Recall

- every polygon can be triangulated
- there is an $O(n \log n)$ algorithm to triangulate polygons and polygonal regions
- this is best possible for polygonal regions
- polygons can be triangulated in O(n) via Chazelle's hard algorithm

Optimizing Polygon Triangulations

We may have criteria to prefer one triangulation over another.
For meshing, angles are often the main issue — more on this later.

Minimum Weight Triangulation ("minimum ink") — minimize the sum of the lengths of the chords used. (length = Euclidean length)

Polynomial Time Algorithm for Min Weight Triangulation of a Polygon (Klincsek 1980)

Idea: dynamic programming

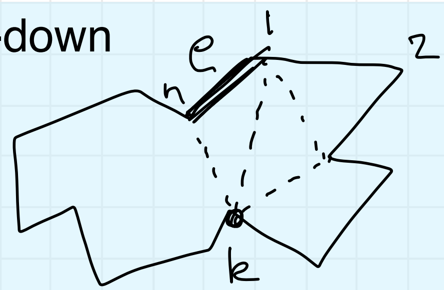
Input: Polygon P on vertices 1, 2, . . . , n

general steps for dynamic programming:

- what are subproblems
- formula for solving a subproblems in terms of smaller ones
- order of solving subproblems.

Polynomial Time Algorithm for Min Weight Triangulation of a Polygon

think top-down



special edge e
and try all possibilities for the triangle on e

subproblems

$\forall\ i<j$ subpolygon P_{ij} vertex $i, i+1, \ldots, j$ + chord ij — $O(n^2)$ subproblems

M_{ij} = min. weight of triangulation of P_{ij}.

$$= \begin{cases} 0 & \text{if } j = i+1 \quad \text{(edges don't count)} \\ \cancel{-1}\ +\infty & \text{if } ij \text{ is not a chord.} \\ d_{ij} + \min_{k,\ i<k<j} \{ M_{ik} + M_{kj} \} \end{cases}$$

d_{ij} ← length of chord ij

ordering: by $j-i = \Delta$

for $\Delta = 2 \ldots n-1$
  for $i = 1 \ldots n-\Delta$
    $j = i + \Delta$

Run Time: $O(n^2)$ subproblems $\times$ $O(n)$ to solve one

Total: $O(n^3)$

Polynomial Time Algorithm for Min Weight Triangulation of a Polygon

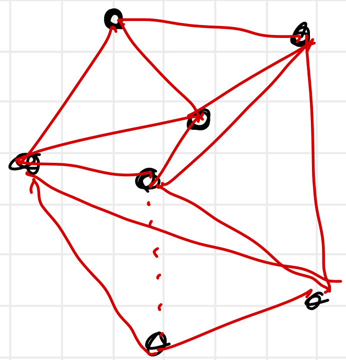
dynamic programming $O(n^3)$

Open Problem. $o(n^3)$ time algorithm for min weight triangulation.

and more generally, there are quite a few dynamic programming algorithms that no one knows how to improve

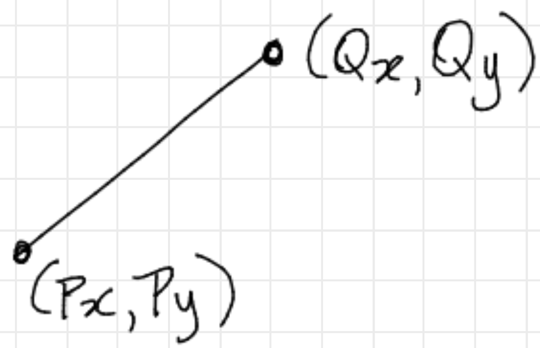
Min weight triangulation of a point set is NP-hard.

Minimum-weight triangulation is NP-hard
W Mulzer, G Rote - Journal of the ACM (JACM), 2008 - dl.acm.org
d https://doi.org/10.1145/1346330.1346336



Real RAM Model of Computation

The above algorithm assumed we can compute Euclidean lengths and compare them at unit cost.



Euclidean length

$$\sqrt{(Q_x - P_x)^2 + (Q_y - P_y)^2}$$

We used a test of the form:

$$\sqrt{a_1} + \sqrt{a_2} + \cdots \sqrt{a_k} \overset{?}{\gtreqless} \sqrt{b_1} + \sqrt{b_2} + \cdots \sqrt{b_\ell}$$

a_i, b_i integer,

OPEN: Measuring bit complexity, can this test be done in polynomial time?

Real RAM Model of Computation

real RAM — random access machine that operates on real numbers.

Allows arithmetic, including square root, at unit cost.
Sometimes also allow k-th roots, trigonometric functions, etc.

This is a basic model of computation used in computational geometry.
In practice, we must address issues of precision.

For an interesting discussion of alternate models of computing for computational geometry, see section 1.2 in:

Finding Closed Quasigeodesics on Convex Polyhedra
ED **Demaine**, AC Hesterberg… - … Geometry (SoCG 2020), 2020 - drops.dagstuhl.de
https://drops.dagstuhl.de/opus/volltexte/2020/12191/

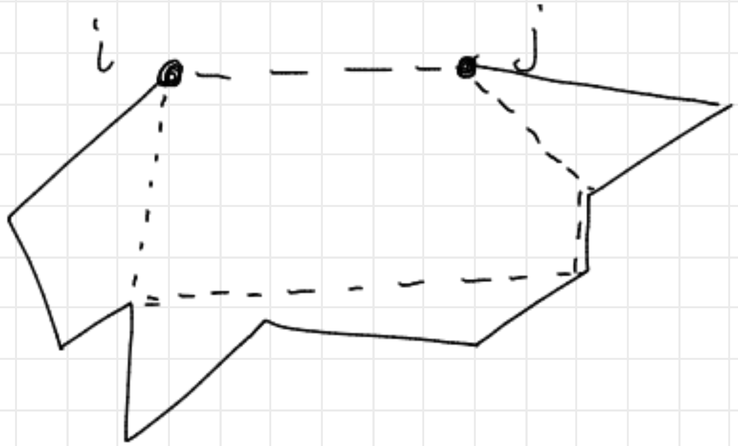
Partitioning Polygons into Convex Pieces

Sometimes we don't really need triangles — convex pieces will do.

Partitioning a polygon into a minimum number of convex pieces by adding chords.

O(n^3 log n) using dynamic programming. Keil 1985.

Decomposing a polygon into simpler components
JM **Keil** - SIAM Journal on Computing, **1985** - SIAM
d https://doi.org/10.1137/0214056



Careful — there can be exponentially many subpolygons.

Partitioning Polygons into Convex Pieces

A faster approximation algorithm
Hertel and Mehlhorn 1983.

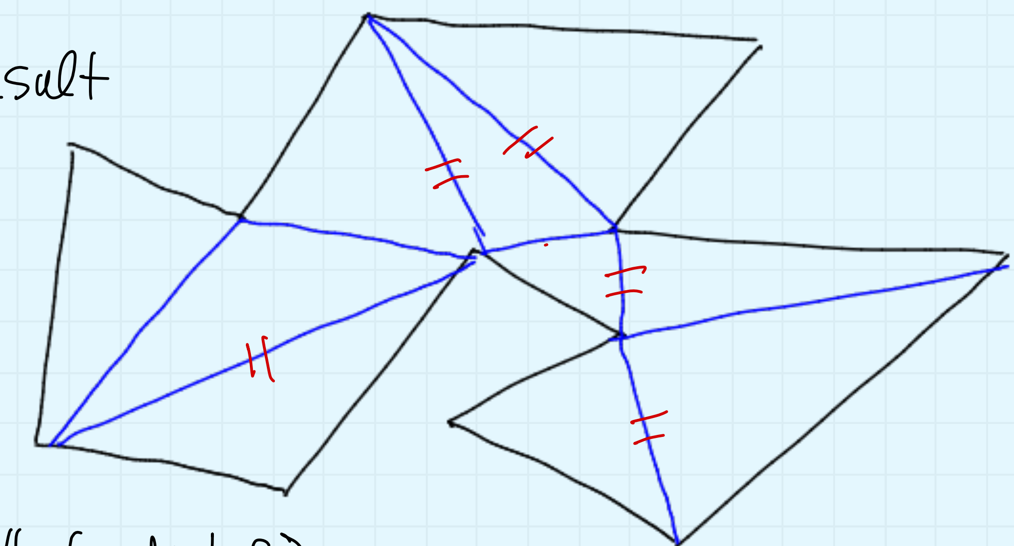
O(n log n) time and finds number of pieces ≤ 4 x min

Start with a triangulation

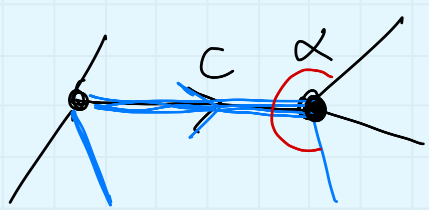
- remove a chord if result leaves convex pieces.

How to get O(n log n)? Ex.

Claim # pieces ≤ 4·min.



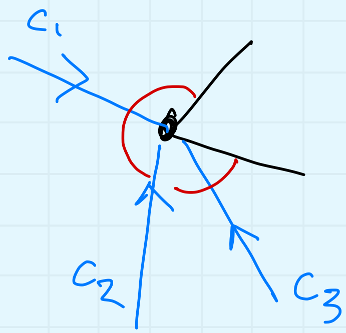
Pf. Analyze # chords left (# pieces = 1 + # chords left)



if chord c cannot be removed then at one endpt. we have a reflex angle α

Next: argue: at most 2 chords point to same vertex

Claim: number of pieces is ≤ 4 x min



impossible for 3 (or more chords) that are left to cause reflex angles.

these are left.

$\#$ chords $\leq 2r$ $\qquad r = \#$ reflex vertices

$\#$ pieces $\leq 2r+1$

On the other hand, every reflex vertex needs at least one chord, and each chord can kill off at most 2 reflex vertices

$$\min \# \text{ chords} \geq \left\lceil \frac{r}{2} \right\rceil$$

$$\min \# \text{ pieces} \geq \frac{r}{2}+1$$

$$\# \text{pieces} \leq 2r+1 \leq 2r+4 \leq 4 \min \# \text{pieces}$$

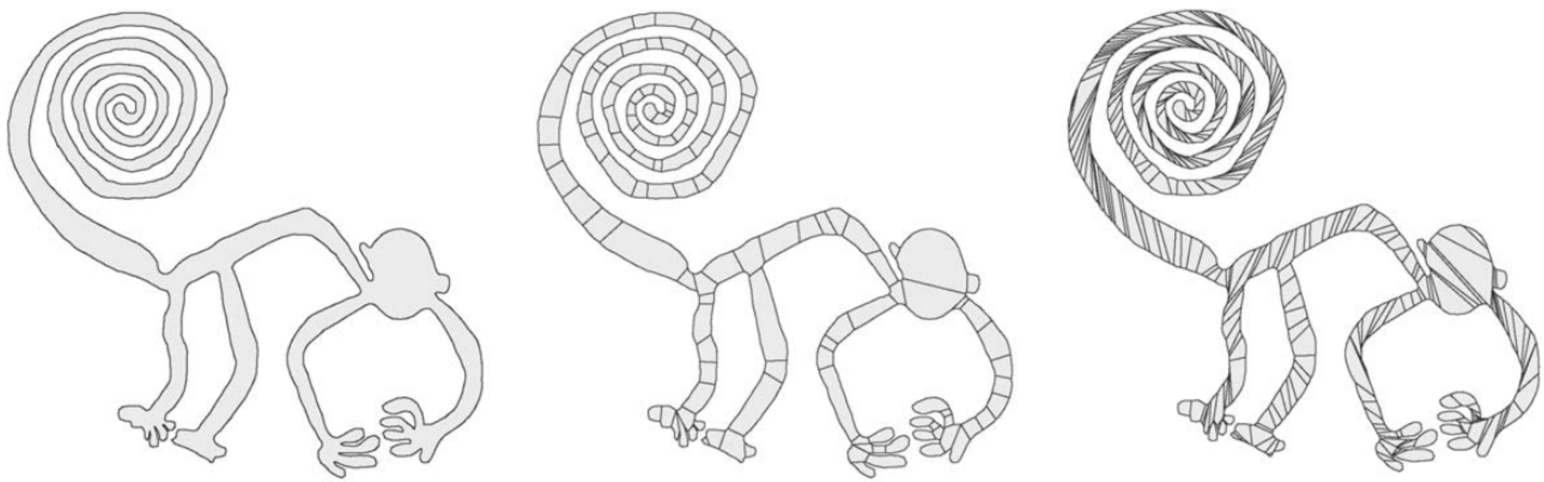
$\boxtimes$

Practical algorithms to decompose into approximately convex pieces

Approximate convex decomposition of polygons https://doi.org/10.1016/j.comgeo.2005.10.005
JM Lien, NM **Amato** - Computational Geometry, 2006 - Elsevier
We propose a strategy to decompose a polygon, containing zero or more holes, into "approximately convex" pieces. For many applications, the approximately convex components of this decomposition provide similar benefits as convex components, while the …

Approximate convex decomposition of polyhedra and its applications https://doi.org/10.1016/j.cagd.2008.05.003
JM Lien, NM **Amato** - Computer Aided Geometric Design, 2008 - Elsevier
Decomposition is a technique commonly used to partition complex models into simpler components. While decomposition into convex components results in pieces that are easy to . . .

J.-M. Lien, N.M. Amato / Computational Geometry 35 (2006) 100–123 101



(a) (b) (c)

Fig. 1. (a) The initial Nazca monkey has 1,204 vertices and 577 notches. The radius of the minimum bounding circle of this model is 81.7 units. Setting the concavity tolerance at 0.5 units, and not allowing Steiner points, (b) an approximate convex decomposition has 126 approximately convex components, an (c) a minimum convex decomposition has 340 convex components.

Dissecting one polygon into another

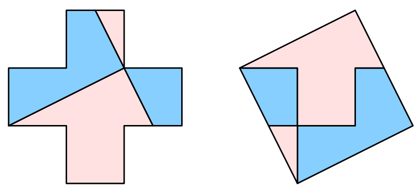


Figure 1: 4-piece dissection of Greek cross to square from 1890 [25].

Any two polygons of the same area have a common dissection (1807).

https://en.wikipedia.org/wiki/Bolyai%E2%80%93Gerwien_theorem

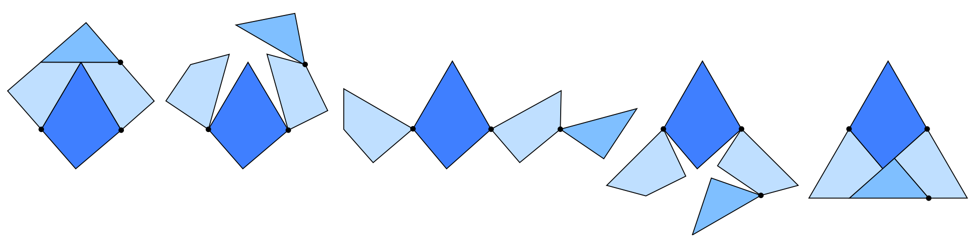


Figure 2: Dudeney's 1902 hinged dissection of a square into a triangle [15].

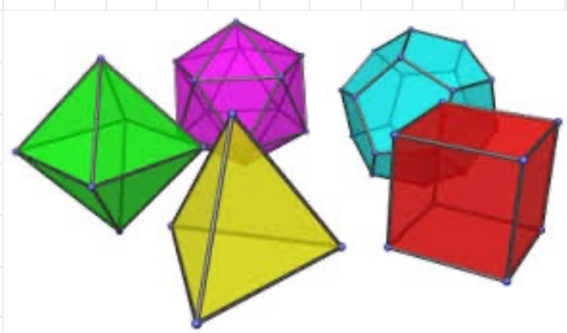
Hinged dissections exist d https://doi.org/10.1007/s00454-010-9305-9
TG Abbott, Z Abel, D Charlton, ED Demaine… - Discrete & …, 2012 - Springer
We prove that any finite collection of polygons of equal area has a common hinged dissection. That is, for any such collection of polygons there exists a chain of polygons hinged at vertices that can be folded in the plane continuously without self-intersection to …

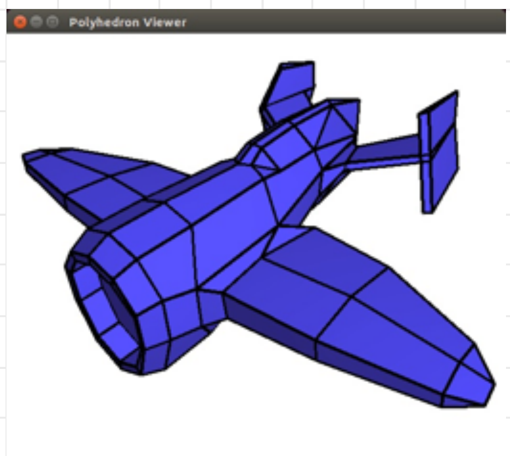
Polyhedra

A ***polyhedron*** consists of a finite connected set of (plane) polygons called ***faces*** such that

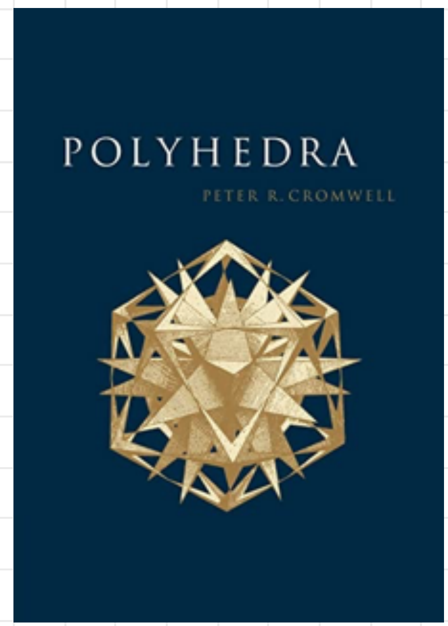
1. if two faces intersect it is only at a common vertex or edge
2. every edge of every face is an edge of exactly one other face
3. the faces surrounding each vertex form a single circuit

W https://en.wikipedia.org/wiki/Polyhedron





https://doc.cgal.org/latest/Polyhedron/index.html



https://tinyurl.com/yy5tt439

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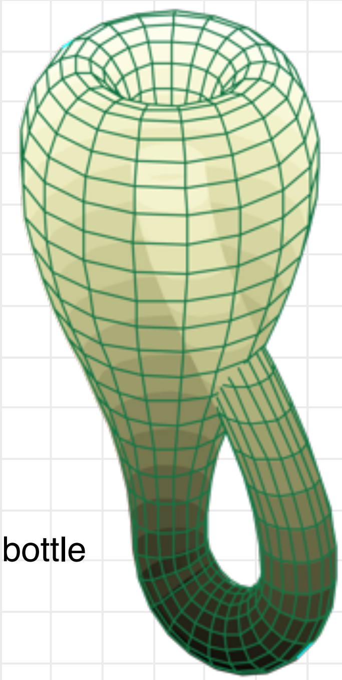
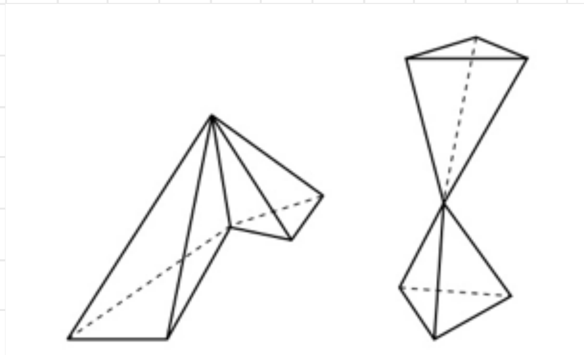
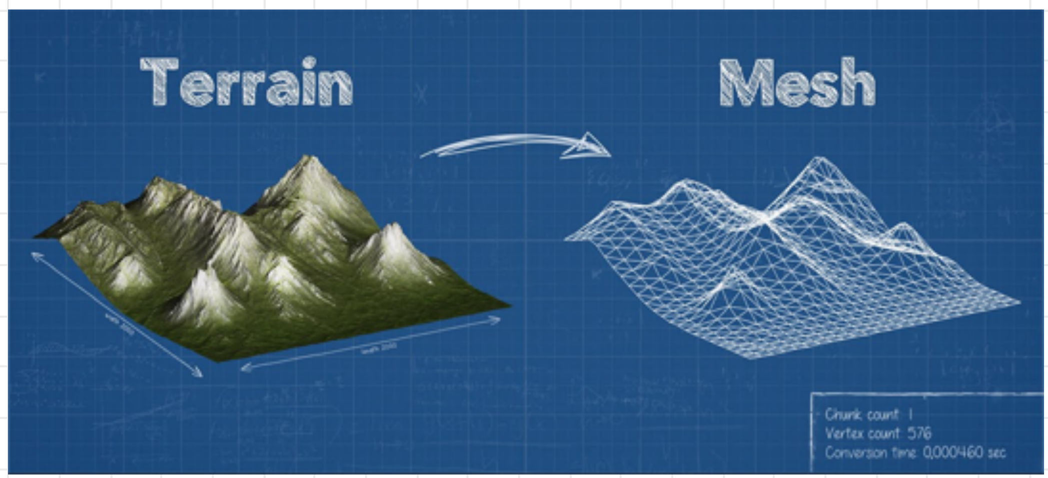
NOT Polyhedra



Klein bottle



terrain



Mesh Materializer

Summary

- optimizing triangulations, dynamic programming technique
- partitioning polygons into convex pieces
- polyhedra
- partitioning polyhedra deferred to next lecture

References

- [O'Rourke] 2.5
- [Handbook] Chapter 30

Further topic

We only discussed partitioning. What about covering (the pieces are allowed to overlap), or Boolean combinations?