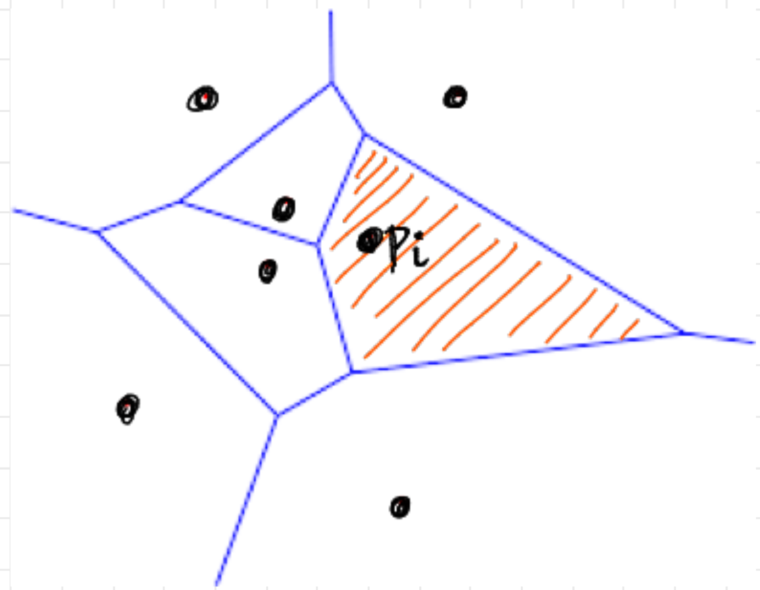
Voronoi diagram

Given points $P = \{p_1, . . . , p_n\}$ in the plane, the ***Voronoi region*** of p_i is

$$V(p_i) = \{x \in \mathbb{R}^2 : d(x, p_i) \leq d(x, p_j) \forall j \neq i\}$$

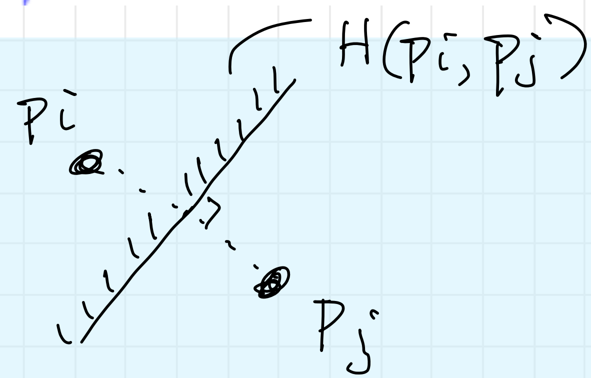
p_i is called a ***site***.

The ***Voronoi diagram*** $\mathcal{V}(P)$ consists of all the Voronoi regions



To see that the Voronoi diagram consists of straight line segments:

$H(p_i, p_j)$ = points closer to p_i than p_j — a half-plane.



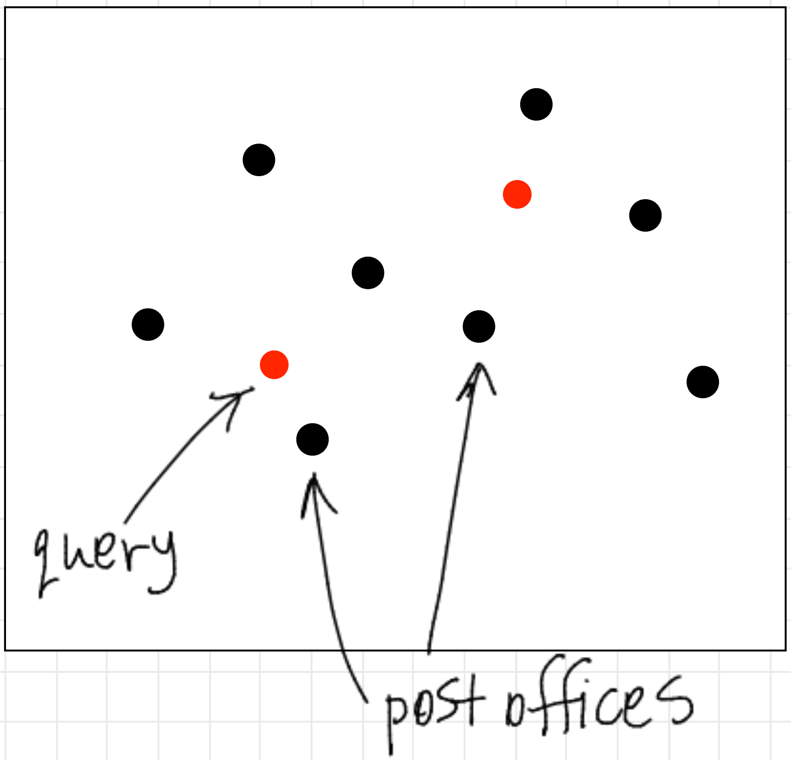
$V(p_i) = \bigcap_{j \neq i} H(p_i, p_j)$

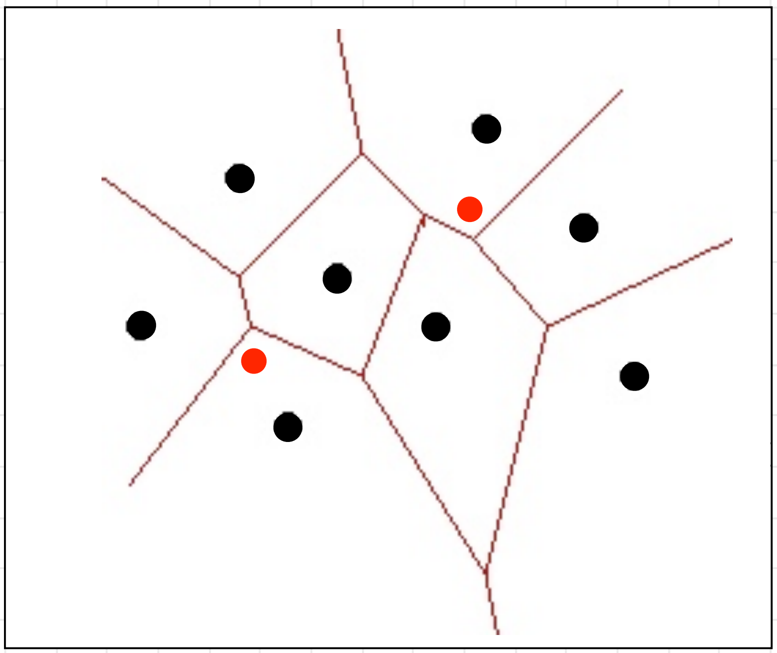
∴ $V(p_i)$ is bounded by a convex polygon (maybe unbounded)

Recall

Application: Post Office Problem — also closest airfield for airplane in flight.

Given *n* points in the plane (post offices) preprocess to handle query:
which post office is closest to query point *p*





Compute the Voronoi diagram.
Query becomes: which region contains *p* ?
This is ***Planar Point Location***.

History

W https://en.wikipedia.org/wiki/Voronoi_diagram

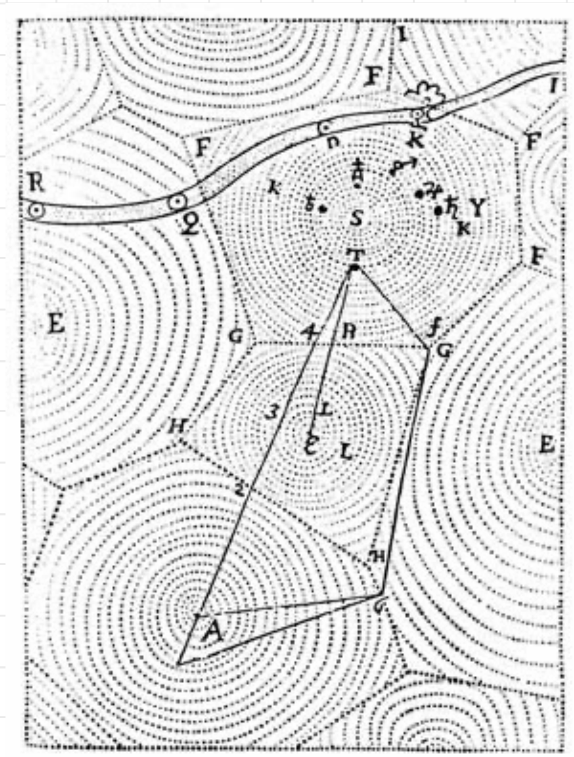
Georgy Voronoy, 1908, Russian/Ukrainian mathematician

but Voronoi diagrams were used before that:
- Descartes 1644
- Dirichlet 1850, “Dirichlet tessellation” - used to prove unique reducibility of quadratic forms

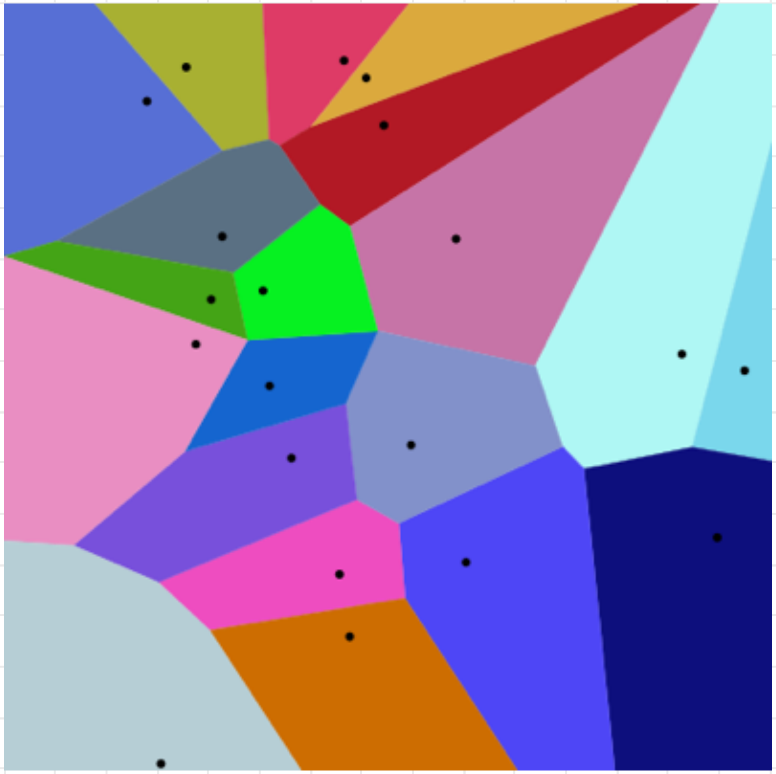
also used in
- crystallography - crystals growing outward from “seeds” form Voronoi regions
- epidemiology, 1854, mapping cholera by proximity to infected water pump in London
- geography and meteorology - “Thiessen polygons” 1911
- condensed matter physics - “ Wigner–Seitz cells”
- natural sciences - “Blum’s transform” = medial axis of polygon, used to describe shapes
- aviation, to identify nearest airfield

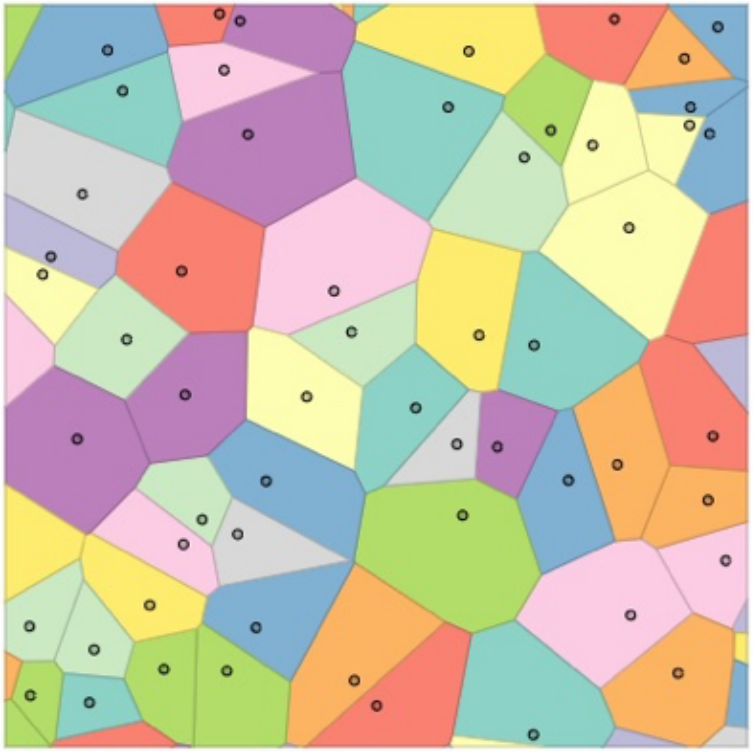
dual structure: ***Delaunay triangulation***

Boris Delone (but used French transliteration “Delaunay”), 1934



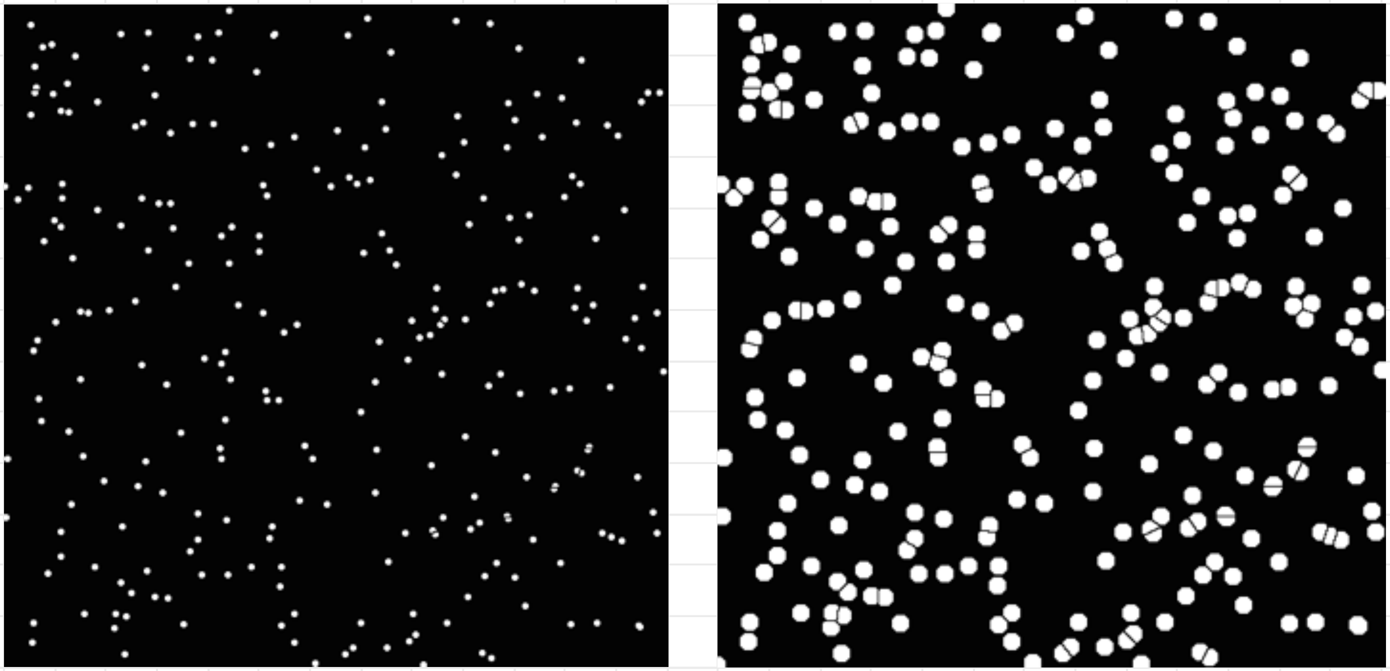
Descartes 1644



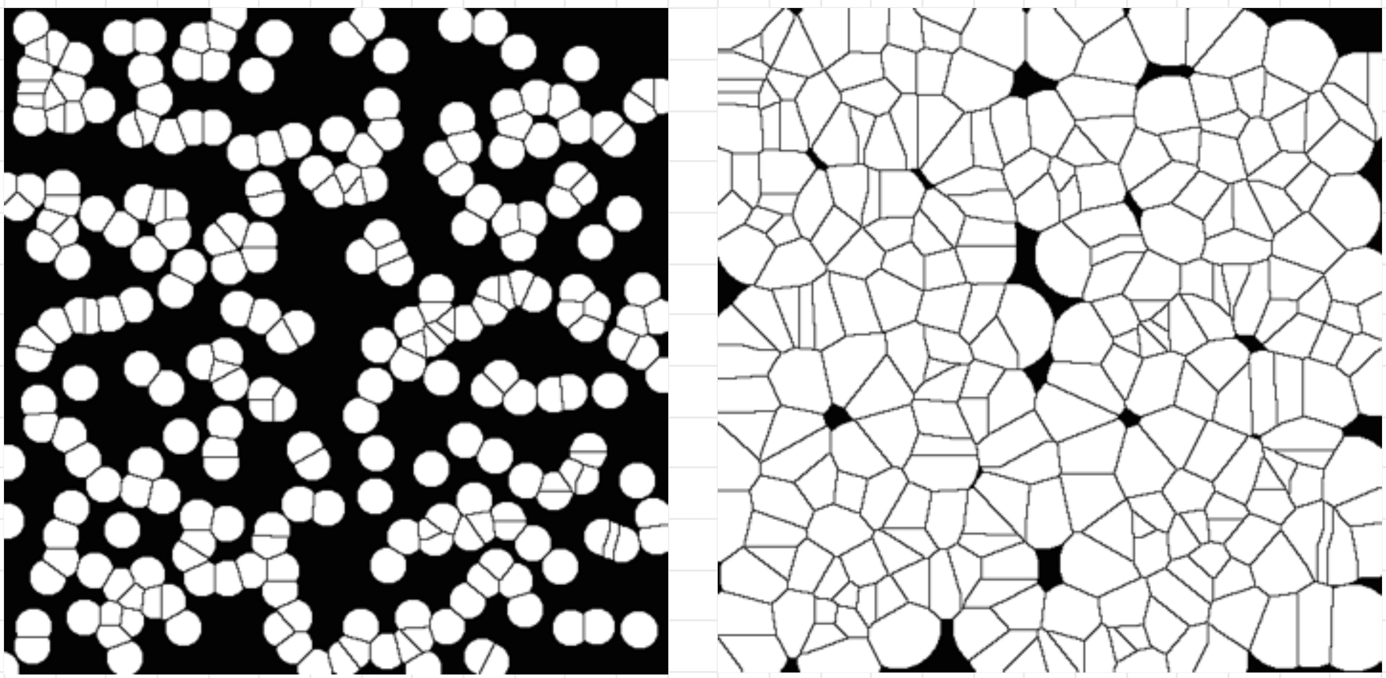


Some nice illustrations, including Voronoi diagram of moving points

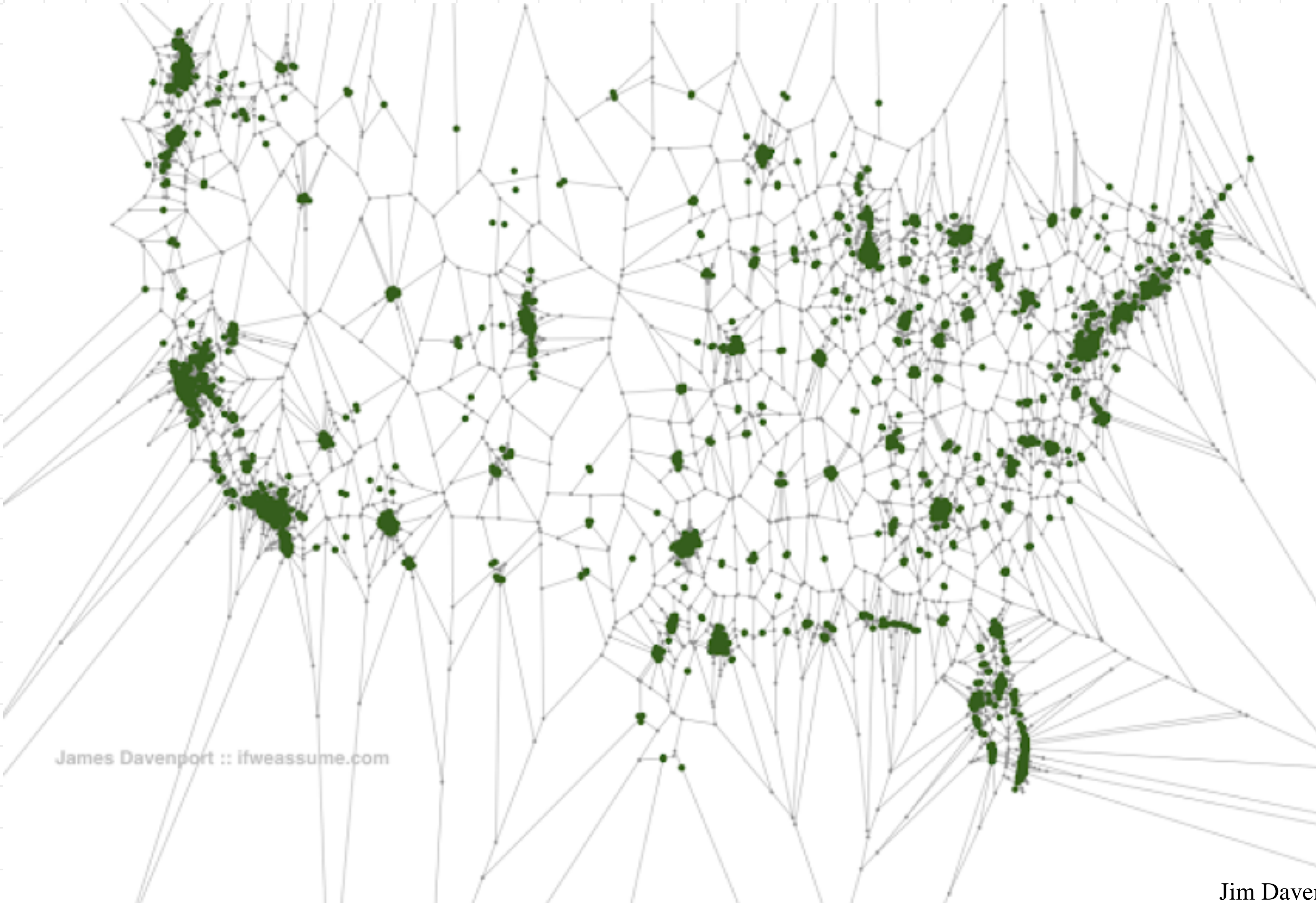
- http://datagenetics.com/blog/may12017/index.html
- http://datagenetics.com/blog/may12017/anim2.gif — many pts moving
- https://www.youtube.com/watch?v=z3yy5aBv0ug — one pt moving



http://i.stack.imgur.com/gcajM.gif

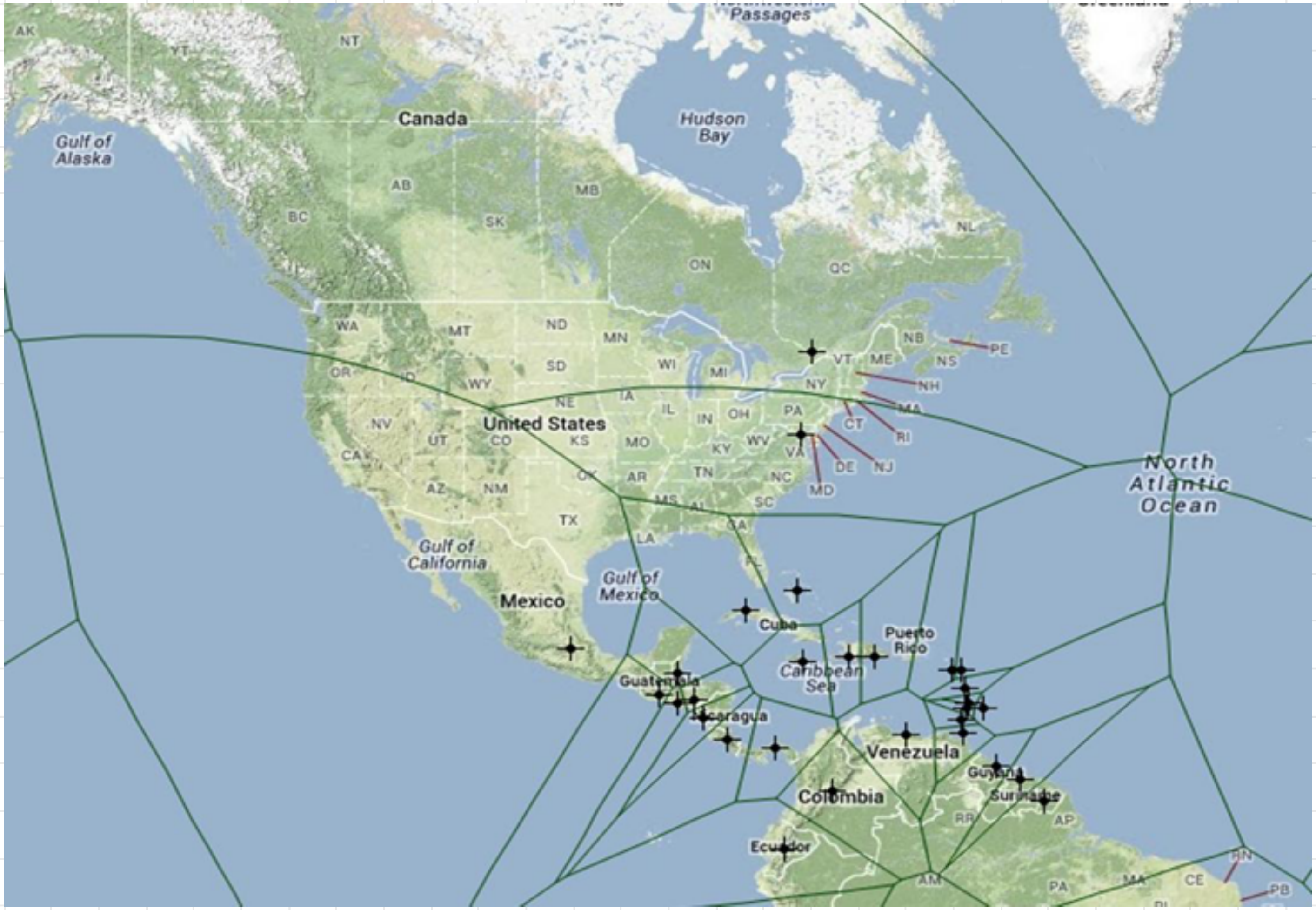


Voronoi diagram of US Starbucks

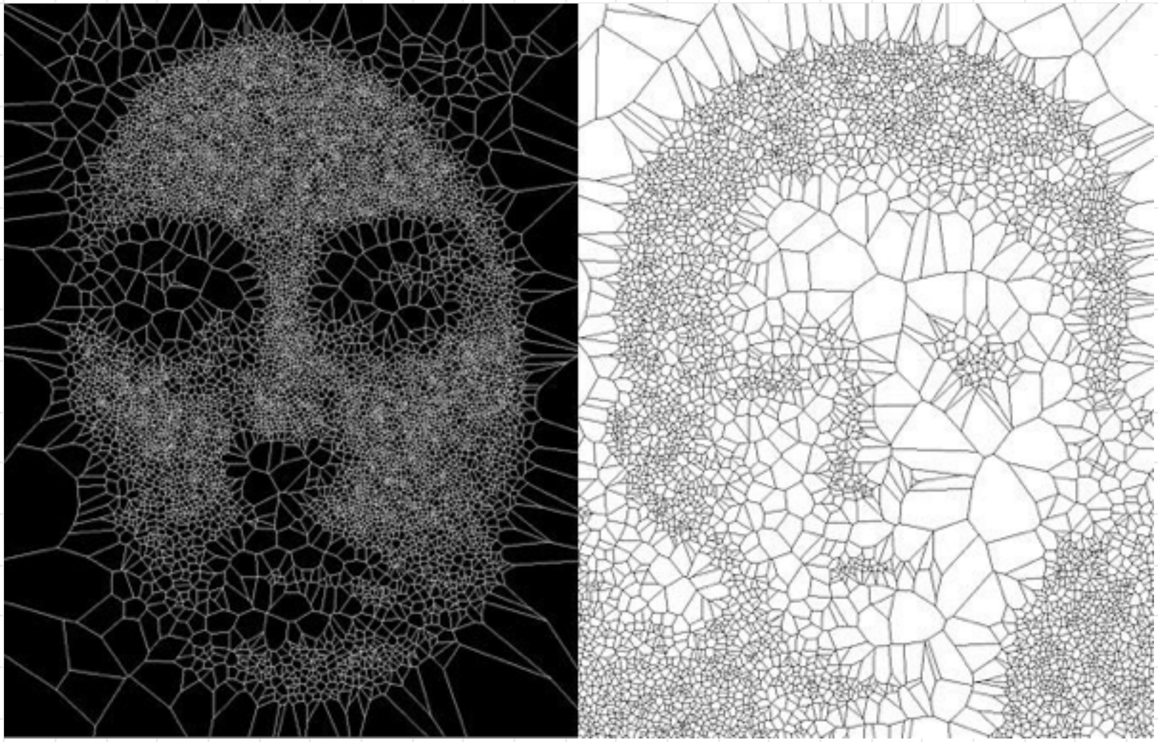


Jim Davenport

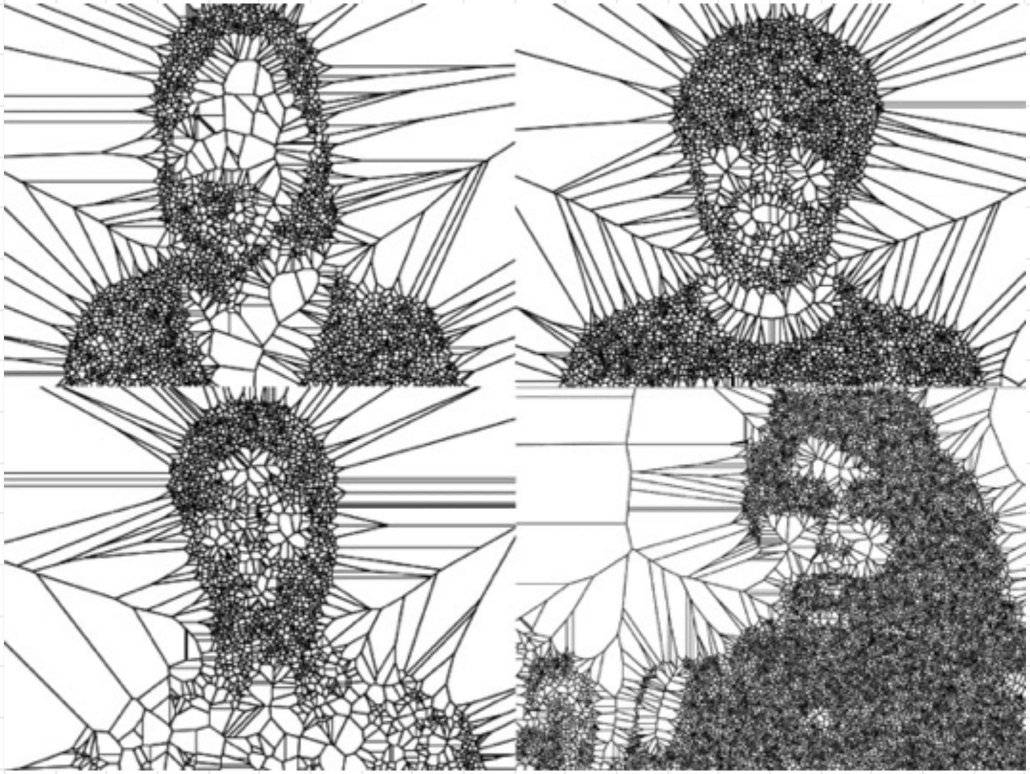
Voronoi diagram of capital cities



or of airports

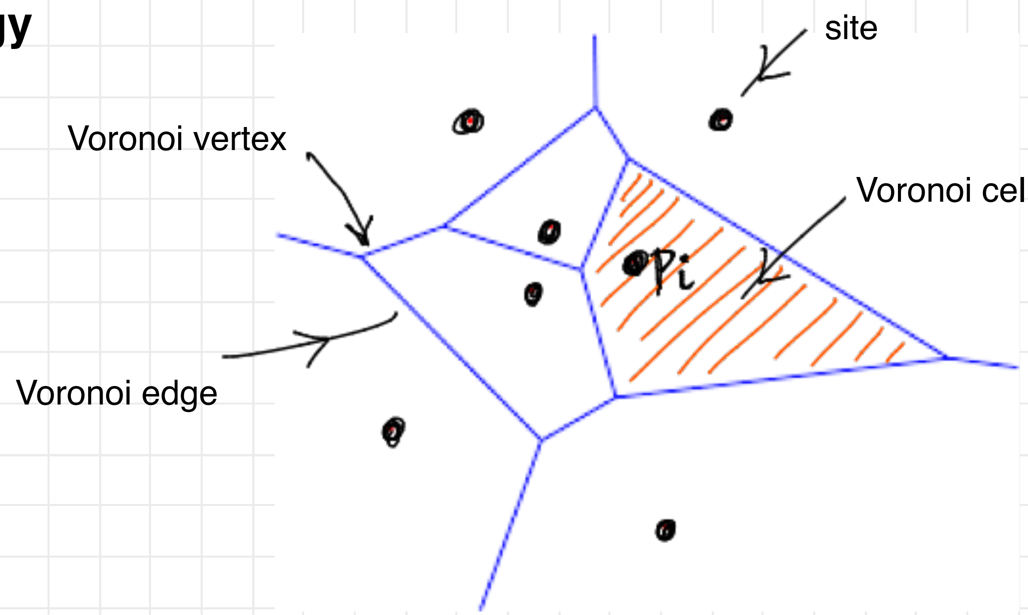


http://www.flong.com/projects/zoo/



http://blog.kleebtronics.com/voronoi/

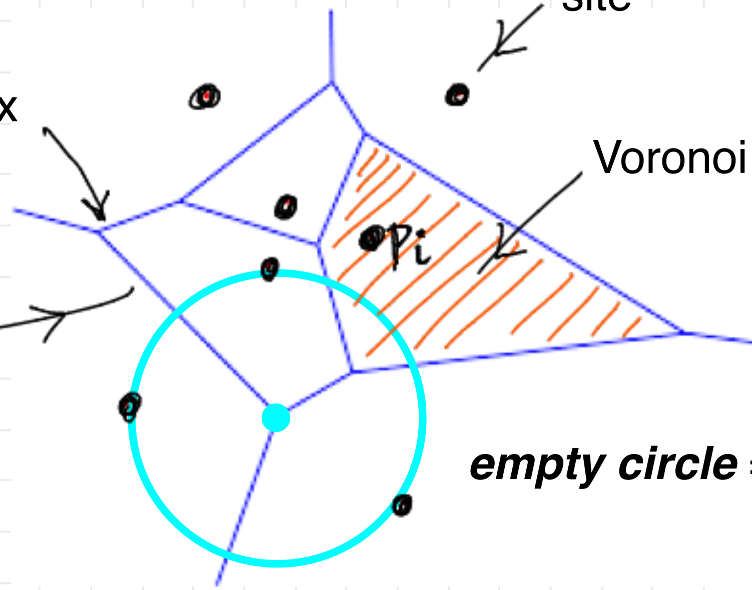
Terminology



Properties

- ***empty circle*** property
- degrees of Voronoi vertices?
- shape of Voronoi cells?
- when is a cell unbounded?
- number of vertices/edges/cells

Terminology



empty circle = no sites inside

Properties

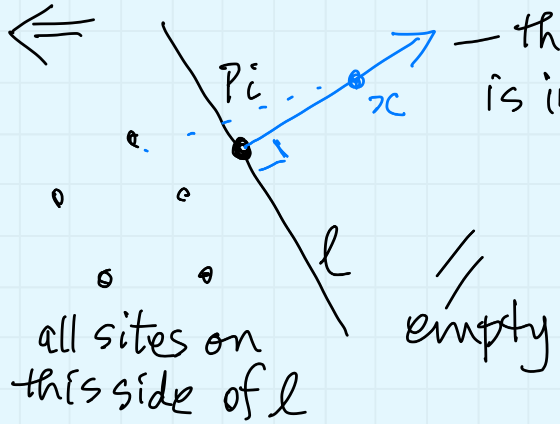
- ***empty circle*** property — there is an empty circle centered at each Voronoi vertex, going through 3 or more sites (#sites = degree of vertex)
- degrees of Voronoi vertices? — we will assume no 3 points on line, no 4 points on circle. Then Voronoi vertices have degree 3
- shape of Voronoi cells? — convex regions bounded by straight lines
- when is a cell unbounded? — iff site on convex hull — proof next
- number of vertices/edges/cells — see next

n

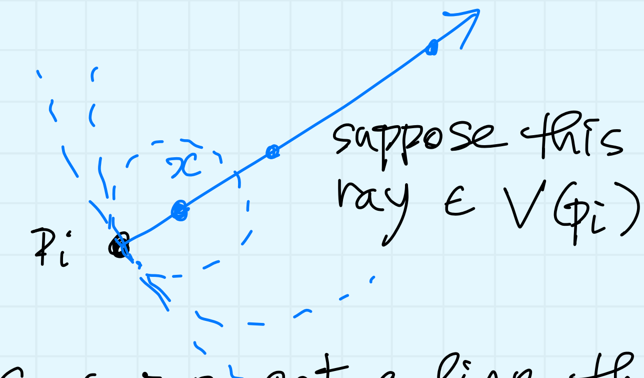
Properties of Voronoi diagram

$V(p_i)$ is unbounded iff p_i is on the convex hull of the sites

$\Leftarrow$



$\Rightarrow$



As $x \to \infty$ get a line through p_i with all p_j on the far side.
$\therefore$ p_i on convex hull.

There are $\leq 2n$ Voronoi vertices and $\leq 3n$ Voronoi edges

Euler $2 = v - e + f$ $\quad f = \#\text{sites} = n$

but some edges are rays — imagine they all meet at a point at infinity

every vertex has degree ≥ 3

$2e = \sum_v \text{degree} \geq 3v$

$2 = v - e + f \leq v - \frac{3}{2}v + n$

$v \leq 2(n-2)$

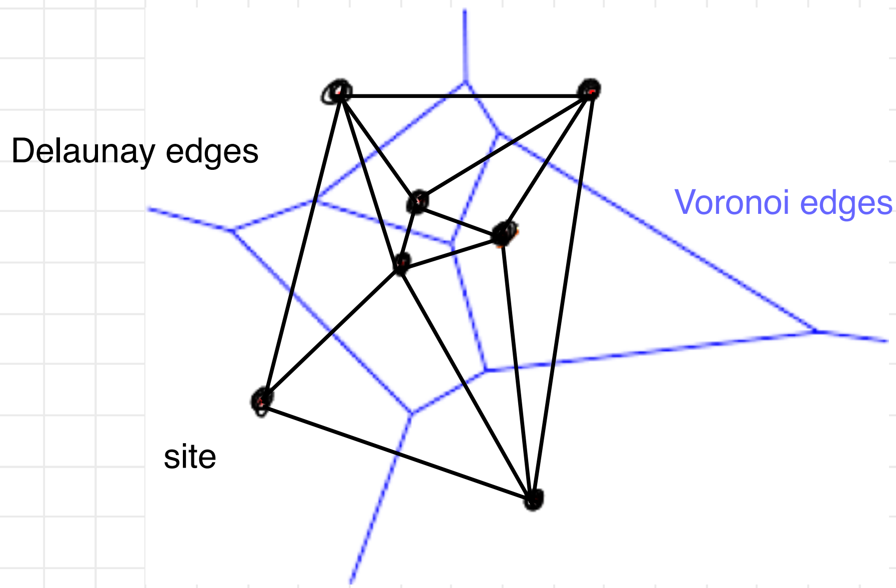
$e \leq 3(n-2)$

Delaunay triangulation

The Voronoi diagram can be captured by a purely combinatorical structure (versus computing coordinates of Voronoi vertices)

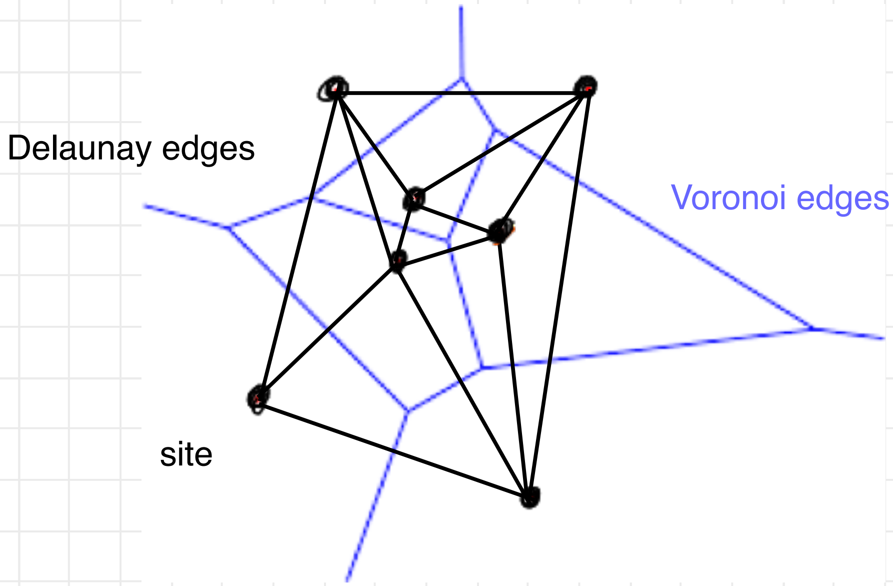
Given points $P = \{p_1, . . . , p_n\}$ in the plane, the ***Delaunay triangulation*** $\mathcal{D}(P)$ is a graph with vertices $p_1, . . . , p_n$ and edge (p_i, p_j) iff $V(p_i)$ and $V(p_j)$ share an edge.

$\mathcal{D}(P)$ is the ***planar dual*** of $\mathcal{V}(P)$



Note: a Voronoi edge and its corresponding Delaunay edge do not always cross.

Delaunay triangulation $\mathcal{D}(P)$



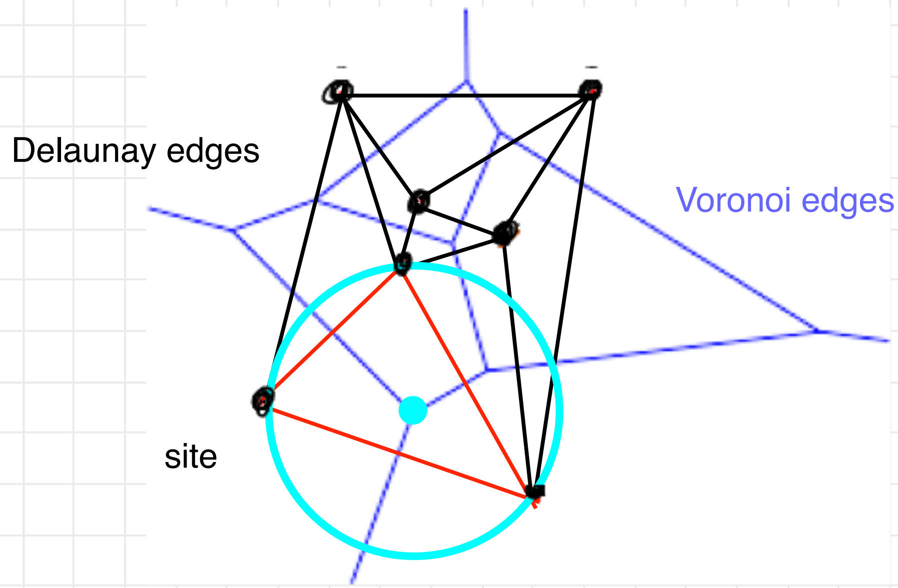
Properties

- it is a triangulation

- edge/face iff empty circle through those sites

- what is the boundary ?

Delaunay triangulation $\mathcal{D}(P)$



Properties

- it is a triangulation

- edge/face iff empty circle through those sites

- what is the boundary ?

see next slides

— the convex hull (to be proved)

Properties of Delaunay triangulations

(p_i, p_j) is an edge iff there is an empty circle through $p_i p_j$

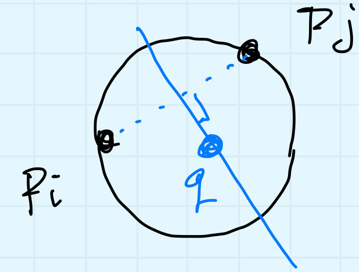
no sites inside

Proof

(p_i, p_j) is an edge of $\mathcal{D}(P)$

iff $V(p_i)$ and $V(p_j)$ share a boundary edge with point q on it

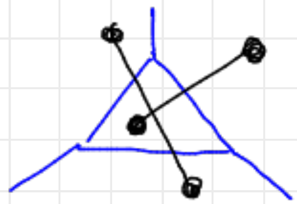
iff circle centered at q is empty circle through p_i and p_j



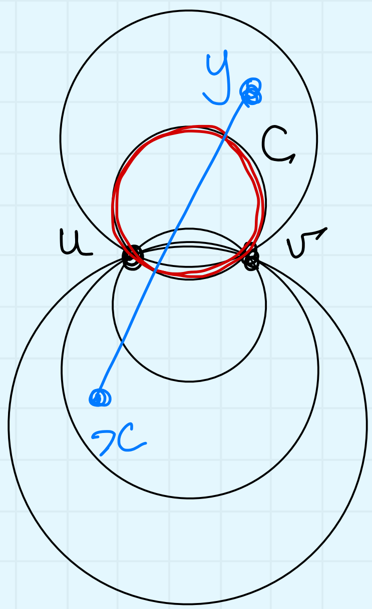
Properties of Delaunay triangulations

No two Delaunay edges cross

Note that this must be proved — it is not true for general planar maps



Proof



Suppose two Delaunay edges (u, v) and (x, y) cross

by previous, ∃ empty circle C through u, v

Edge xy crosses uv and x and y are outside C

Then ∄ empty circle through x and y

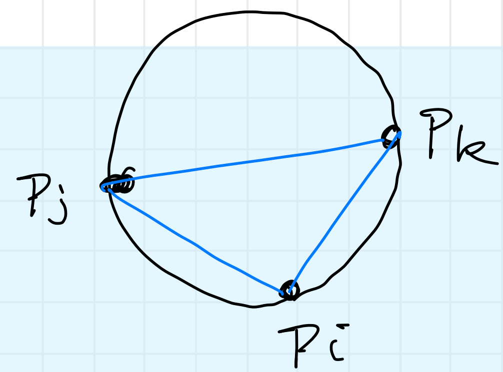
(if x and y are on C then 4 pts on circle — NO.
else any circle through x y contains u or v)

Properties of Delaunay triangulations

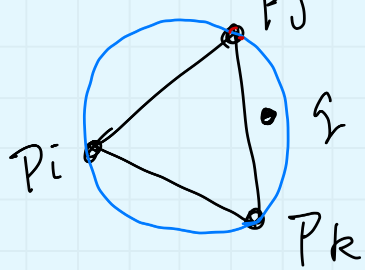
Sites form a face iff there is an empty circle through them
i.e., (since we assumed no 4 sites on a circle)
$p_i \; p_j \; p_k$ form a face iff there is an empty circle through them

Proof

$\Leftarrow$ suppose empty circle
by previous, get 3 edges
and nothing inside since
no edges cross.



$\Rightarrow$ take circle C through Pi Pj Pk
Can there be a site q in C?
then $\nexists$ empty circle through
Pj Pk (for this q)



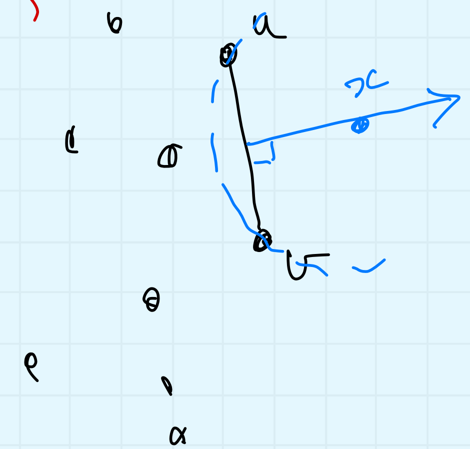
Properties of Delaunay triangulations

The boundary of the Delaunay triangulation is the convex hull of the sites

Proof

i.e. a Delaunay edge has no triangle on one side iff the edge is on CH of sites.

⇐



uv on CH

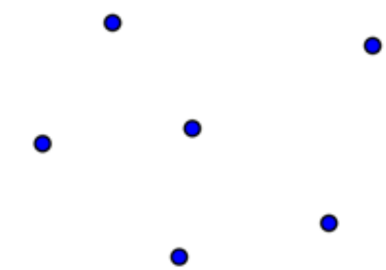
⇒ inf. ray and any pt x on ray is center of empty circle through uv

⇒ no triangle on that side (~~it~~ it's empty circle would have to go through another site)

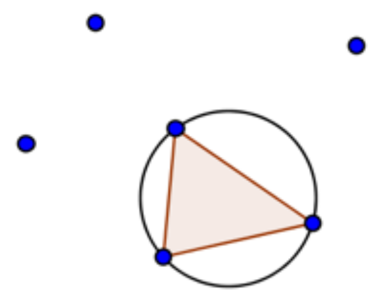
⇒ follow same proof backwards.

An alternative definition of Delaunay triangulation:

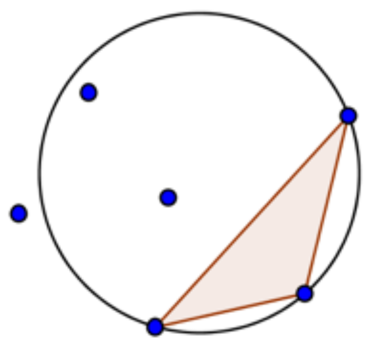
for each empty circle through 3 points, add a triangle



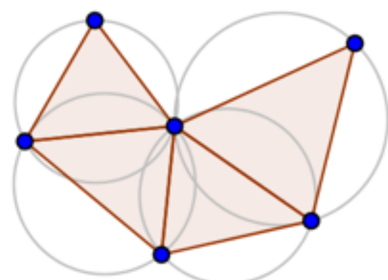
A collection of points



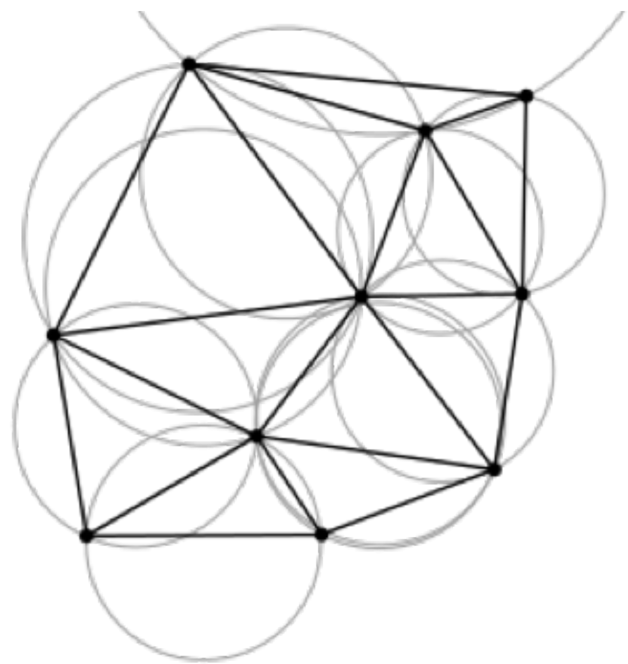
A Delaunay triangle



A non-Delaunay triangle



(part of)
A Delaunay triangulation

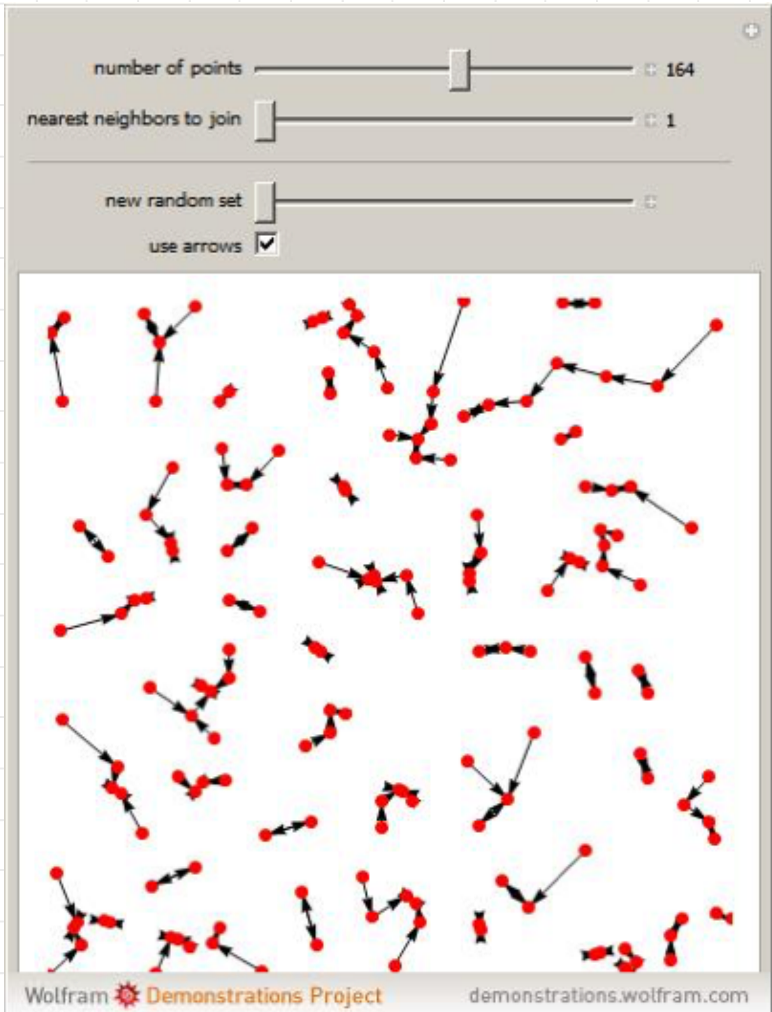


Delaunay triangulation

and from the Delaunay triangulation we can construct Voronoi diagram.

Application of Delaunay triangulations: finding all nearest neighbours

Given n points in the plane find, for each point, its nearest neighbour — gives ***nearest neighbour graph***, a directed graph of out-degree 1.



Many applications, e.g.

in statistical analysis:
find hierarchical clusters using nearest neighbour chain algorithm

More on this in the next lecture.

https://demonstrations.wolfram.com/NearestNeighborNetworks/

Summary

- Voronoi diagram and Delaunay triangulation
  definitions, relationships, properties

References

- [CGAA] Chapters 7, 9
- [Zurich notes] Chapters 5, 7 (they start with Delaunay)
- [O'Rourke] Chapter 5
- [Devadoss-O'Rourke] Chapter 4.