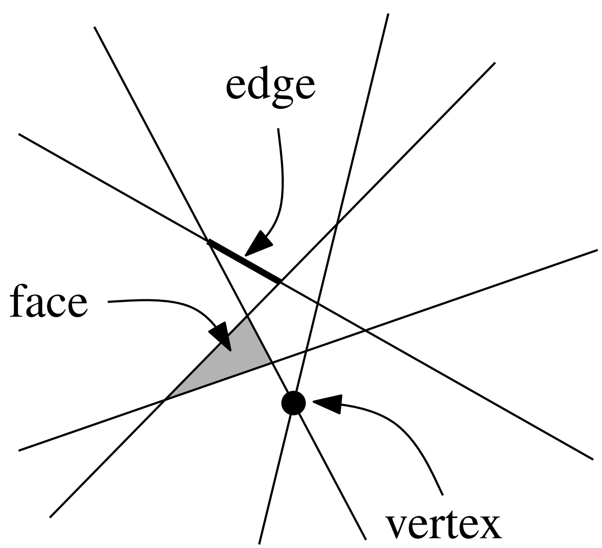
Recall a problem we considered before: given n points, are there 3 (or more) collinear

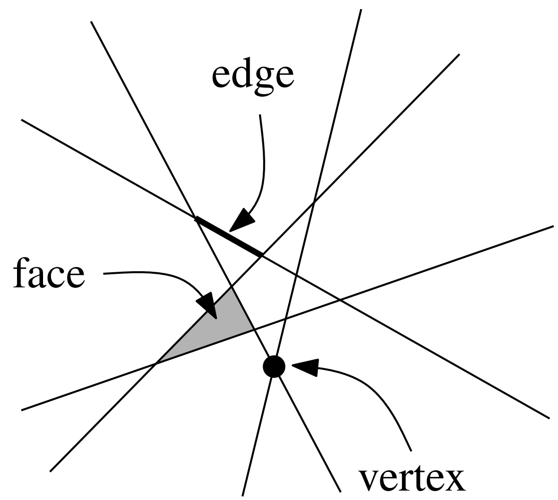
By duality (points <—> lines) this becomes:
given n lines, do 3 of them intersect at a point.

To get an $O(n^2)$ algorithm, we study ***line arrangements***.

A set of n lines in the plane partitions the plane into faces (cells), edges, vertices, called the *arrangement*.



How many vertices, edges, faces for n lines?



* in general case
$f_n \le \binom{n+1}{2} + 1$

A ***degeneracy*** is parallel lines or >2 lines through one point.

Exact bounds if there are no degeneracies (these decrease in case of degeneracy):

n lines

\# vertices $= \binom{n}{2}$ because every 2 lines intersect.
general case $\le \binom{n}{2}$

\# edges — each line crossed by n-1 others $\Rightarrow$ n edges per line
$= n^2$ general case $\le n^2$

\# faces $f_n = f_{n-1} + n$ — new line cuts n old faces in two

$f_0 = 1$ $f_n = n + n-1 + n-2 \cdots + 1 + \underset{1}{f_0} = \binom{n+1}{2} + 1$

*

Constructing arrangements

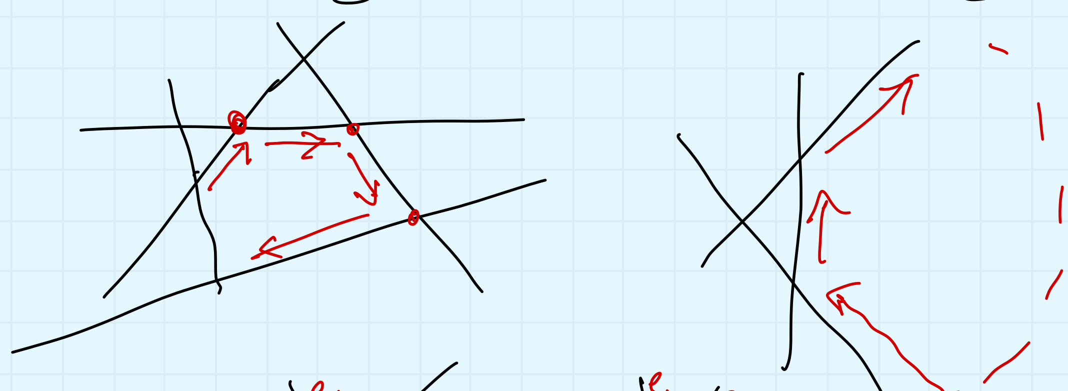
input: n lines
output: list of faces, edges, vertices and all incidence relationships
(note: size is Theta(n^2))

Plane sweep would take O(n^2 log n) (because n^2 events, and log n per update)

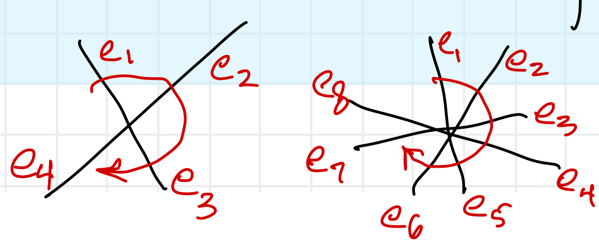
There is an O(n^2) time algorithm (deterministic, not randomized)

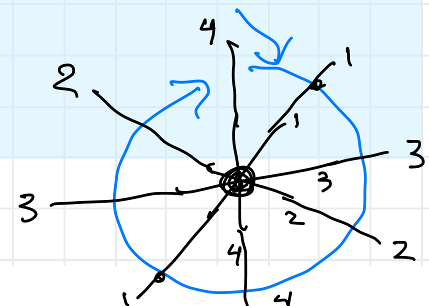
Idea: insert lines one by one
update each time.

Maintain cyclic order of edges around vertices



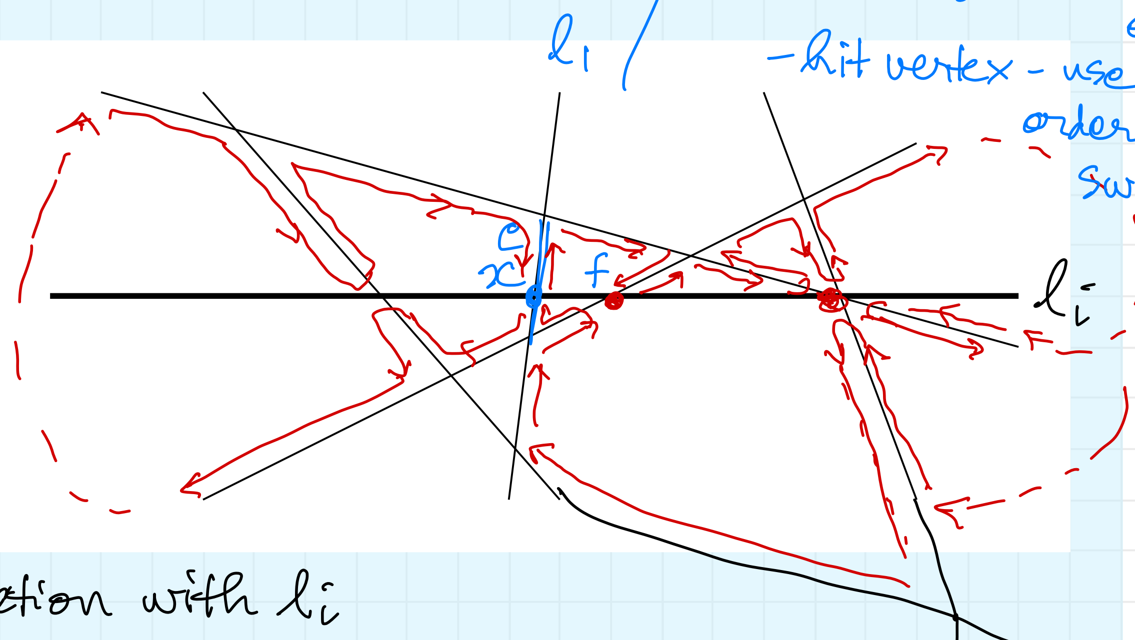
also keep order at circle at infinity





How to update after adding line ℓ_i to the line arrangement:

- find intersection x of ℓ_i and ℓ_1
- and edge e on ℓ_1 and a face f containing e



2 kinds of updates
- hit ℓ_i - switch faces incident to current edge
- hit vertex - use cyclic order to switch edges

- walk around f to get next intersection with ℓ_i
- hop to adjacent face . . . continue ... until we return to e
- update all info about the arrangement as we go.

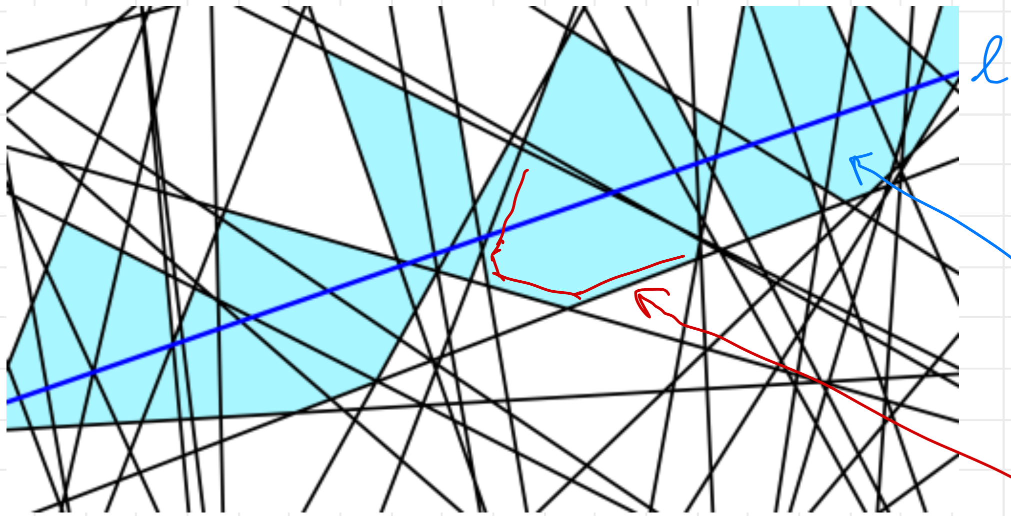
Time: - initialize (find x, e, f) $O(n)$
- walking and updating: constant time per edge visited

$O(\#\text{ edges in faces cut by } \ell_i)$

We will show $O(i)$

How to update after adding line ℓ_i to the line arrangement:

To bound the run time we need the Zone Theorem

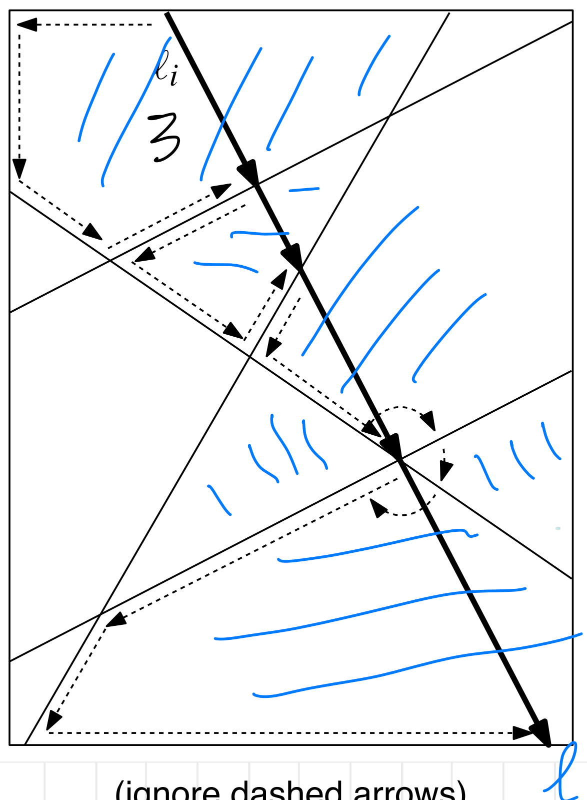


the zone of the blue line David Dumas

Definitions. Let *A* be an arrangment, and ℓ be a line not in A.
The ***zone*** of ℓ in arrangement *A* is $Z_A(\ell)$ = {faces of A cut by ℓ }.
The ***size*** of the zone is $z_A(\ell) = \sum$ { # edges in face f : $f \in Z_A(\ell)$ }
$z_n = \max\{z_A(\ell)$: over all possible ℓ, A of n lines }

Zone Theorem. z_n is O(n).

Example



(ignore dashed arrows)

ignore rectangle boundary
= circle at ∞

Zone of ℓ has 6 faces

$$z(\ell) = 3 + 3 + 4 + 3 + 2 + 3$$
$$= 18$$

Zone Theorem. z_n is $O(n)$. (For non-degenerate case, $z_n \leq 6n$.) 10n for degenerate case

[Chazelle, Guibas, Lee, '85, Edelsbrunner, O'Rourke, Seidel, '86, correct proof for dimensions ≥ 3, Edelsbrunner, Seidel, Sharir, '93]

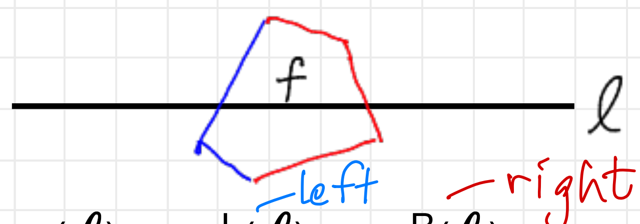
Consequence: the incremental algorithm takes time $O(n^2)$.

Proof

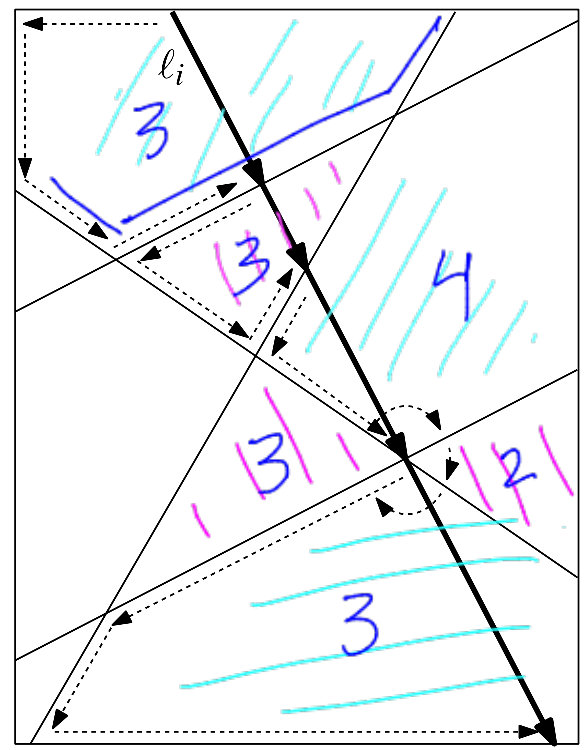
We will bound $z_A(\ell) = \sum \{ \#\text{ edges in face } f : f \in Z_A(\ell) \}$

Rotate so ℓ is horizontal. Perturb so no other line is horizontal (this only increases the zone size).

Any face f in $Z_A(\ell)$ has left and right boundary edges.



$z_A(\ell) = z(\ell) = z^L(\ell) + z^R(\ell)$

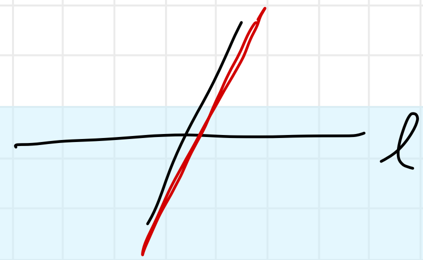


Claim. $z^L(\ell) \leq 5n$ [$\leq 3n$ for non-degenerate case]

This will prove the Zone Theorem.

Claim. $z^L(\ell) \leq 5n$ [$\leq 3n$ for non-degenerate case]

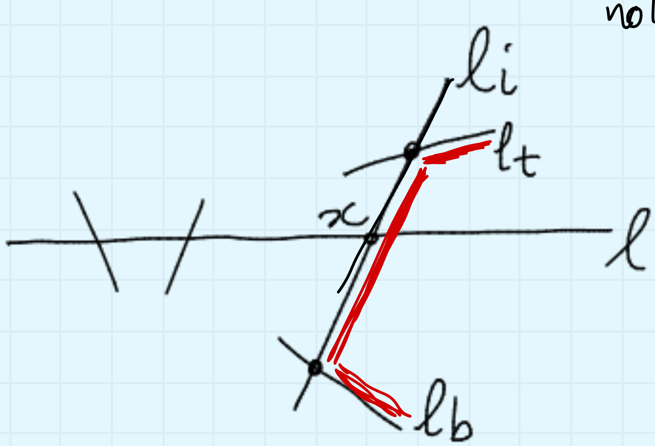
Proof By induction on n. Basis



$n=2$
$z^L(\ell)=1$

$n=1$
$z^L(\ell)=0$

note ℓ_t has + slope
ℓ_b has − slope



Let x = rightmost intersection with ℓ say with ℓ_i

Find first intersection on ℓ_i above/below
ℓ_t ℓ_b

How may new left edges are created by adding ℓ_i?

claim: 1 left edge on ℓ_i (none above ℓ_t or below ℓ_b)

- left edge on ℓ_t splits in 2
- left edge on ℓ_b splits in 2

} no other left edge is cut by ℓ_i

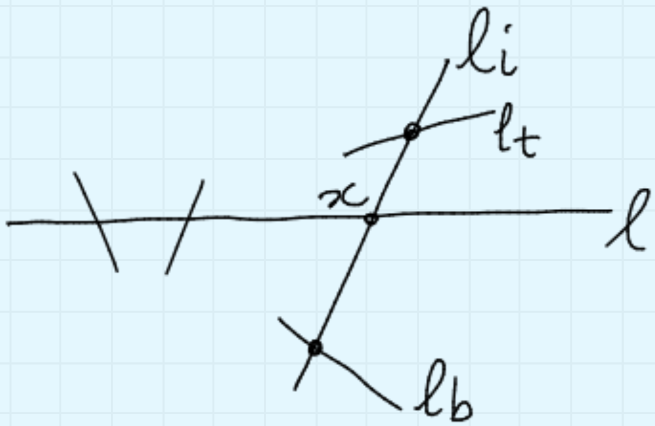
Increase in # left edges is 3. (non-degenerate case)

So $\leq 3n$ left edges in non-degenerate case

~~degeneracies allowed~~

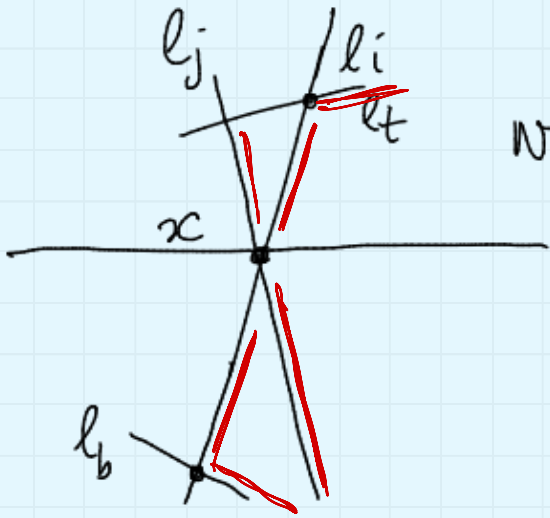
Claim. $z^L(\ell) \leq 5n$ [$\leq 3n$ for non-degenerate case]

Proof



ignore

What happens in case there is degeneracy?



what if another line ℓ_j goes though x?

still get left edge on ℓ_t split in two $+1$

ℓ_b " " " $+1$

on ℓ_i get two new left edges $+2$

on ℓ_j one left edge splits in two $+1$

$+5$

So # left edges $\leq 5n$

(if even more lines go through x, the bound goes down)

Arrangements in higher dimensions

For an arrangement of n hyperplanes in R^d

- the number of cells is $O(n^d)$
- Zone Theorem. The zone of a hyperplane has complexity $O(n^{d-1})$

In 3D, for n planes, there are $O(n^3)$ cells, and a zone has complexity $O(n^2)$.

Next lecture:

- applications of arrangements
- testing collinearity and the 3-SUM problem.

Summary

- arrangements
- size of parts of arrangements and the zone theorem

References

- [CGAA] Chapter 8
- [Zurich notes] Chapter 8