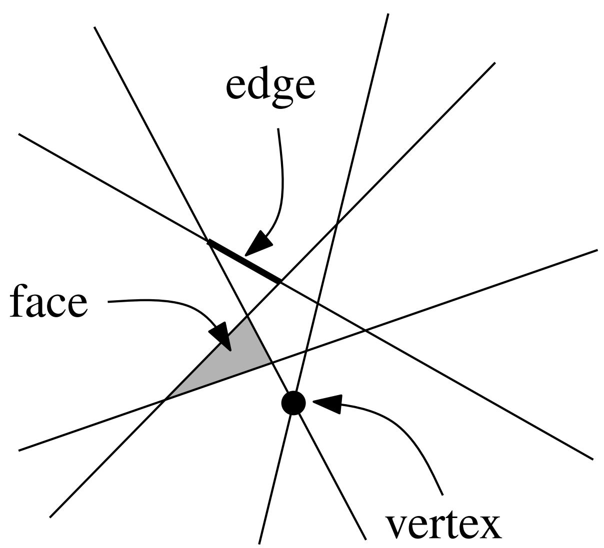
Recall

Recall a problem we considered before: given n points, are there 3 (or more) collinear

By duality (points <—> lines) this becomes:
given n lines, do 3 of them intersect at a point.

To get an $O(n^2)$ algorithm, we study ***line arrangements***.

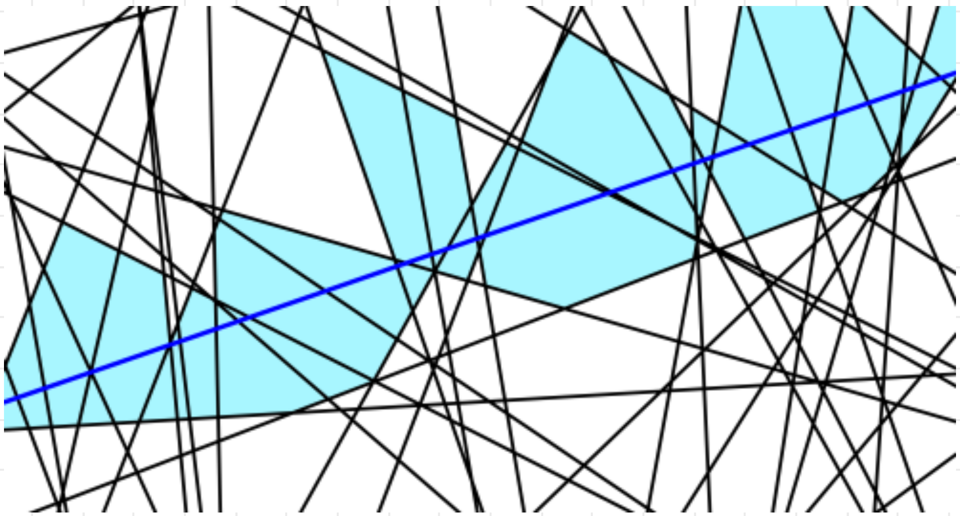
A set of n lines in the plane partitions the plane into faces (cells), edges, vertices, called the *arrangement*.



Recall

How to update after adding line $\ell_i$ to the line arrangement:

To bound the run time we need the Zone Theorem



the zone of the blue line David Dumas

**Definitions.** Let *A* be an arrangment, and $\ell$ be a line not in A.
The ***zone*** of $\ell$ in arrangement *A* is $Z_A(\ell)$ = {faces of A cut by $\ell$ } .
The ***size*** of the zone is $z_A(\ell) = \sum$ { # edges in face f : $f \in Z_A(\ell)$ }
$z_n$ = max{$z_A(\ell)$: over all possible $\ell$, A of n lines }

**Zone Theorem.** $z_n$ is O(n).

Recall

**Arrangements in higher dimensions**

For an arrangement of n hyperplanes in $R^d$

- the number of cells is $O(n^d)$
- Zone Theorem. The zone of a hyperplane has complexity $O(n^{d-1})$

In 3D, for n planes, there are $O(n^3)$ cells, and a zone has complexity $O(n^2)$.

## Application: Aspect Graph

What are all the combinatorially distinct viewpoints of an object?

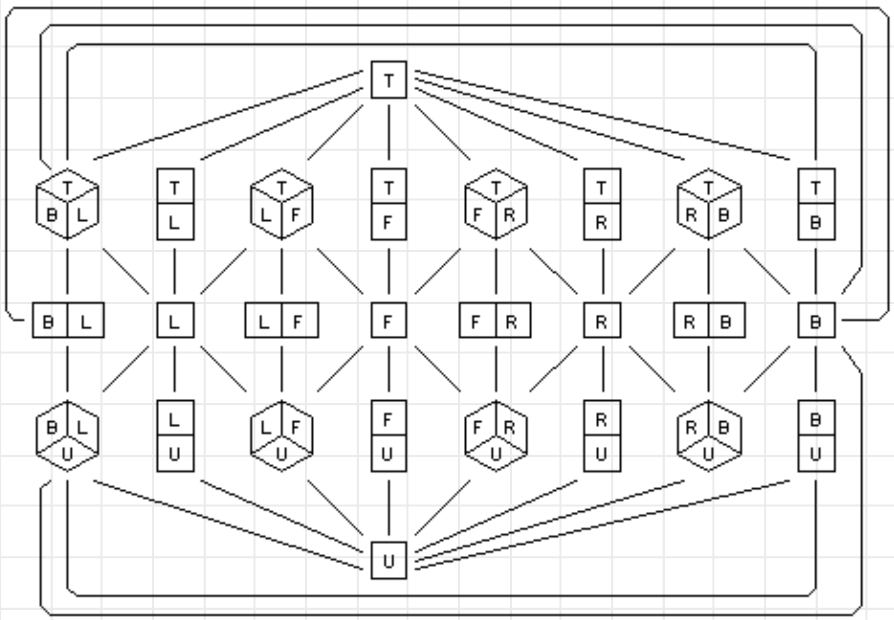
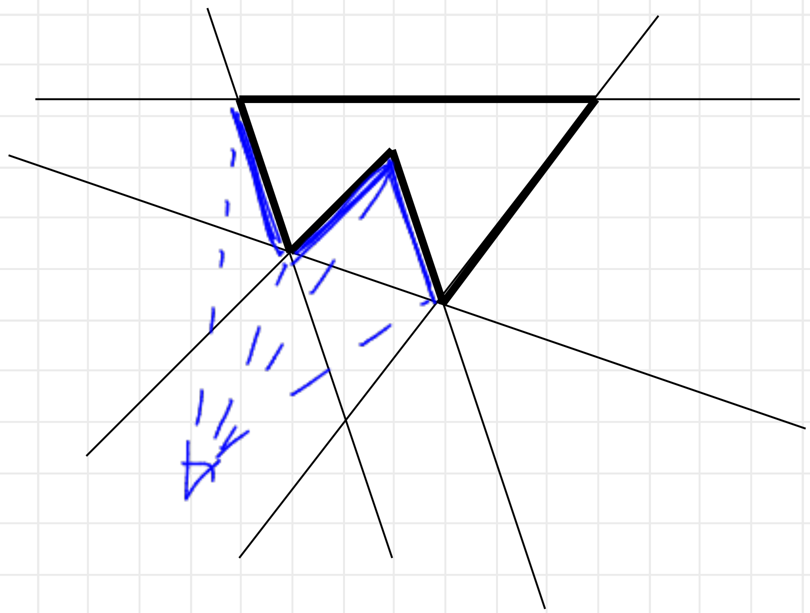


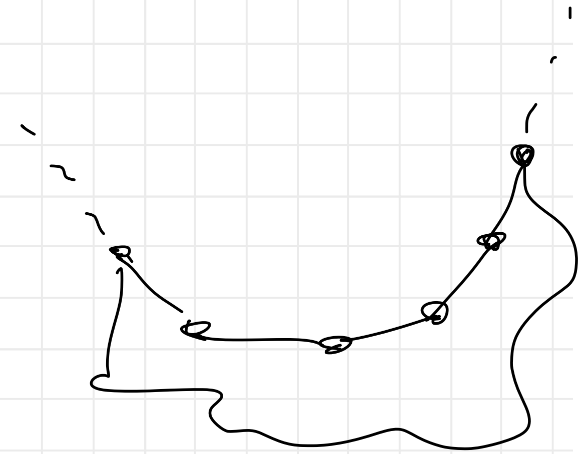

Figure 7: Aspect graph of a cube. The front, left, right, back, top and under sides of the cube are denoted by the letters F, L, R, B, T and U respectively.

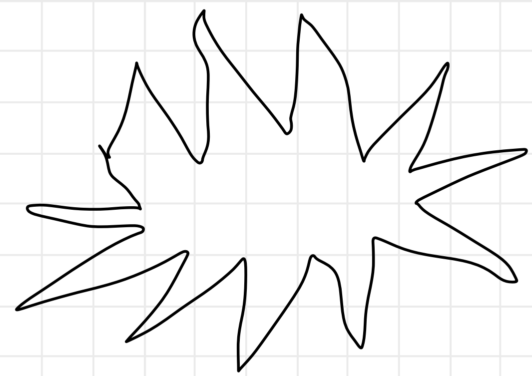
http://im-possible.info/english/articles/animation/animation.html

## Application: Aspect Graph



aspect graph of a polygon





area with same viewpoint = cell in arrangement of lines through pairs of visible points

$n^2$ lines, so $n^4$ cells

← can we really get $\Omega(n^2)$ lines

convex polygon — $\Theta(n)$ lines

**Application: Aspect Graph**

Aspect graph of convex polyhedron in 3D with n vertices

O(n) faces ⇒ $O(n^3)$ cells

for non-convex polyhedron Theta($n^9$) cells

we need planes through every 3 points (in worst case)
so $O(n^3)$ planes ⇒ $O(n^9)$ cells

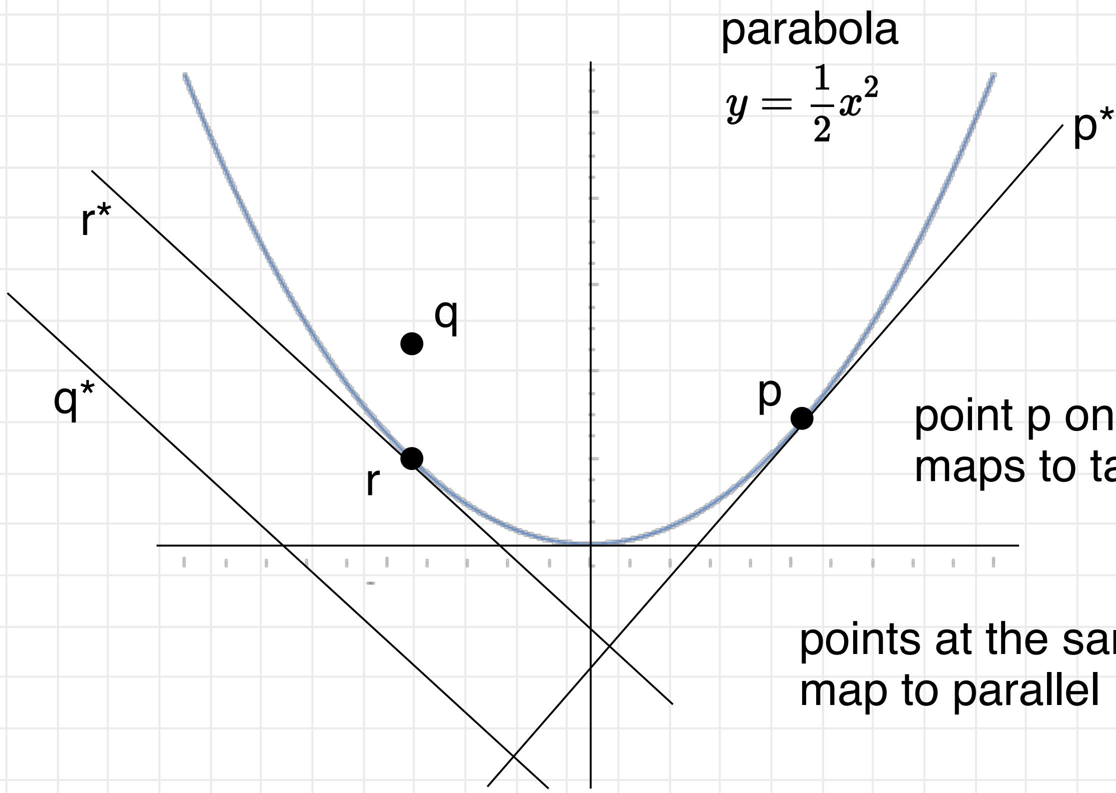
The aspect graph can be used to:

- find the viewpoint seeing the maximum number of faces
- find a “nice” projection
- figure out where a robot is, based on what it sees.

## Recall Duality Map

point in $\mathbb{R}^2$ ⟷ (*) line in $\mathbb{R}^2$

$p = (p_1, p_2)$ ⟷ $p^*$: $y = p_1 x - p_2$ (not defined for vertical lines)




when is $p$ on $p^*$

$p_2 = p_1^2 - p_2$

$2p_2 = p_1^2$

point p on the parabola maps to tangent line

points at the same x-coordinate map to parallel lines

**Lemma.** point p lies on/above/below line q* iff point q lies on/above/below line p*.

**Corollary.** Some points lie on a line iff their dual lines go through a point.

Properties of duality map

**Lemma.** point p lies on/above/below line q* iff point q lies on/above/below line p*.

**Corollary.** Some points lie on a line iff their dual lines go through a point.

**Proof.** $p = (a, b)$ $\quad p^*: \; y = ax - b$

$q = (c, d)$ $\quad q^*: \; y = cx - d$

p lies on (above) $q^*$ $\quad b = ca - d$ ($>$)

q lies on (above) $p^*$ $\quad d = ac - b$ ($>$)

} same. same.

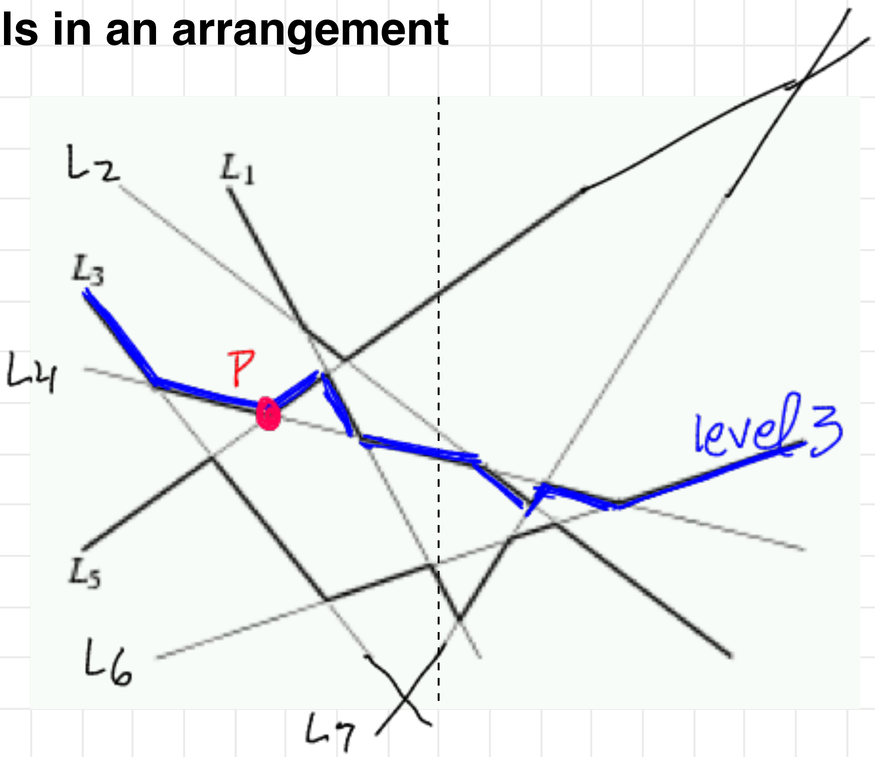
**Application: Collinear points.**

Given n points, are there 3 (or more) collinear?

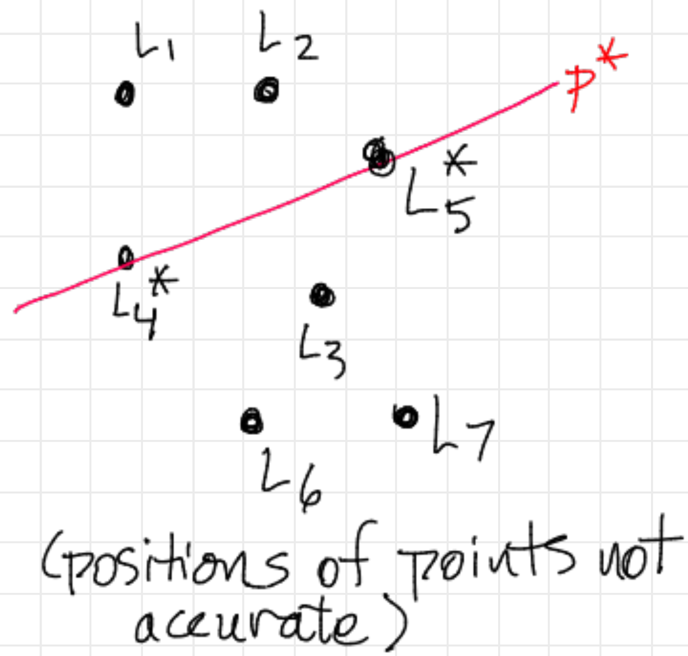
**Solution.** Apply duality. There are 3 collinear points iff the dual has 3 lines through a point. Construct the arrangement, and check for this. $O(n^2)$

Is there a faster algorithm? — see below

**Levels in an arrangement**




dual




Any vertical line not through vertices orders the edges top to bottom.

Level $L_1$ = all edges that appear first (topmost) along such a vertical line: $L_1\ L_2\ L_5\ L_7$
Level $L_i$ = all edges that appear in i-th place along such a line

**Claim.** Levels can be constructed in $O(n^2)$ time.

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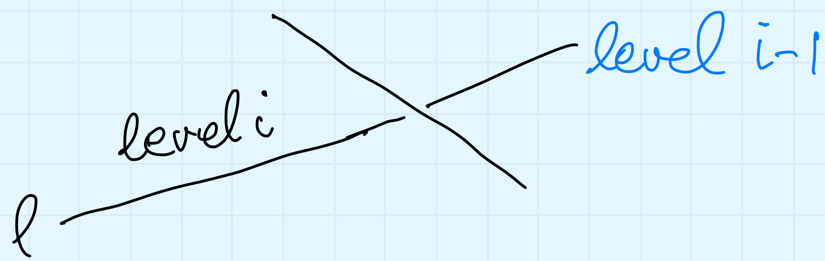
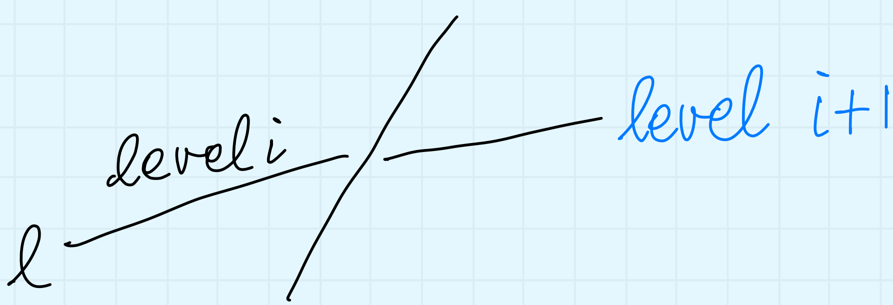
compute arrangement $O(n^2)$

sort lines by slope — this gives edges in each level at $x = -\infty$

$O(n \log n)$

Trace line $\ell$ through the arrangement




Total time $O(n^2)$.

## Levels in an arrangement

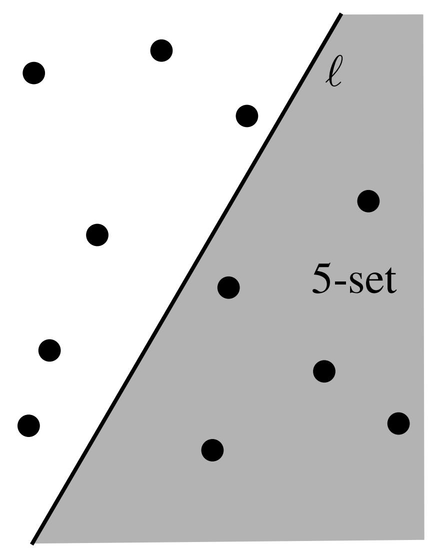
**Open problem:** what is the complexity of level $L_k$?
i.e. what is the worst case number of edges in level $L_k$?

Dual: given a set of points, how many subsets of size k can be cut away with a line?
For k = n/2, how many *halving lines* can there be?

Best known bounds:

$\Omega(n \log k)$ 1973, Erdös et al., raised a bit by Toth, 2001

$O(nk^{1/3})$ Tamal Dey, 1997

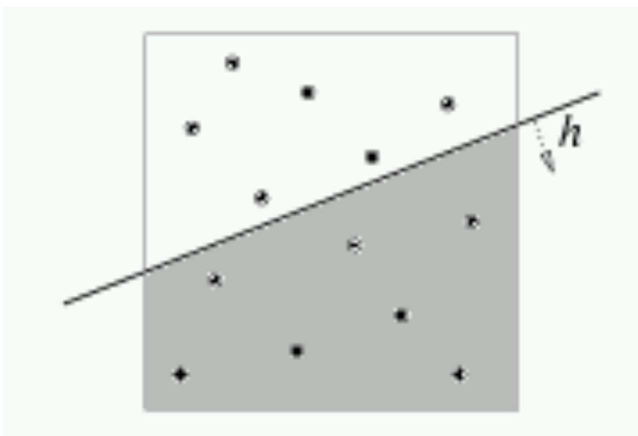



Also: find level $L_k$ (without constructing whole arrangement)

**Application: Discrepancy problem.**

Given n points in a unit square, do they provide a reasonable random sample?

***discrepancy*** of half-plane h = | area of square below h – fraction of points below h |




example:
7 points in shaded area; 13 points total
so fraction of points below h is 7/13

Given n points, find the maximum discrepancy of any half-plane.

Arrangements give an $O(n^2)$ time algorithm for this.

nice presentation: http://www.ams.org/samplings/feature-column/fc-2011-12

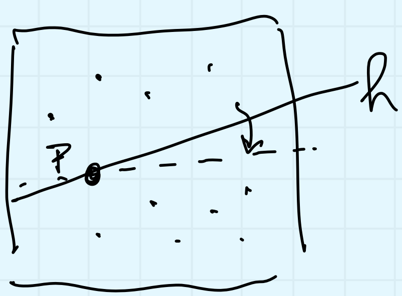
**Application: Discrepancy problem.**

**Lemma.** Maximum discrepancy occurs

1. at line h through 2 points, or
2. at line h through 1 point and the point is the midpoint of the segment h ∩ unit square

Proof.

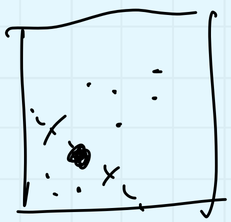
- if h goes through 0 points we can slide h up or down to increase discrepancy.
- if h goes through 1 point p




increase discrepancy by rotating h around p unless p is midpoint.

**Solving the discrepancy problem via arrangements.**

– type 2 lines $h$ can be checked brute force



$O(n)$ per point

$O(n^2)$ total.

– type 1 points – $h$ thru 2 points

use dual arrangement

point $h^*$ ~~through~~ at intersection of 2 lines

test all vertices of arrangement

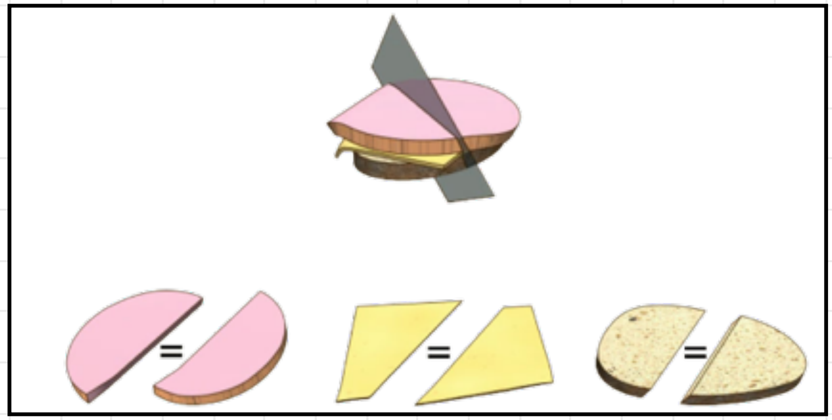
– the level gives # points above.

Total $O(n^2)$ (area below $h$ takes $O(1)$)

**Application: Ham sandwich theorem.**

**Theorem.** Given a set of red points and a set of blue points in the plane, there exists a line that cuts both sets in half.

**General Ham-Sandwich Theorem**
(from ’30’s - 40’s).
In $R^d$, any d measurable objects can be cut in half by one (d-1) dimensional hyperplane.




For discrete version in plane, there is an O(n) time algorithm to find the halving line.

Can be viewed in terms of arrangements.

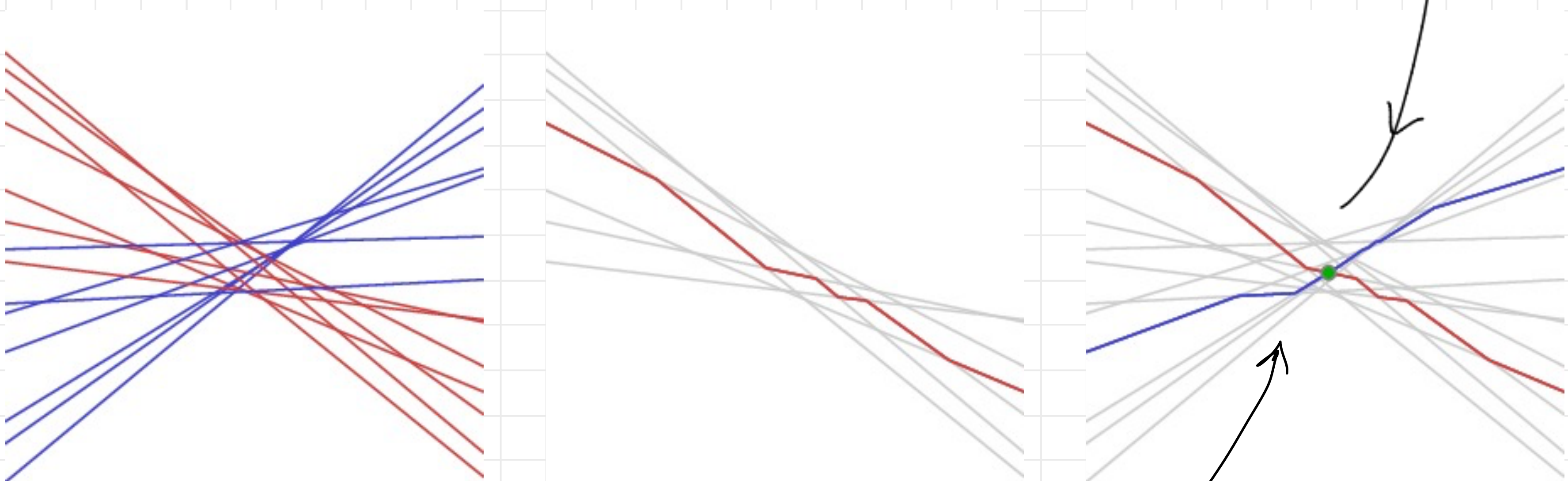
**Application: Ham sandwich theorem.**

**Theorem.** Given a set of red points and a set of blue points in the plane, there exists a line that cuts both sets in half.

Standard proof idea uses a rotating line.

In terms of duality and arrangements:

the two $\frac{n}{2}$-levels intersect at a point ⟷ halving line.



the n/2 level of red lines

David Austin

this point p ⟹ line $p^*$ is solution to ham sandwich.

**3-SUM hardness** W https://en.wikipedia.org/wiki/3SUM

Can we test for 3 collinear points (for point set in the plane) faster than $O(n^2)$?

It is a "3-SUM-hard" problem, one of a large class of "equivalent" problems that all seem(ed) to need $O(n^2)$ time.

**3-SUM problem**: Given n numbers, are there 3 that sum to 0? (repetition is allowed)

**Exercise.** Find an $O(n^2)$ time algorithm for 3-SUM.
[This is not too hard. Start by sorting the points.]

**Lemma.** If we could test for 3 collinear points in $o(n^2)$, then we could solve 3-SUM in $o(n^2)$.

**Proof.** Given n numbers as input to 3-SUM, map each number x to the point $(x,x^3)$.
**Claim.** 3 numbers a,b,c sum to 0 iff the corresponding points are collinear.

points collinear iff slopes $(a,a^3)$ to $(b,b^3)$ = slope $(b,b^3)$ to $(c,c^3)$

iff $\frac{b^3-a^3}{b-a} = \frac{c^3-b^3}{c-b}$ iff $b^2+a^2+ab = c^2+b^2+cb$

iff $b(a-c) = c^2-a^2$ iff $b = \frac{c^2-a^2}{c-a}$ $\quad b = -(a+c)$

iff $a+c+b=0$

recent breakthrough on 3-SUM:

an algorithm with run time $O(n^2/(\log n/\log\log n)^{2/3})$

Grønlund, Allan, and Seth Pettie. "Threesomes, degenerates, and love triangles." Journal of the ACM, 2018. (conference version 2014)

d https://doi.org/10.1145/3185378

improved by Timothy Chan, 2018

very recent paper on "fine-grained" complexity lower bounds:

Hardness for Triangle Problems under Even More Believable Hypotheses: Reductions from Real APSP, Real 3SUM, and OV
TM Chan, VV Williams, Y Xu - arXiv preprint arXiv:2203.08356, 2022 - arxiv.org

The 3-SUM hypothesis, the APSP hypothesis and SETH are the three main hypotheses in fine-grained complexity. So far, within the area, the first two hypotheses have mainly been about integer inputs in the Word RAM model of computation. The "Real APSP" and "Real 3-SUM" hypotheses, which assert that the APSP and 3-SUM hypotheses hold for real-valued inputs in a reasonable version of the Real RAM model, are even more believable than their integer counterparts. Under the very believable hypothesis that at least one of the …

## Summary

- applications of arrangements
- testing collinearity and the 3-SUM problem.

## References

- [CGAA] Chapter 8
- [Zurich notes] Chapter 8