Triangulations of point sets/polygons. Recall what we've seen:

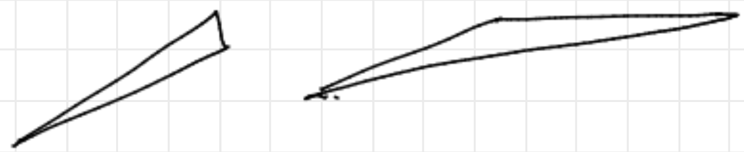
- Delaunay triangulation of point set in R^d, O(n log n) algorithm in R^2.
- O(n) algorithm to triangulate any polygon in R^2 (Chazelle's hard algorithm)

Applications and criteria (this is the outline for the next lectures)

- angle criteria - for meshing
- length criteria: minimum weight triangulation
- constrained triangulations (when certain edge must be included)
- meshing - triangulations with Steiner points
- flip distance
- morphing
- curve and surface reconstruction
- medial axis and straight skeleton

Angle conditions for triangulations

The motivation is meshing for finite element methods (more on this later) where small and large angles are bad.



bad triangles

Problems:

1. given a point set, find a triangulation that maximizes the min. angle
   The Delaunay triangulation does this.

2. given a point set, find a triangulation that minimizes the max. angle

EX. Show that these two can be different.

There is a poly time algorithm to find a triangulation that minimizes the maximum angle.

It uses a solution for the case of triangulating a polygon (via dynamic programming).

Bern, M., Edelsbrunner, H., Eppstein, D., Mitchell, S. and Tan, T.S., 1993. Edge insertion for optimal triangulations. *Discrete & Computational Geometry*, *10*(1), pp.47-65.

d https://doi.org/10.1007/BF02573962

Length conditions: **Minimum weight triangulation**

Given a point set find a triangulation that minimizes the sum of the lengths of the edges.

Solved by dynamic programming for triangulations of a simple polygon.

For point sets, proved NP-hard in 2008 (had been open since 1979).
Note: not known to be in NP because of square root computations.

Mulzer, Wolfgang, and Günter Rote. "Minimum-weight triangulation is NP-hard." *Journal of the ACM (JACM)* 55.2 (2008): 11.

d https://doi.org/10.1145/1346330.1346336

Approximations (how do various triangulations compare to min weight)

- approximation ratio of Delaunay triangulation: Theta(n)
- approximation ratio of greedy triangulation (add edges in order of weight): Theta(\sqrt(n))

- quasi-poly time approximation scheme:

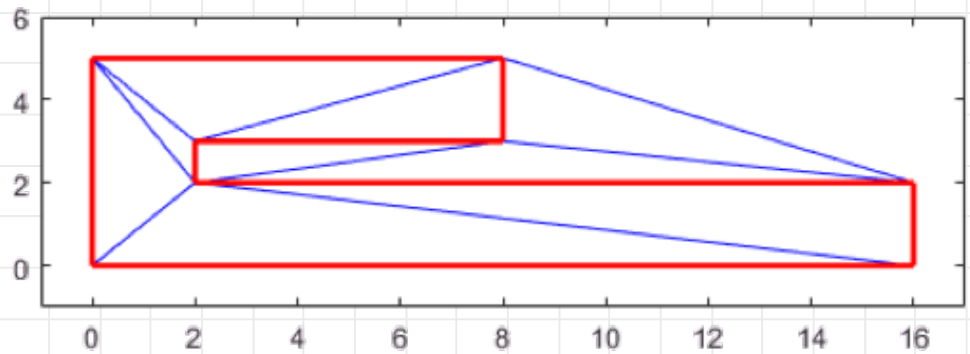
Remy, Jan, and Angelika Steger. "A quasi-polynomial time approximation scheme for minimum weight triangulation." *Journal of the ACM (JACM)* 56.3 (2009): 15. d https://doi.org/10.1145/1516512.1516517

Constrained triangulations

Given points P in R^2 and some non-crossing edges F (the “fixed” edges), add more edges to get a triangulation optimizing some criterion.

This generalizes polygon triangulation (though note that the above problem asks to triangulate the inside AND the outside of a polygon).

F = red edges

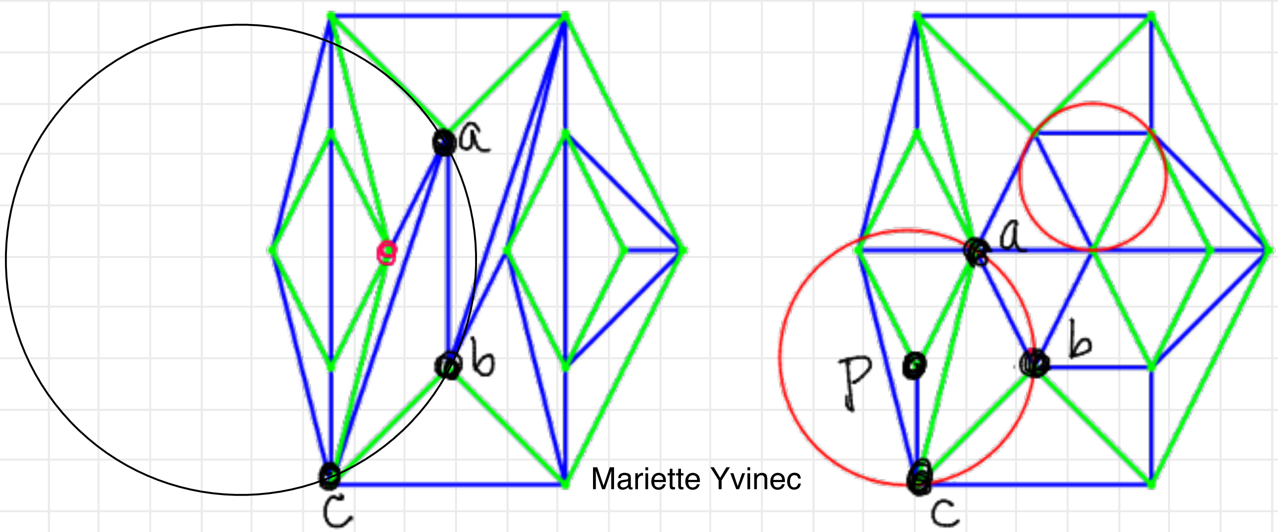


The **Constrained Delaunay triangulation (CDT)** consists of triangles abc not crossed by any edge of F such that Circle(a,b,c) contains no point of P visible from inside triangle abc.

← without crossing fixed edges.

Must be proved that these triangles form a triangulation.

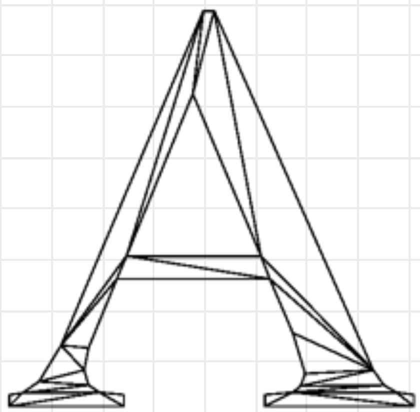
F = green edges

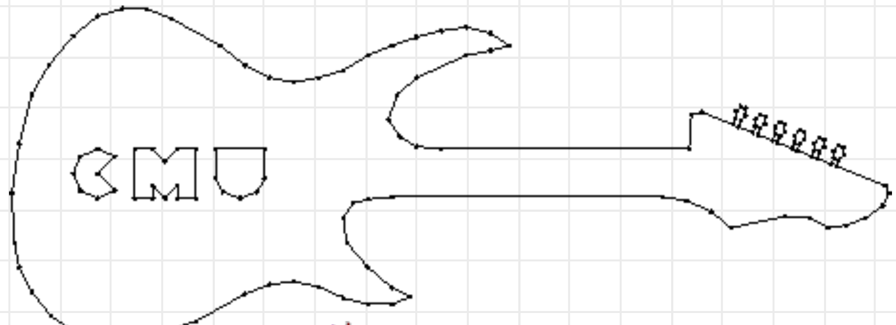


NOT Constrained Del. Constrained Del.

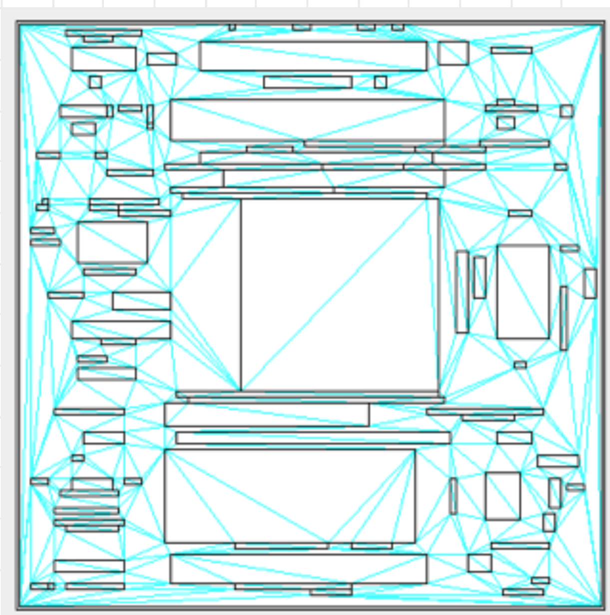
p ∈ Circle(a,b,c) is ok because edge (a,c) ∈ F blocks it from △abc

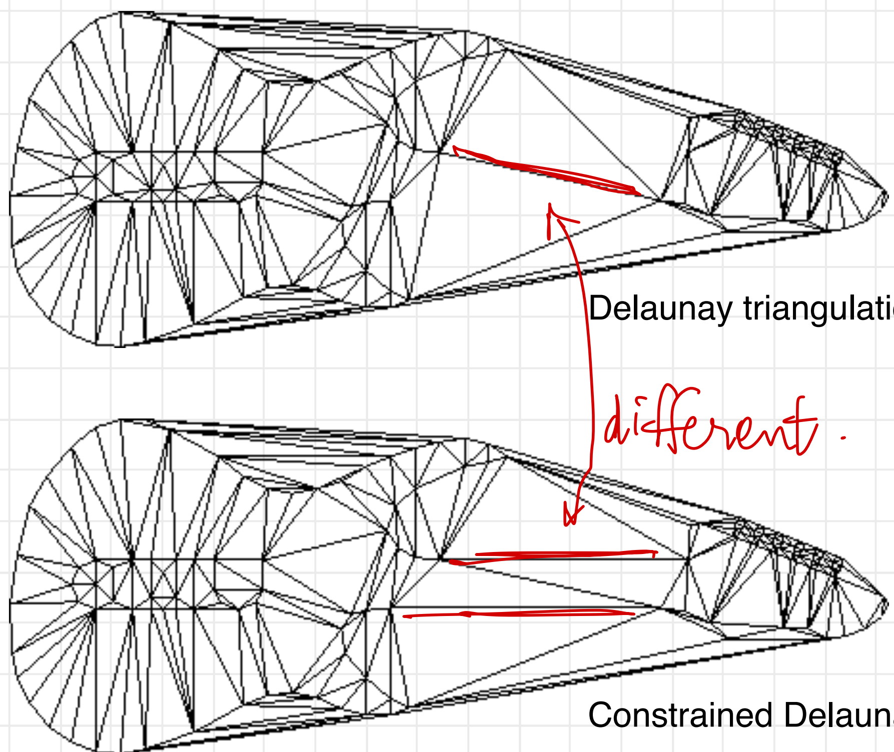
Examples of Constrained Delaunay triangulations





https://www.cs.cmu.edu/~quake/tripaper/triangle3.html





Delaunay triangulation

Constrained Delaunay triangulation

Most results for Delaunay triangulations carry over to Constrained Delaunay triangulations.

The edge empty circle condition carries over:
(a,b) is an edge of the Constrained Delaunay triangulation iff no edge of F crosses (a,b) and there is a circle through a and b that does not contain any point p in P visible to a point on edge (a,b).

Illegal edge flipping carries over.

There is an O(n log n) time algorithm to compute the Constrained Delaunay triangulation.

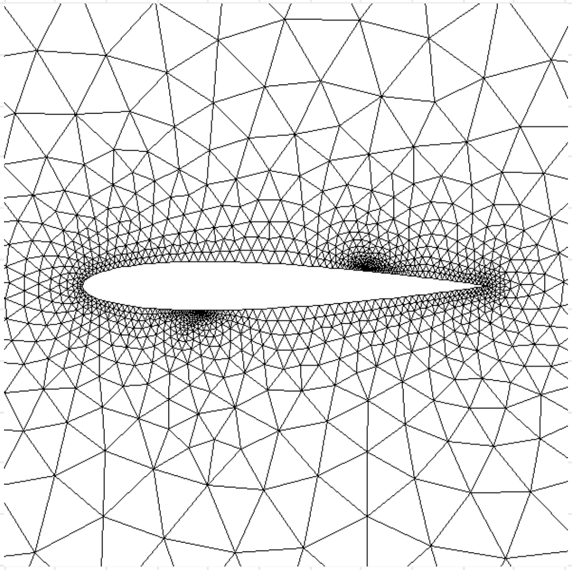
The Constrained Delaunay triangulation maximizes the min angle (among all constrained triangulations).

Triangulations for finite element methods

Example problem: find how a solid body deforms under stress.
Requires solving partial differential equations, which is done by approximating on a "mesh" (often a triangulation)

Meshing. Given a region of R^2 with a polygonal boundary, subdivide it into disjoint triangles meeting edge-to-edge and *conforming* to the boundary, i.e. every boundary edge is a union of triangle edges. Use "nicely shaped" triangles.

Note: can add new points called "Steiner points"



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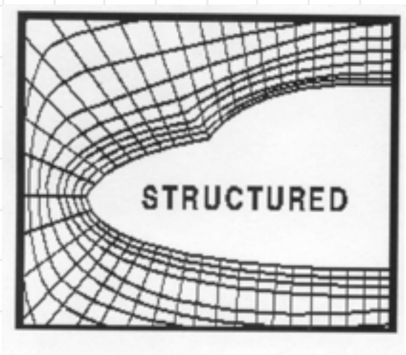
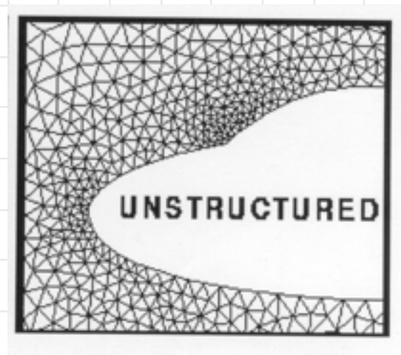




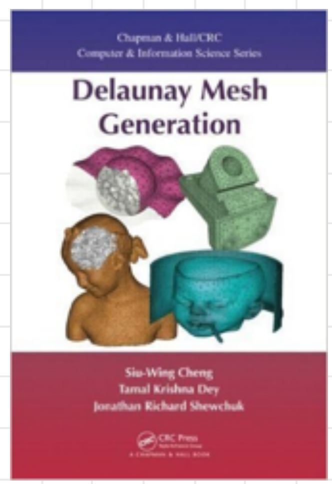
We concentrate on unstructured meshes.

Shewchuk, Jonathan Richard. "Unstructured mesh generation." *Combinatorial Scientific Computing* (2011): 259-297.

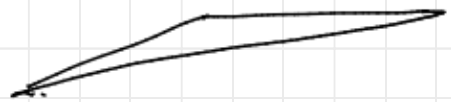
https://people.eecs.berkeley.edu/~jrs/papers/umg.pdf

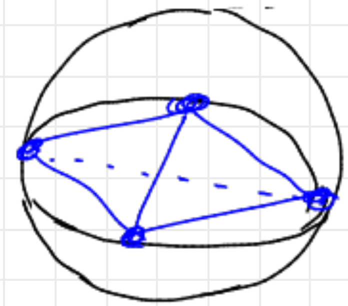
Theoretically Guaranteed Delaunay Mesh Generation–In Practice (slides)

https://people.eecs.berkeley.edu/~jrs/papers/imrtalk.pdf



bad for finite element methods:



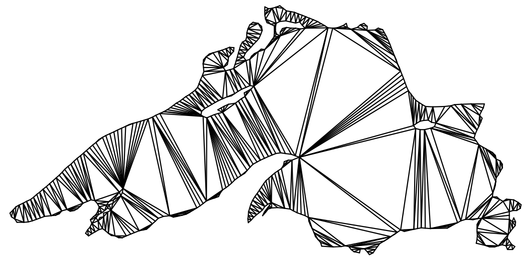


Also want as few triangles as possible, but this conflicts with angle constraints.

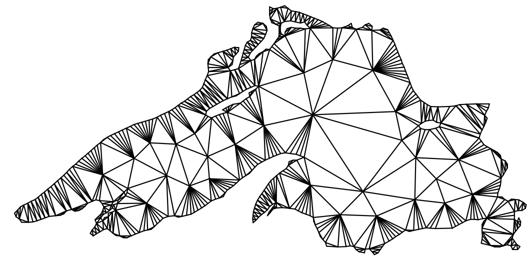
Well–Shaped Elements vs. Few Elements

somewhat contradictory goals

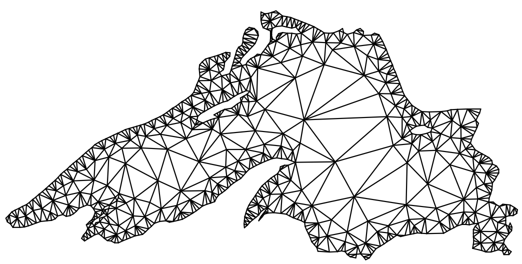
Lake Superior



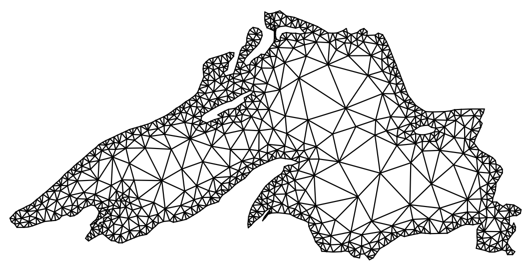
No minimum angle
518 elements



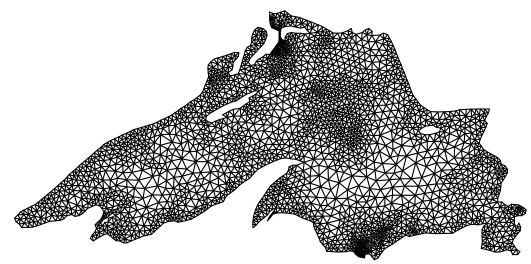
5° minimum angle
593 elements



15° minimum angle
917 elements



25° minimum angle
1,427 elements



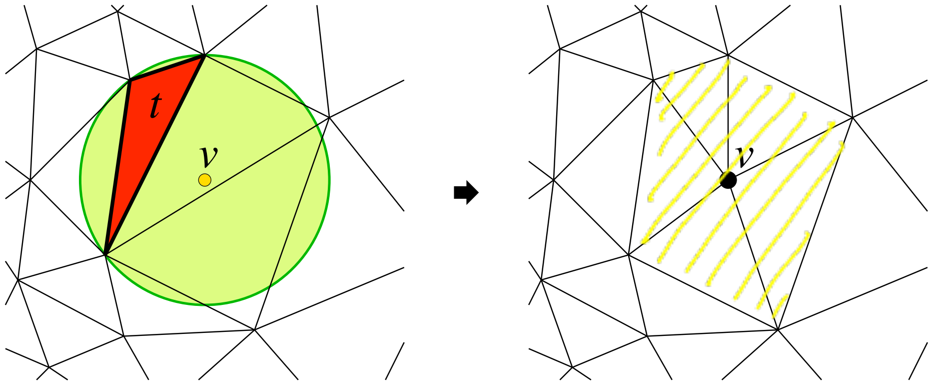
34.2° minimum angle
4,886 elements

These meshes generated by Ruppert's Delaunay refinement algorithm.

Jonathan Shewchuk

Delaunay refinement algorithm

Start with Delaunay triangulation and add more points to improve angles.



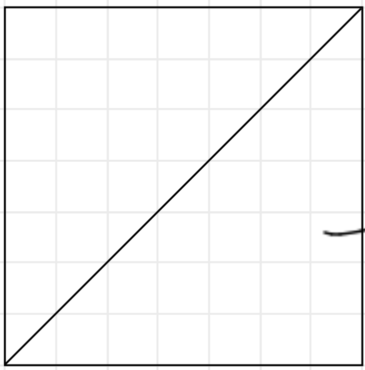
kill skinny triangle by adding point v at center of circumcircle

Ruppert, Jim. "A Delaunay refinement algorithm for quality 2-dimensional mesh generation." Journal of algorithms 18.3 (1995): 548-585.

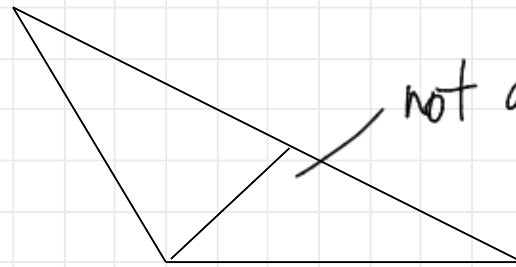
https://doi.org/10.1006/jagm.1995.1021

A related puzzle problem (Martin Gardner)

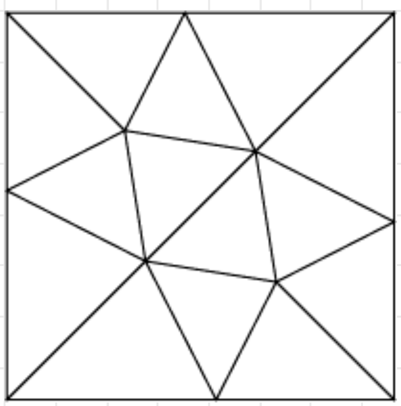
Given a square or an obtuse triangle, dissect into smallest number of acute triangles.



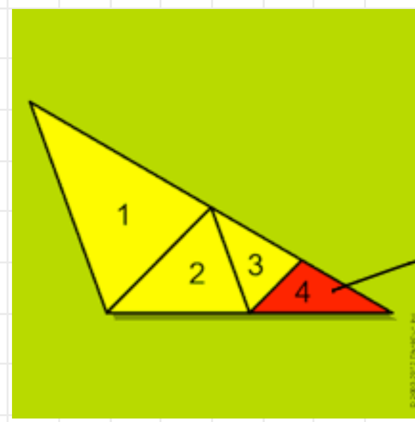
not acute



not acute



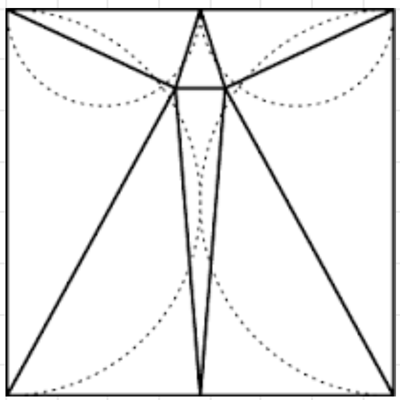
Solution with 14 triangles



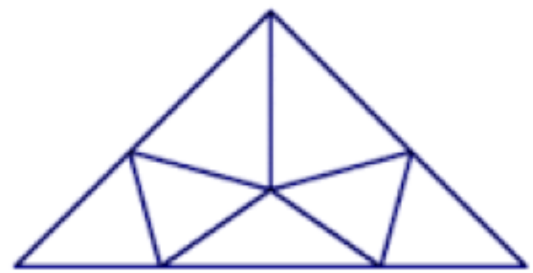
not acute

A related puzzle problem (Martin Gardner)

Given a square or an obtuse triangle, dissect into smallest number of acute triangles.



min is 8



min is 7

Flip distance

Reconfiguration problem: changing one structure to another via discrete steps

Examples:
- edit distance of strings
- sorting via swaps
- solving Rubik's cube
- pivot operations for simplex method of linear programming

Questions:
- can we get from every configuration to every other one?
- worst case bound on number of steps?
- how many steps between a given pair of configurations?

Yes

20 (God's number)

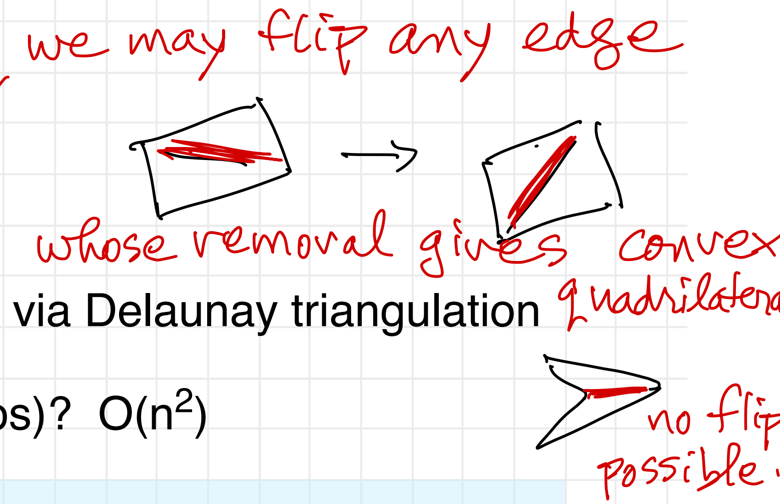
These can be viewed as connectivity and shortest path questions in a ***reconfiguration graph*** — vertex for each configuration, edge for each step

Reconfiguration graphs are large, so we don't explore them explicitly.

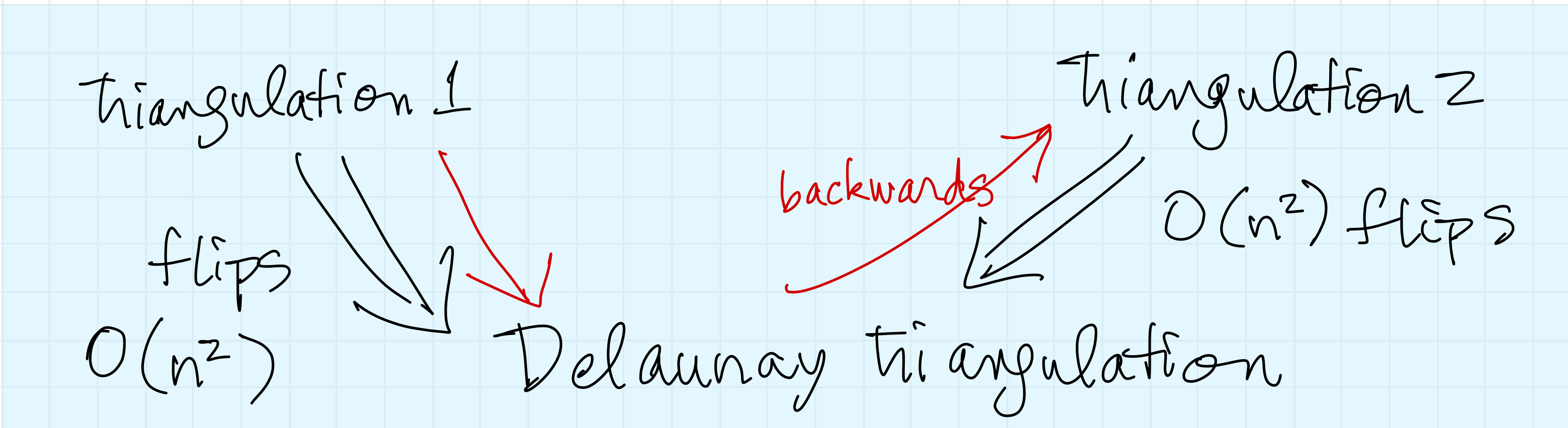
43×10^{18}

Flip distance

Reconfiguring triangulations of a given point set via flips



- can we get from any triangulation to any other? Yes, via Delaunay triangulation
- what is the worst case ***flip distance*** (= number of flips)? $O(n^2)$

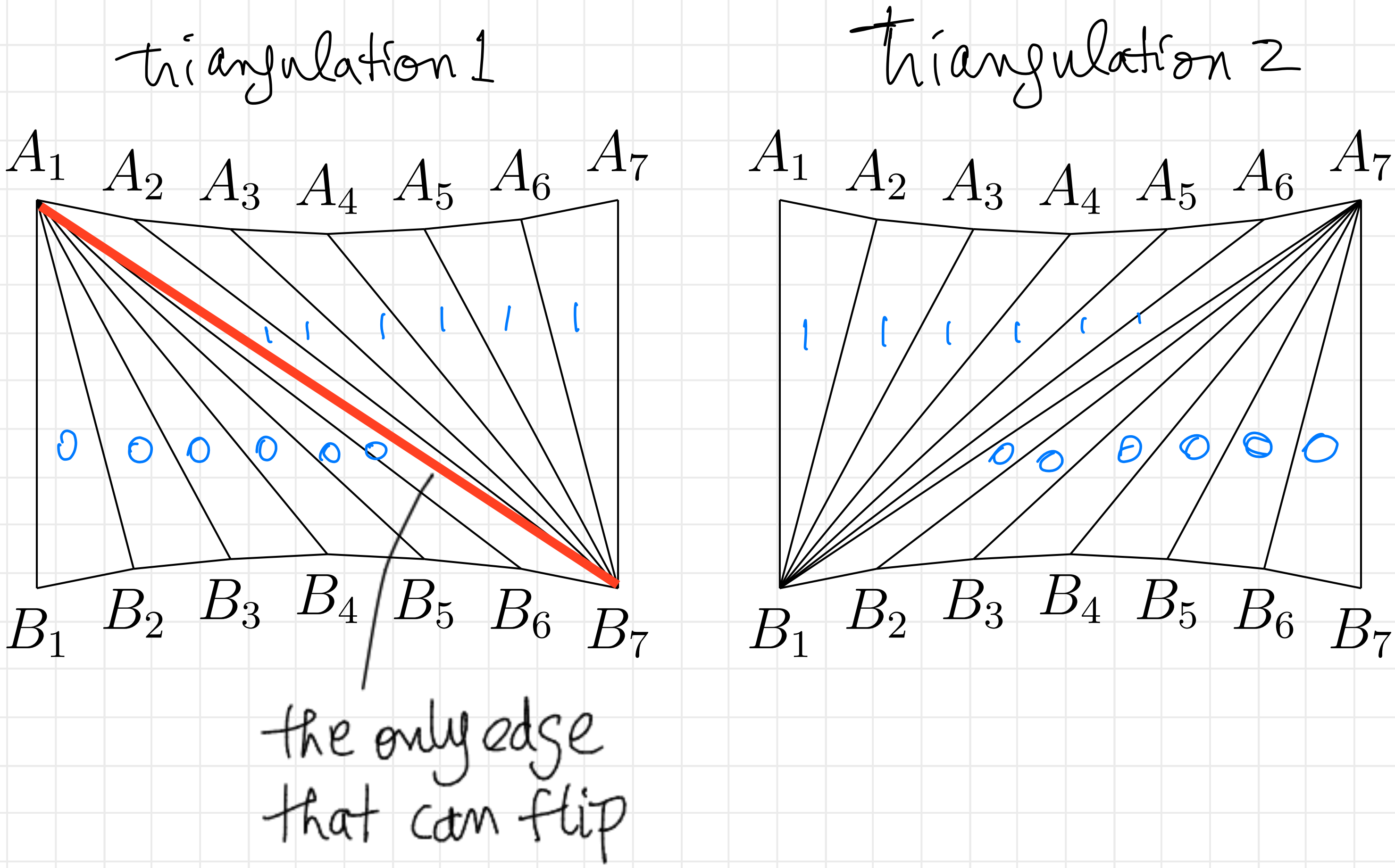


- can we find the flip distance between two given triangulations?

  This is NP-complete, but OPEN for the case of convex polygons = rotation distance between binary trees.

Flip distance lower bound of Omega(n^2)

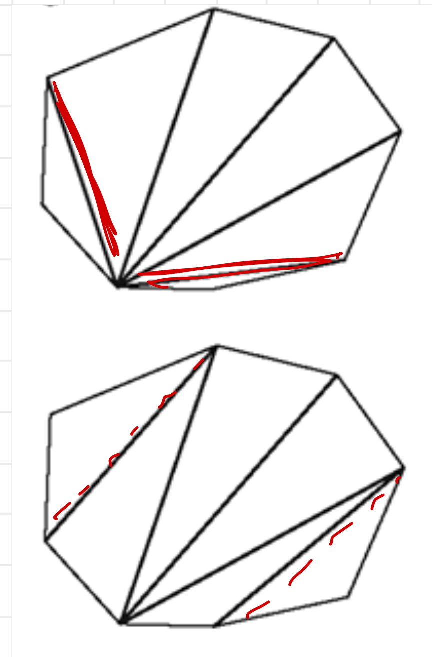
This example (generalized from 7 to n) shows that n^2 flips may be needed to get from one triangulation to another.



get from 0000.. 11111...1 ⟶ 11111.. 00000

$\frac{n}{2}$ 0's — each one must move $\frac{n}{2}$ steps to right.

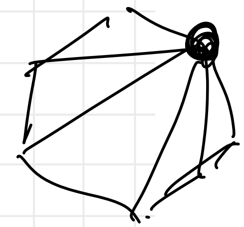
Flip distance for triangulations of a convex polygon.



In this case the flip distance is O(n) (easy) 2n-6
and there is a lower bound of 2n-6
(hard! — due to Sleator, Tarjan, Thurston, 1986).

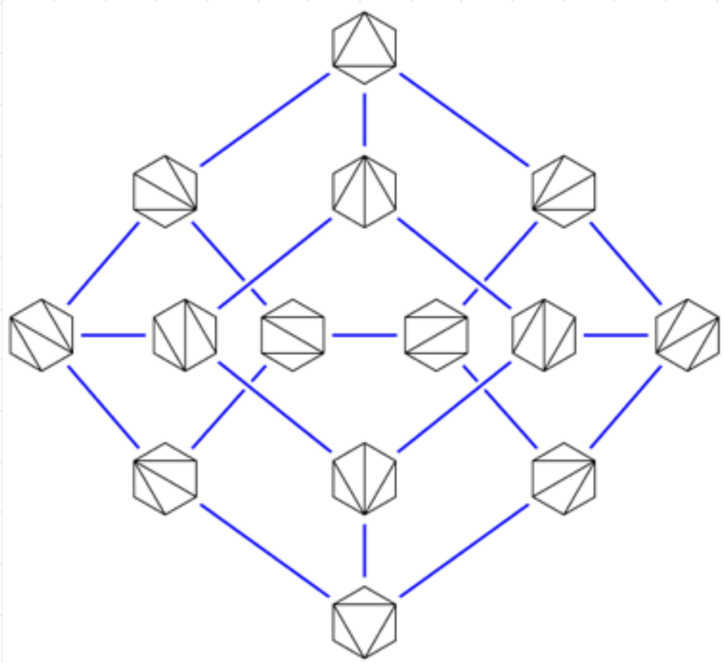
more recent version Pournin
– good project.

use as hub:



"star triangulation"

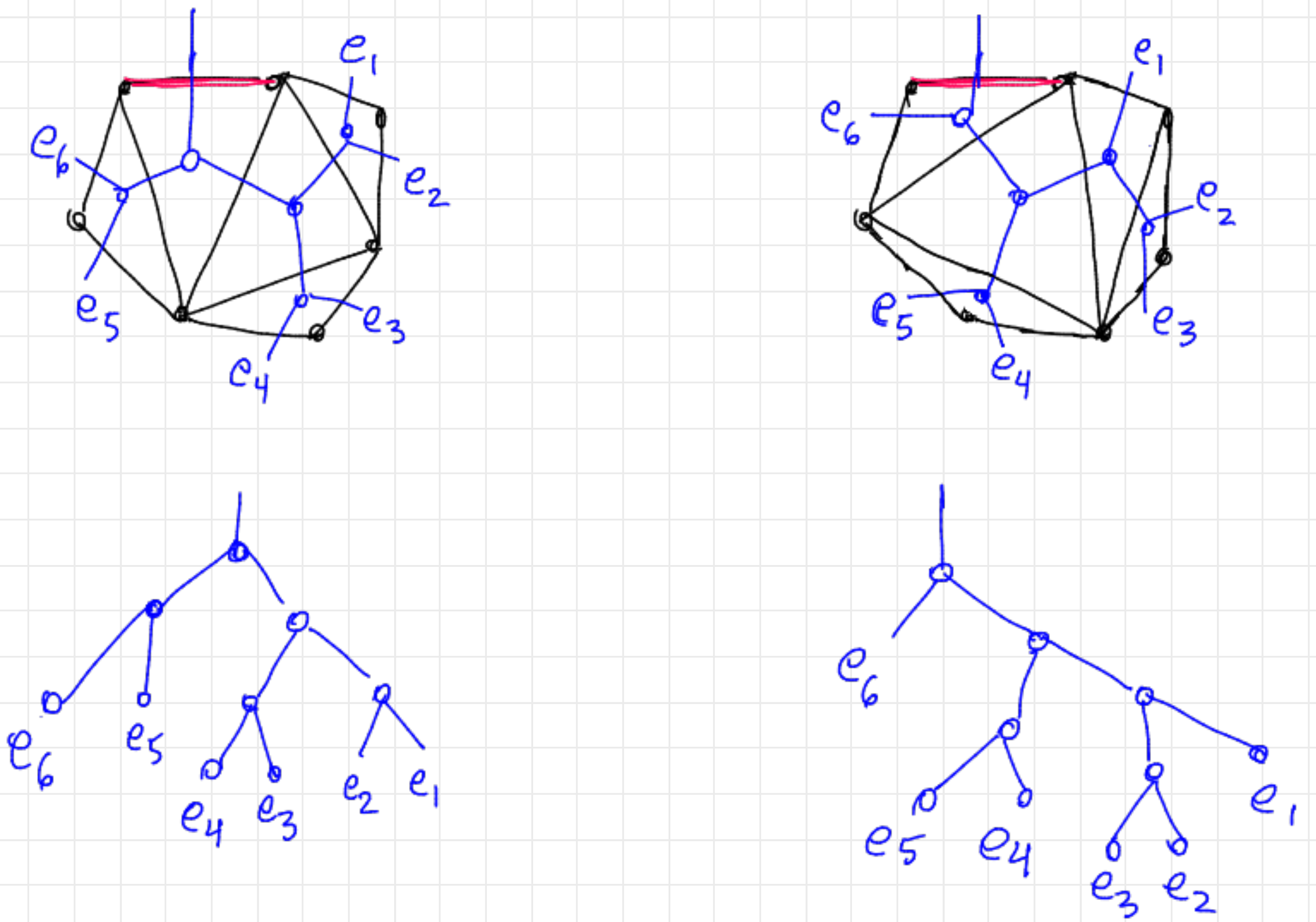
The reconfiguration graph is the "associahedron"



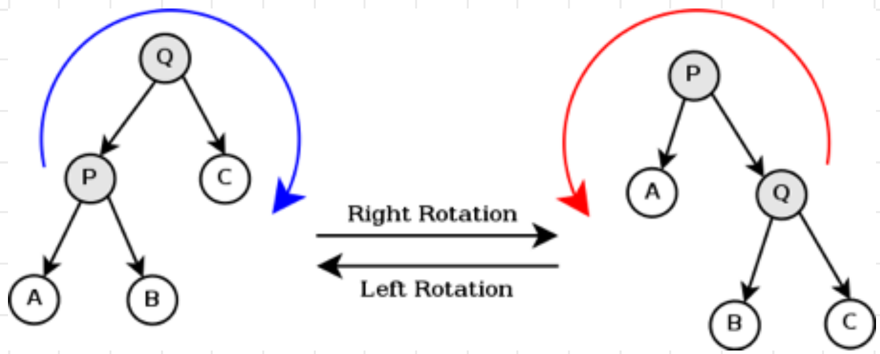
The reconfiguration graph of triangulations of a hexagon.

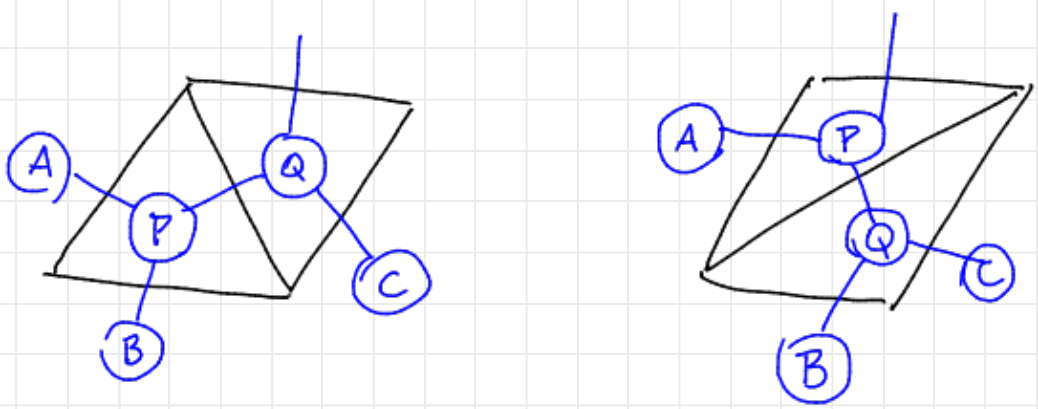
see Devadoss O'Rourke book

Flip distance for triangulations of a convex polygon
= rotation distance for binary search trees.



a flip corresponds to a rotation





OPEN. Is the following problem NP-complete or in P? Given number k and two triangulations of a convex polygon, is their flip distance <= k?

Why is rotation distance interesting?

- dynamic optimality conjecture for splay trees: splay trees perform within a constant factor of any offline rotation-based search tree algorithm
- distance between phylogenetic trees

Summary

- triangulations of point sets, possibly with fixed edges (“constrained”)
- angles, meshing, lengths, flipping, reconfiguration

References

- papers and books listed throughout