Voronoi diagram

Given points $P = \{p_1, \ldots, p_n\}$ in the plane, the **Voronoi region** of $p_i$ is

$$V(p_i) = \{x \in \mathbb{R}^2 : d(x, p_i) \leq d(x, p_j) \forall j \neq i\}$$

$p_i$ is called a **site**.

The **Voronoi diagram** $\mathcal{V}(P)$ consists of all the Voronoi regions.

To see that the Voronoi diagram consists of straight line segments:

$$H(p_i, p_j) = \text{points closer to } p_i \text{ than } p_j$$

$$V(p_i) = \bigcap_{j \neq i} H(p_i, p_j)$$

$\therefore V(p_i)$ is bounded by a convex polygon.
Application: Post Office Problem — also closest airfield for airplane in flight.

Given $n$ points in the plane (post offices) preprocess to handle query:
which post office is closest to query point $p$

Compute the Voronoi diagram. Query becomes: which region contains $p$?
This is **Planar Point Location**.
History


Georgy Voronoy, 1908, Russian/Ukrainian mathematician

but Voronoi diagrams were used before that:
- Descartes 1644
- Dirichlet 1850, “Dirichlet tessellation” - used to prove unique reducibility of quadratic forms

also used in
- crystallography - crystals growing outward from “seeds” form Voronoi regions
- epidemiology, 1854, mapping cholera by proximity to infected water pump in London
- geography and meteorology - “Thiessen polygons” 1911
- condensed matter physics - “Wigner–Seitz cells”
- natural sciences - “Blum’s transform” = medial axis of polygon, used to describe shapes
- aviation, to identify nearest airfield

dual structure: *Delaunay triangulation*

Boris Delone (but used French transliteration “Delaunay”), 1934
Descartes 1644
Some nice illustrations, including Voronoi diagram of moving points

- [http://datagenetics.com/blog/may12017/index.html](http://datagenetics.com/blog/may12017/index.html)
- [http://datagenetics.com/blog/may12017/anim2.gif](http://datagenetics.com/blog/may12017/anim2.gif)
- [https://www.youtube.com/watch?v=z3yy5aBv0ug](https://www.youtube.com/watch?v=z3yy5aBv0ug)
Voronoi diagram of US Starbucks
Voronoi diagram of capital cities

or of airports
http://blog.kleebtronics.com/voronoi/
Terminology

- site
- Voronoi vertex
- Voronoi edge
- Voronoi cell

Properties

- empty circle property
- degrees of Voronoi vertices?
- shape of Voronoi cells?
- when is a cell unbounded?
- number of vertices/edges/cells
Terminology
- Voronoi vertex
- Voronoi edge
- site

Properties
- empty circle property
- degrees of Voronoi vertices?
- shape of Voronoi cells?
- when is a cell unbounded?
- number of vertices/edges/cells

empty circle = no sites inside

there is an empty circle centered at each Voronoi vertex, going through 3 or more sites
we will assume no 3 points on line, no 4 points on circle
then Voronoi vertices have degree 3
convex regions bounded by straight lines
iff site on convex hull — proof next

n

see next
Properties of Voronoi diagram

\( V(p_i) \) is unbounded iff \( p_i \) is on the convex hull of the sites

There are \( \leq 2n \) Voronoi vertices and \( \leq 3n \) Voronoi edges

Euler \( \, 2 = v - e + f \)

but some edges are rays — imagine they all meet at a point at infinity

\( \sum_{v_i} \text{degree} \geq 3 \)\n
\( 2e = \sum_{v_i} \text{degree} \geq 3v \)

\( 2 = v - e + f \leq v - \frac{3}{2}v + n \)

\( v \leq 2(n-2) \)

\( e \leq 3(n-2) \)
**Delaunay triangulation**

The Voronoi diagram can be captured by a purely combinatorial structure (versus computing coordinates of Voronoi vertices).

Given points $P = \{p_1, \ldots, p_n\}$ in the plane, the **Delaunay triangulation** $\mathcal{D}(P)$ is a graph with vertices $p_1, \ldots, p_n$ and edge $(p_i, p_j)$ iff $V(p_i)$ and $V(p_j)$ share an edge.

$\mathcal{D}(P)$ is the *planar dual* of $\mathcal{V}(P)$.

Note: a Voronoi edge and its corresponding Delaunay edge do not always cross.
Delaunay triangulation \( \mathcal{D}(P) \)

Properties

- it is a triangulation

- edge/face iff empty circle through those sites

- what is the boundary?
Delaunay triangulation $\mathcal{D}(P)$

**Properties**

- it is a triangulation
- edge/face iff empty circle through those sites
- what is the boundary?

see next slides

- the convex hull (to be proved)
Properties of Delaunay triangulations

\((p_i, p_j)\) is an edge iff there is an empty circle through \(p_i, p_j\)

Proof

\((p_i, p_j)\) is an edge of \(D(P)\) iff \(V(p_i)\) and \(V(p_j)\) share a boundary edge with point \(q\) on it iff circle centered at \(q\) is empty circle through \(p_i\) and \(p_j\)
Properties of Delaunay triangulations

No two Delaunay edges cross

Note that this must be proved — it is not true for general planar maps.

Proof

Suppose two Delaunay edges \((u, v)\) and \((x, y)\) cross by previous, \(\exists\) empty circle \(C\) through \(u, v\).

Edge \(x y\) crosses \(uv\) and \(x\) and \(y\) are outside \(C\).

Then \(\nexists\) empty circle through \(x\) and \(y\) (if \(x\) and \(y\) are on \(C\) then 4 pts on circle — NO. else any circle through \(x y\) contains \(u\) or \(v\)).
Properties of Delaunay triangulations

Sites form a face iff there is an empty circle through them
i.e., (since we assumed no 4 sites on a circle)
$p_i, p_j, p_k$ form a face iff there is an empty circle through them

Proof

$\Leftarrow$ Suppose empty circle
by previous, get 3 edges
and nothing inside since
no edges cross.

$\Rightarrow$

Can there be a site $q$ in $C$?
then $\exists$ empty circle through
$p_i, p_j, p_k$ (for this $I$)
Properties of Delaunay triangulations

The boundary of the Delaunay triangulation is the convex hull of the sites

Proof

i.e. a Delaunay edge has no triangle on one side iff the edge is on CH of sites.

\[ \Rightarrow \inf \text{ ray and any pt } x \text{ on ray is center of empty circle through } uv \]

\[ \Rightarrow \text{ no triangle on that side (it would have to go through another site) } \]

\[ \Rightarrow \text{ follow same proof backwards. } \]
An alternative definition of Delaunay triangulation:

for each empty circle through 3 points, add a triangle.
Application of Delaunay triangulations: finding all nearest neighbours

Given n points in the plane find, for each point, its nearest neighbour — gives nearest neighbour graph, a directed graph of out-degree 1.

Many applications, e.g.

in statistical analysis:
find hierarchical clusters using nearest neighbour chain algorithm

More on this in the next lecture.
Summary

- Voronoi diagram and Delaunay triangulation definitions, relationships, properties

References

- [CGAA] Chapters 7, 9

- [Zurich notes] Chapters 5, 7 (they start with Delaunay)

- [O’Rourke] Chapter 5

- [Devadoss-O’Rourke] Chapter 4.